# Geometry Developed by the Electromagnetic Tensor Field (*). 

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Summary. - The well-known algebraic classification of the electromagnetic tensor field is used to provide the space-time manifold of general relativity with the latest technique of differential geometry.

## 1. - Introduction.

Consider the 4 -dimensional space-time manifold $V_{4}$ of general relativity, with the gravitational symmetric tensor field $h$ of rank four. This tensor is said to be of index of inertia 0,2 or 4 according as its signature ( ${ }^{1}$ ) is $(++--),(+++-)$ or ( ++++ ). Let ' $\boldsymbol{F}$ be the electromagnetic tensor field (skew symmetric) of type $(0,2)$. Using $h$, one can get a tensor field $F$ of type (1, 1) defined by

$$
\begin{equation*}
h(F X, Y)=' F(X, Y) \tag{1a}
\end{equation*}
$$

where $X$ and $I$ are arbitrary vector fields. Let us put

$$
\begin{equation*}
4 K=\operatorname{trace}\left(F^{2}\right), \quad k=\operatorname{det}(F), \quad D=K^{2}-k \tag{1b}
\end{equation*}
$$

It is well-known [5] that $F$ satisfies its own characteristic equation (minimum recurrent relation of $F$ ):

$$
\begin{equation*}
F^{4}+2 K F^{2}+k I=0 \tag{1c}
\end{equation*}
$$

$I$ is the identity operator. $V_{4}$ or $F$ is said to be of the
(a) first class if $k \neq 0$,
(b) second class if $k=0$ and $K \neq 0$,
(c) third class if $k=0, K=0$ and $F^{2} \neq 0$,
(d) fourth class if $k=0, K=0$ and $F^{2}=0$.
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${ }^{(1)}$ Physically, $h$ is necessarily of signature ( +++- ).

It is easy to see that ( $1 c$ ) reduces to $F^{3}+2 K F=0, F^{3}=0$ or $F^{2}=0$ according as $V_{4}$ is of the second, third or fourth class. $F$ is said to be non-null or null field according as it belongs to the first two classes or the last two classes.

For the non-null field there are exactly four linearly independent null eigenvectors $e_{\alpha}(1 \leqslant \alpha \leqslant 4)$ of $F$ which satisfy

$$
\begin{align*}
& F e_{1}=i \sqrt{\sqrt{D}+k} e_{1}, \quad F e_{2}=-i \sqrt{\sqrt{D}+k} e_{2}, \quad i=\sqrt{-1} \\
& F e_{3}=\sqrt{\sqrt{D}-k} e_{3}, \quad F e_{4}=-\sqrt{\sqrt{D}-k} e_{4} \tag{2}
\end{align*}
$$

For the third class, since all roots of the characteristic equation (1c) of $F$ are equal to zero, any set of four eigenvectors can not be linearly independent. However, it is known [6] that for this class, there exists a set of four linearly independent vectors $U_{\alpha}\left(U_{4}\right.$ is the only eigenvector belonging to the single eigenvalue 0 ) which give rise to a nonholonomic frame $\left.{ }^{(2}\right)$ such that $U_{1}, U_{2}$ are complex conjugate null vectors and $U_{3}, U_{4}$ are real null vectors. They satisfy the following relations.

$$
\begin{gather*}
\sqrt{2} F X=u^{3}(X)\left\{U_{1}+U_{2}\right\}+\left\{u^{1}(X)+u^{2}(X)\right\} U_{4}, \\
X=u^{\alpha}(X) U_{\alpha} . \tag{3a}
\end{gather*}
$$

Consequently,

$$
\begin{align*}
& \sqrt{2} F U_{1}=\sqrt{2} F U_{2}=U_{4}, \quad \sqrt{2} F U_{3}=U_{1}+U_{2}  \tag{3b}\\
& F^{2} X=u^{3}(X) U_{4}, \quad F U_{4}=0, \quad u^{3} F=0 \tag{3c}
\end{align*}
$$

where $\left\{u^{\alpha}\right\}$ is the dual of $\left\{U_{\alpha}\right\}$. Rank $F=4$ or 2 according as $F$ is of the first or second class. For the third class rank $F=2$ and rank $F^{2}=1$. Physically, the fourth class does not exist as $h$ must be of index 2 which means that $F^{2} \neq 0$. However, for the geometric interpretations, all the four classes can be discussed.

In this paper, we create a differentiable structure, called almost contingent [11-14] on $V_{4}$ by the help of $F$ and $h$. The importance of this structure is bourne out by the fact that it inherits the properties of now well-known structures such as almost complex [ 1,7$]$, almost product [4], globally framed [10], $G_{r}-[9]$ and almost tangent [8] all are obtained as its particular cases. The wealth of these structures has contributed many new and exciting results which we could not have obtained by classical way. The objective of this paper is to enrich the space-time manifold $V_{4}$ with this inherited geometry of substantial meaning by the technique of contingent structures.
$\left.{ }^{(2}\right)$ Hlavaty [3] showed the existence of such a frame by the use of line geometry.

## 2. - Almost contingent structures.

(I) Non-null kind. Consider a 4-dimensional differential manifold $M$ on which there exists a tensor field $J$ of type (1, 1), two linearly independent vector fields $U$ and $V$, two 1 -forms $u$ and $v$ and two arbitrary non-zero scalars $\lambda$ and $\varepsilon$ such that

$$
\begin{align*}
& J^{2} X+\lambda^{2} X+\varepsilon u(X) U+\varepsilon v(X) V=0 \\
& J U=P V, \quad J V=-P U, \quad P=\sqrt{\lambda^{2}+\varepsilon}  \tag{4a}\\
& 2 \leqslant \operatorname{rank} J \leqslant 4
\end{align*}
$$

for arbitrary vector fields $X$ and $Y$ on $M$.
In the above case, we say that $M$ is endowed with an almost contingent structure of non-null kind [12], briefly denoted by ( $J, U, V, u, v, \lambda, \varepsilon$ )-structure. Following relations can be deduced from (4a):

$$
\begin{equation*}
u J=-P v, \quad v J=P u, \quad u(U)=v(V)=1, \quad u(V)=v(U)=0 \tag{4b}
\end{equation*}
$$

We shall say that, $M$ is almost contingent metric manifold, when Riemannian metric ( ${ }^{3}$ ) $h$ on $M$ satisfies:

$$
\begin{align*}
& u(X)=h(U, X), \quad v(X)=h(V, X)  \tag{5a}\\
& h(J X, J Y)=\lambda^{2} h(X, Y)+\varepsilon u(X) u(Y)+\varepsilon v(X) v(Y) \tag{5b}
\end{align*}
$$

From defining equations (4) minimum recurrent relation of $J$ is:

$$
\begin{equation*}
J^{4}+\left(\lambda^{2}+P^{2}\right) J^{2}+\lambda^{2} P^{2} I=0 \tag{5c}
\end{equation*}
$$

$M$ or $J$ is said to be of the
A) first class if $P \neq 0$,
$B$ ) second class if $P=0$.
It is obvious that for the second class (5c) reduces to $J^{3}+\lambda^{2} J=0$. If we set $P=-2 \lambda$ and $\tilde{J}=J+\lambda(u(X) V-v(X) U)$, then it is easy to see that $\widetilde{J}^{2}=-\lambda^{2} I$. Thus $M$ has an underlying almost complex [1, 7] or an almost product [4] structure according as $\lambda^{2}=1$ or $\lambda^{2}=-1$ and $\operatorname{rank} \tilde{J}=4$. If $\lambda^{2}=1, \varepsilon=-1$ and rank $J=2$, then the equations (4) define a globally framed structure [10] on $M$.

Consequently, we assume that $(J, U, V, u, v, \lambda, \varepsilon)$-structure would inherit the geometry developed by above mentioned well-known structures.
$\left(^{3}\right)$ In general, $h$ may be of index 0,2 or 4.
(II) Null-find. $M$ is said to have an almost contingent structure of nullkind [12], briefly $\left(J, \xi_{1}, \xi_{2}, \eta^{1}, \eta^{2}, \sigma, \gamma\right)$-structure, if there exists on $M$ a tensor field ${ }^{(4)} J$ of type ( 1,1 ), two vector fields $\xi_{1}$ and $\xi_{2}$, two 1 -forms $\eta^{1}$ and $\eta^{2}$ and two arbitrary scalars $\sigma$ and $\gamma$ such that:

$$
\begin{align*}
& J^{2} X+\sigma^{2} X+\gamma \eta^{1}(X) \xi_{1}+\gamma \eta^{2}(X) \xi_{2}=0  \tag{6a}\\
& J \xi_{1}=0, \quad J \xi_{2}=0, \quad \eta^{1} J=0, \quad \eta^{2} J=0, \quad \operatorname{rank} J=2
\end{align*}
$$

Minimum recurrent relation for the powers of $J$ is:

$$
\begin{equation*}
J^{3}+\sigma^{2} J=0 \tag{6b}
\end{equation*}
$$

Riemannian metric $h$ on $M$ satisfies conditions similar to ( $5 a b$ ). $M$ or $J$ is said to be of the
(C) third class if $\sigma=0$ and $\gamma \neq 0$,
( $D$ ) fourth class if $\sigma=0$ and $\gamma=0$.
It is obvious that ( $6 b$ ) reduces to $J^{3}=0$ or $J^{2}=0$ according as $M$ is of the third or fourth class. As examples, $M$ has a $G_{2}$-structure [9] or an almost tangent structure [8] according as it belongs to the third or the fourth class. In general, the structures of this nature for an arbitrary nilpotent operator $J$ satisfying $J^{r+1}=0$ are called $G_{r}$-structures [9]. Thus, we assume that the ( $\left.J, \xi_{1}, \xi_{2}, \eta^{1}, \eta^{2}, 0, \gamma\right)$-structure would inherit the geometry developed by the $G_{r}$-structures. For further details on the almost contingent structures, we refer to [11-14].

## 3. - Central idea.

In this section, we show how the differential geometric objects which define an almost contingent structure can be related to the gravitational and the electromagnetic tensor fields of general relativity. Assume that $M$ is the space-time manifold $V_{4}$ with $h$ and ' $F$ satisfying all the relations (1)-(3). Let $h$ be used as the fundamental metric tensor of $V_{4}$. It is evident from (1a) that there exists a tensor field $F$ of type $(1,1)$ on $V_{4}$. We wish to show that $F$ gives rise to an almost contingent structure on $\nabla_{4}$.
(I) Non-null fields. Let us consider two operators $l$ and $\tilde{l}$ defined by

$$
\begin{equation*}
2 \sqrt{D} l=-F^{2}-(K-\sqrt{D}) I, \quad 2 \sqrt{D} \tilde{l}=F^{2}+(K+\sqrt{D}) I \tag{7a}
\end{equation*}
$$

[^0]$I$ being the identity operator, to the tangent space $T_{x}$ at a point $x$ of $V_{4}$. These are complementary projection operators. Indeed, $l+\tilde{l}=I$ and using (1c) one can show that $\tilde{l}=0$. Thus, there exist complementary distributions $L$ and $\tilde{L}$ corresponding to $l$ and $\tilde{l}$ such that $\operatorname{dim} L=\operatorname{dim} \tilde{L}=2$. Assume that $L$ is parallelizable [10] which allows us to take an ordered set of vector fields $U$ and $V$ spanning $L$ at each point. Thus, there exists uniquely an ordered set of 1 -forms $u$ and $v$ such that $l(X)=u(X) U+v(X) V, u(U)=v(V)=1, u(V)=v(U)=0, F^{2} U=-(K+\sqrt{D}) U$ and $F^{2} V=-(K+\sqrt{D}) V$. Using these results and the first equation of (7a), we get
\[

$$
\begin{align*}
& F^{2} X+(K-\sqrt{D}) X+2 \sqrt{D}\{u(X) U+v(X) V\}=0 \\
& F U=\sqrt{K+\sqrt{D}} \nabla, \quad F V=-\sqrt{K+\sqrt{D}} U, \quad 2 \leqslant \operatorname{rank} F \leqslant 4 \tag{7b}
\end{align*}
$$
\]

Now assume that $\tilde{L}$ is parallelizable which is spanned by vector fields $\tilde{U}$ and $\bar{V}$ and $\tilde{u}, \tilde{v}$ are respectively 1 -forms such that $\tilde{l}(X)=\tilde{u}(X) \tilde{U}+\tilde{v}(X) \tilde{V}, \tilde{u}(\tilde{U})=\tilde{v}(\tilde{V})=1$, $\tilde{u}(\tilde{V})=\tilde{v}(\widetilde{U})=0, F^{2} \widetilde{U}=-(K-\sqrt{D}) \tilde{U}$ and $F^{2} \tilde{V}=-(K-\sqrt{D}) \tilde{V}$. Using these results and the second equation of (7a), we get

$$
F^{2} X+(K+\sqrt{D}) X-2 \sqrt{D}\{\tilde{u}(X) \tilde{U}+\tilde{v}(X) \tilde{V}\}=0
$$

$$
\begin{equation*}
F \tilde{U}=\sqrt{K-\sqrt{D}} \tilde{V}, \quad F \tilde{V}=-\sqrt{K-\sqrt{D}} \tilde{U} \quad 2 \leqslant \operatorname{rank} F \leqslant 4 \tag{7c}
\end{equation*}
$$

Comparing ( $7 b$ ) and ( $7 c$ ) with ( $4 a$ ), we conclude that, in general, there exist two almost contingent structures of non-null kind defined by ( $F, U, V, u, v, K, \sqrt{D}$ ) or ( $F, \tilde{U}, \tilde{V}, \tilde{u}, \tilde{v}, K, \sqrt{D}$ ) according as $P=\sqrt{K+\sqrt{D}}$ or $P=\sqrt{K-\sqrt{D}}$. This allows us to replace ( $1 a$ ) by:

$$
\begin{equation*}
h(J X, Y)==^{\prime} F(X, Y) \tag{8a}
\end{equation*}
$$

Now we wish to find an explicit relationship (locally) between $U, V, \tilde{U}, \tilde{V}$ and the eigenvectors of $F$. The existence of complementary distributions $L$ and $\tilde{L}$ allows us to adjust so that $\{U, V\}$ and $\{\tilde{U}, \tilde{V}\}$ are in the plane of $\left\{e_{1}, e_{2}\right\}$ and $\left\{e_{3}, e_{4}\right\}$ respectively. Thus, comparing (2) with (7bc), we get

$$
\begin{array}{llll}
U=\frac{e_{1}+e_{2}}{\sqrt{2}}, & V=\frac{i\left(e_{1}-e_{2}\right)}{\sqrt{2}}, & \tilde{U}=\frac{e_{3}+e_{4}}{\sqrt{2}}, & \tilde{V}=\frac{i\left(e_{3}-e_{4}\right)}{\sqrt{2}} \\
e_{1}=\frac{U-i V}{\sqrt{2}}, & e_{2}=\frac{U+i V}{\sqrt{2}}, & e_{3}=\frac{\tilde{U}-i \tilde{V}}{\sqrt{2}}, & e_{4}=\frac{\tilde{U}+i \tilde{V}}{\sqrt{2}} \tag{8b}
\end{array}
$$

Thus, (7) and (8) relate the differential geometric objects ( $J, h, U, V, \tilde{U}, \tilde{V}$ ) defining almost contingent structures of non-null kind to the gravitational field $h$, the electromagnetic field ' $F$ and the eigenvectors of $F$.
(II) Null fields. Let us assume that $F$ is of class three. This means that $F^{3}=0, K=0, k=0, \operatorname{rank} F=2$ and rank $F^{2}=1$. We apply two operators $\pi$ and $\tilde{\pi}$ defined by

$$
\begin{equation*}
\gamma \pi=F^{2}+\gamma I, \quad \gamma \tilde{\pi}=-F^{2} \tag{9a}
\end{equation*}
$$

$\gamma$ is an arbitrary non-zero scalar, to the tangent space $T_{x}$ at a point $x$ of $V_{4}$. $\pi+\tilde{\pi}=I, \pi F=F \pi=F$ and $\tilde{\pi} F=F \tilde{\pi}=0$. Thus there exist two complementary spaces $C=\operatorname{Image}(F)$ and $\tilde{C}=\operatorname{Image}\left(F^{2}\right)=\operatorname{Ker}(F)$ corresponding to $\pi$ and $\tilde{\pi}$ respectively. Let us assume that there exists a vector field $\xi$ and a 1 -form $\eta$ of $\tilde{C}$ such that $\tilde{\pi}(X)=\eta(X) \xi$. Using this in (9a) $F \tilde{\pi}=0$ we get

$$
\begin{equation*}
F^{2}(X)+\gamma \eta(X) \xi=0, \quad F \xi=0, \quad \eta F=0, \quad \operatorname{rank} F=2 \tag{9b}
\end{equation*}
$$

Comparing (9b) with (3c) and ( $6 a$ ), we conclude that ( $F, \eta, \xi, \gamma$ ) describes an almost contingent structure of class three on $V_{4}$ such that $F$ is identified with $J, \xi=\xi_{1}=U_{4}$ is the only eigenvector of $F^{F}$ belonging to its eigenvalue $0, \eta=u^{3}, \gamma=-1$ and $\xi_{2}=0$.

Now let us assume that $F$ is of class four. This means that $F^{2}=0$ and rank $F=2$. Thus, $F$ of the fourth class admits an almost contingent structure of the same class.
4. - (i) Index of inertia 4. Since $h$ is of signature $(++++)$, $\operatorname{det}(h)$ is positive. This means that $k \geqslant 0$ and $K>0$. Therefore, only the first two classes exist. For the first class $D$ may be non-zero or zero. If $D \neq 0$ then there are two almost contingent structures defined by (7b) and (7c). If $D=0$ then ( $7 b c$ ) reduces to $F^{2} X+K X=0$ which defines a generalized almost complex structure [2]. For the second class $k=0$ and $D=K^{2}>0$ provides only one almost contingent structure defined by (7c).
(ii) Index of inertia 2. Since $h$ is of signature $(+++-)$, $\operatorname{det}(h)$ is negative. This means that $k \leqslant 0$ and $D \geqslant 0$. For the first class $D>0$ which provides two structures defined by ( $7 b$ ) and ( $7 c$ ). For the second class $D>0$ and, therefore, the defining equations for the structure are ( $7 b$ ) or ( $7 c$ ) according as $K<0$ or $K>0$. For the third class, the structure is defined by (9b). Fourth class does not exist.
(iii) Index of inertia 0 . Since $h$ is of signature $(++--)$, $\operatorname{det}(h)$ is positive and $k \geqslant 0$. As there is no restriction on $K$ and $D$, all the four classes exist. The properties of first and second class are identical with the respective classes of (i) and (ii).

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[^0]:    $\left.{ }^{( }{ }^{4}\right)$ For the sake of simplicity, $J$ is used for both kinds.

