On the Oscillatory and Asymptotic Behavior of the Bounded Solutions of Differential Equations with Deviating Arguments (*) (**).

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Summary. – *This paper is dealing with the oscillatory and asymptotic behavior of the bounded solutions of n -th order ($n > 1$) differential equations with deviating arguments involving the so called r -derivatives $D_r^{(i)}x$ ($i = 0, 1, \dots, n$) of the unknown function x defined by*

$$D_r^{(0)}x = x, \quad D_r^{(i)}x = r_i(D_r^{(i-1)}x)' \quad (i = 1, 2, \dots, n-1) \quad \text{and} \quad D_r^{(n)}x = (D_r^{(n-1)}x)',$$

where r_i ($i = 1, 2, \dots, n-1$) are positive continuous functions on the interval $[t_0, \infty)$. The fundamental purpose is to find a necessary and sufficient condition in order to have at least one (bounded nonoscillatory) solution whose the limit at ∞ exists in $\mathbf{R} - \{0\}$.

1. – Introduction.

Let r_i ($i = 0, 1, \dots, n$) be positive continuous functions on the interval $[t_0, \infty)$. For a real-valued function h on $[T, \infty)$, $T \geq t_0$, and any $\mu = 0, 1, \dots, n$ the μ -th r -derivative of h is defined by the formula

$$D_r^{(\mu)}h = r_\mu \left(r_{\mu-1} \left(\dots \left(r_1 (r_0 h)' \right)' \dots \right)' \right) ' .$$

Then we obviously have

$$D_r^{(0)}h = r_0 h \quad \text{and} \quad D_r^{(i)}h = r_i (D_r^{(i-1)}h)' \quad (i = 1, 2, \dots, n).$$

If $D_r^{(n)}h$ is defined on $[T, \infty)$, then the function h is called n -times r -differentiable and, if in addition $D_r^{(n)}h$ is continuous, h is called n -times continuously r -differentiable. We note that in the case where $r_0 = r_1 = \dots = r_n = 1$ the above notion of r -differentiability specializes to the usual one.

Now, we consider the n -th order ($n > 1$) differential equation with deviating arguments of the form

$$(E) \quad (D_r^{(n)}x)(t) + a(t) \Phi \{ x \langle g_0(t) \rangle, (D_r^{(1)}x) \langle g_1(t) \rangle, \dots, (D_r^{(N)}x) \langle g_N(t) \rangle \} = b(t),$$

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where $r_0 = r_n = 1$, N is an integer with $0 \leq N \leq n-1$ and

$$(D_r^{(i)}x)\langle g_i(t) \rangle = ((D_r^{(i)}x)[g_{i1}(t)], (D_r^{(i)}x)[g_{i2}(t)], \dots, (D_r^{(i)}x)[g_{im_i}(t)]) ,$$

$$g_i = (g_{i1}, g_{i2}, \dots, g_{im_i}) \quad (i = 0, 1, \dots, N) .$$

The continuity of the real-valued functions a , g_{ij} ($j = 1, 2, \dots, m_i$; $i = 0, 1, \dots, N$) and b on $[t_0, \infty)$ and Φ on $\mathbf{R}^{m_0} \times \mathbf{R}^{m_1} \times \dots \times \mathbf{R}^{m_N}$ as well as sufficient smoothness for the existence of solutions of (E) on an infinite subinterval of $[t_0, \infty)$ will be assumed without mention. In what follows the term « solution » is always used only for such solutions $x(t)$ of (E) which are defined for all large t . The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T_0, \infty)$ is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

Furthermore, the conditions (i) and (ii) below are assumed to hold throughout the paper:

(i) For $i = 0, 1, \dots, N$ and every $j = 1, 2, \dots, m_i$

$$\lim_{t \rightarrow \infty} g_{ij}(t) = \infty .$$

(ii) The function Φ has the sign property

$$\begin{cases} (\forall j = 1, 2, \dots, m_0) y_{0j} > 0 \Rightarrow \Phi(y_0, y_1, \dots, y_N) > 0 \\ (\forall j = 1, 2, \dots, m_0) y_{0j} < 0 \Rightarrow \Phi(y_0, y_1, \dots, y_N) < 0 , \end{cases}$$

where $y_i \in \mathbf{R}^{m_i}$ ($i = 0, 1, \dots, N$) and $y_0 = (y_{01}, y_{02}, \dots, y_{0m_0})$.

For such differential equations involving the operator $D_r^{(n)}$ there is recently an increasing interest in studying the oscillatory and asymptotic behavior of the solutions. We choose to refer here to the papers by KUSANO and ONOSE [2, 3], PHILOS [4, 5, 6], PHILOS and STAIKOS [7, 8], STAIKOS and PHILOS [11, 12, 13] and TRENCH [14]. As one can see in the book of COPPEL [1], the class of operators $D_r^{(n)}$ properly contains the disconjugate operators L ,

$$Lx = x^{(n)} + p_1 x^{(n-1)} + \dots + p_{n-1} x + p_n \quad (p_i \text{ continuous functions}).$$

2. - Main results.

In order to obtain our first theorem (Theorem 1) we shall apply the fixed point technique, by using the following Schauder's theorem (see SCHAUDER [9]).

THE SCHAUDER THEOREM. - *Let E be a Banach space and X any nonempty convex and closed subset of E . If S is a continuous mapping of X into itself and SX is relatively compact, then the mapping S has at least one fixed point (i.e. there exists an $x \in X$ with $x = Sx$).*

A set $\mathcal{F}$ of real-valued functions defined on the interval $[T, \infty)$ is said to be *equi-convergent at ∞* if all functions in $\mathcal{F}$ are convergent in $\mathbf{R}$ at the point ∞ and, **more-over**, for every $\varepsilon > 0$ there exists a $T' \geq T$ such that, for all $f \in \mathcal{F}$,

$$t \geq T' \Rightarrow |f(t) - \lim_{s \rightarrow \infty} f(s)| < \varepsilon.$$

Let now $B([T, \infty))$ be the Banach space of all continuous and bounded real-valued functions on the interval $[T, \infty)$, endowed with the usual sup-norm $\| \cdot \|$. We need the following compactness criterion for subsets of $B([T, \infty))$, which is a corollary of the Arzela-Ascoli theorem. For a proof of this criterion we refer to STAIKOS [10].

COMPACTNESS CRITERION. — *Let $\mathcal{F}$ be an equicontinuous and uniformly bounded subset of the Banach space $B([T, \infty))$. If $\mathcal{F}$ is equiconvergent at ∞ , it is also relatively compact.*

THEOREM 1. — *Consider the differential equation (E) subject to the conditions (i), (ii) and:*

(C₁) *There exists an n -times continuously r -differentiable function w on $[t_0, \infty)$ with $D_r^{(n)}w = b$ and such that*

$$A = \max_{0 \leq i \leq N} \limsup_{t \rightarrow \infty} |(D_r^{(i)}w)(t)| < \infty.$$

$$(C_2) \quad \int_{s_1}^{\infty} \frac{1}{r_1(s_1)} \int_{s_2}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |a(s)| ds ds_{n-1} \cdots ds_2 ds_1 < \infty.$$

Then for every real number L with $|L| > 2A$ there exists a (bounded nonoscillatory) solution x of the equation (E) with

$$\lim_{t \rightarrow \infty} [x(t) - w(t)] = L$$

and

$$\lim_{t \rightarrow \infty} [(D_r^{(i)}x)(t) - (D_r^{(i)}w)(t)] = 0 \quad (i = 1, 2, \dots, n-1).$$

PROOF. — Let L be a real number with $|L| > 2A$. Without loss of generality, we assume that L is positive, since the substitution $z = -x$ transforms (E) into an equation of the same form satisfying the assumptions of the theorem. Moreover, we suppose that $L > 2B$, where

$$B = \max_{0 \leq i \leq N} \sup_{t \geq T_0} |(D_r^{(i)}w)(t)|$$

for some $T_0 \geq t_0$. Furthermore, we set

$$K_0 = \left\{ y_0 = (y_{01}, y_{02}, \dots, y_{0m_0}) \in \mathbf{R}^{m_0} : (\forall j = 1, 2, \dots, m_0) \frac{L}{2} \leq y_{0j} \leq \frac{3L}{2} \right\}$$

and, if $N > 0$, for $i = 1, 2, \dots, N$

$$K_i = \left\{ y_i = (y_{i1}, y_{i2}, \dots, y_{im_i}) \in \mathbf{R}^{m_i} : (\forall j = 1, 2, \dots, m_i) |y_{ij}| \leq \frac{L}{2} \right\}.$$

Then, because of (ii), we have

$$M = \max_{(y_0, y_1, \dots, y_N) \in K_0 \times K_1 \times \dots \times K_N} \Phi(y_0, y_1, \dots, y_N) > 0.$$

Next, by conditions (i) and (C_2) , we choose a $T \geq T_0$ so that

$$g_{ij}(t) \geq T_0 \quad \text{for every } t \geq T \quad (j = 1, 2, \dots, m_i; i = 0, 1, \dots, N)$$

and for any i , $0 \leq i \leq N$,

$$(1) \quad \begin{cases} \int_T^\infty |a(s)| ds \leq \frac{1}{2M} (L - 2B), & \text{if } i = n - 1 \\ \int_T^\infty \frac{1}{r_{i+1}(s_{i+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \dots ds_{i+1} \leq \frac{1}{2M} (L - 2B), \\ & \text{if } i < n - 1. \end{cases}$$

Now, let E be the Banach space of all functions on the interval $[T, \infty)$ which have bounded continuous N -th r -derivative on $[T, \infty)$, endowed with the norm $\| \cdot \|$ defined by

$$\|h\| = \begin{cases} \|h\|, & \text{if } N = 0 \\ \|D_r^{(N)} h\| + \sum_{i=0}^{N-1} |(D_r^{(i)} h)(T)|, & \text{if } N > 0. \end{cases}$$

Furthermore, let the subset X of E consisting of all functions $x \in E$ with

$$|x(t) - L| \leq \frac{L}{2} \quad \text{for every } t \geq T$$

and, if $N > 0$,

$$|(D_r^{(i)} x)(t)| \leq \frac{L}{2} \quad \text{for every } t \geq T \quad (i = 1, 2, \dots, N).$$

The set X is obviously *nonempty* and, as it is easy to see, it is *convex* and *closed*.

Let $x \in X$. By the definition of the set X , for every $t \geq T_0$ we obviously have

$$\begin{cases} \frac{L}{2} \leq x^*(t) \leq \frac{3L}{2} \\ |(D_r^{(i)} x)^*(t)| \leq \frac{L}{2} \quad (i = 1, 2, \dots, N), \text{ provided that } N > 0, \end{cases}$$

where for $i = 0, 1, \dots, N$

$$(D_r^{(i)} x)^*(t) = \begin{cases} (D_r^{(i)} x)(t), & \text{if } t \geq T \\ (D_r^{(i)} x)(T), & \text{if } T_0 \leq t \leq T. \end{cases}$$

So, from (ii) and the definition of the constant M it follows that

$$(2) \quad 0 < \Phi(x^* \langle g_0(t) \rangle, (D_r^{(1)} x)^* \langle g_1(t) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(t) \rangle) \leq M$$

for all $t \geq T$. Thus, because of condition (C_2) , for any $x \in X$ and every $t \geq T$

$$\begin{aligned} & \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ & \quad \int_{s_{n-1}}^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)} x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_2 ds_1 < \infty \end{aligned}$$

and hence it is easy to see that the formula

$$\begin{aligned} (Sx)(t) = L + w(t) + (-1)^{n-1} & \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ & \int_{s_{n-1}}^\infty a(s) \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)} x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_2 ds_1, \quad t \geq T \end{aligned}$$

defines a mapping $S: X \rightarrow E$. The mapping S satisfies the assumptions of the Schauder theorem. Namely, it satisfies the following:

$$\alpha) \quad SX \subseteq X.$$

In fact, taking into account (2) and (1), for any $x \in X$ and every $t \geq T$ we obtain

$$\begin{aligned} |(Sx)(t) - L| & \leq |w(t)| + \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ & \quad \int_{s_{n-1}}^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)} x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_2 ds_1 \\ & \leq \sup_{t \geq T} |w(t)| + M \int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ & \quad \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \dots ds_2 ds_1 \leq B + M \frac{1}{2M} (L - 2B) = \frac{L}{2}. \end{aligned}$$

Furthermore, if $N > 0$, then, by virtue again of (2) and (1), for $x \in X$, $t \geq T$ and any i , $1 \leq i \leq N$, we get

$$\begin{aligned} |(D_r^{(i)}(Sx))(t)| &\leq |(D_r^{(i)}w)(t)| + \int_t^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)}x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)}x)^* \langle g_N(s) \rangle) ds \\ &\leq \sup_{t \geq T} |(D_r^{(i)}w)(t)| + M \int_T^\infty |a(s)| ds \leq B + M \frac{1}{2M} (L - 2B) = \frac{L}{2}, \end{aligned}$$

if $i = n - 1$, and

$$\begin{aligned} |(D_r^{(i)}(Sx))(t)| &\leq |(D_r^{(i)}w)(t)| + \int_t^\infty \frac{1}{r_{i+1}(s_{i+1})} \dots \int_{s_{n-1}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ &\quad \int_{s_{n-1}}^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)}x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)}x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_{i+1} \\ &\leq \sup_{t \geq T} |(D_r^{(i)}w)(t)| + M \int_T^\infty \frac{1}{r_{i+1}(s_{i+1})} \dots \int_{s_{n-1}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ &\quad \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \dots ds_{i+1} \leq B + M \frac{1}{2M} (L - 2B) = \frac{L}{2}, \quad \text{if } i < n - 1. \end{aligned}$$

β) SX is relatively compact.

If $x \in X$ and $f = D_r^{(N)}(Sx) - D_r^{(N)}w$, then, because of the fact that $SX \subseteq X$ and the definition of X , for every $t \geq T$ we have

$$|f(t)| \leq |(D_r^{(N)}(Sx))(t)| + |(D_r^{(N)}w)(t)| \leq \frac{3L}{2} + B.$$

So, the set

$$\mathcal{F} = \{D_r^{(N)}(Sx) - D_r^{(N)}w : x \in X\}$$

is an uniformly bounded subset of the Banach space $B([T, \infty))$. Furthermore, for any function f in $\mathcal{F}$ and every $t \geq T$

$$(3) \quad |\hat{f}(t)| \leq \begin{cases} M \int_t^\infty |a(s)| ds, & \text{if } N = n - 1 \\ \int_t^\infty \frac{1}{r_{N+1}(s_{N+1})} \dots \int_{s_{n-1}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \dots ds_{N+1}, & \text{if } N < n - 1, \end{cases}$$

where

$$\hat{f}(t) = \begin{cases} f(t) - L, & \text{if } N = 0 \\ f(t), & \text{if } N > 0. \end{cases}$$

Indeed, if $f = D_r^{(N)}(Sx) - D_r^{(N)}w$, $x \in X$, for every $t \geq T$ we obtain

$$|f(t)| = \begin{cases} |[(Sx)(t) - w(t)] - L|, & \text{if } N = 0 \\ |(D_r^{(N)}(Sx))(t) - (D_r^{(N)}w)(t)|, & \text{if } N > 0 \end{cases}$$

$$\leq \begin{cases} \int_t^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)}x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)}x)^* \langle g_N(s) \rangle) ds, & \text{if } N = n-1 \\ \int_t^\infty \frac{1}{r_{N+1}(s_{N+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)}x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)}x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_{N+1}, & \text{if } N < n-1 \end{cases}$$

and hence, by virtue of (2), (3) follows immediately. From (3) and condition (C_2) it follows that the set $\mathcal{F}$ is equiconvergent at ∞ . Now, by using again (2), for any function $f \in \mathcal{F}$ and every t_1, t_2 with $T \leq t_1 \leq t_2$ we can easily obtain

$$|f(t_2) - f(t_1)| \leq \begin{cases} M \int_{t_1}^{t_2} |a(s)| ds, & \text{if } N = n-1 \\ M \int_{t_1}^{t_2} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1}, & \text{if } N = n-2 \\ M \int_{t_1}^{t_2} \frac{1}{r_{N+1}(s_{N+1})} \int_{s_{N+1}}^\infty \frac{1}{r_{N+2}(s_{N+2})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \dots ds_{N+2} ds_{N+1}, & \text{if } N < n-2. \end{cases}$$

By the last inequality, it is easy to verify that $\mathcal{F}$ is equicontinuous. Finally, by the compactness criterion, we conclude that the set $\mathcal{F}$ is relatively compact in $B([T, \infty))$. This, because of the definition of the norm $||$ in E and the boundedness of the function $D_r^{(N)}w$, implies the relative compactness of the set SX .

γ) *The mapping S is continuous.*

Let $x \in X$ and (x_ν) be an arbitrary sequence in X with

$$|| - \lim x_\nu = x.$$

Then it is easy to verify that for every $t \geq T_0$

$$\lim_{\nu} (D_r^{(i)}x_\nu)^*(t) = (D_r^{(i)}x)^*(t) \quad (i = 0, 1, \dots, N).$$

On the other hand, by (2), for all ν and every $t \geq T$ we have

$$|a(t) \Phi(x_\nu^* \langle g_0(t) \rangle, (D_r^{(1)} x_\nu)^* \langle g_1(t) \rangle, \dots, (D_r^{(N)} x_\nu)^* \langle g_N(t) \rangle)| \leq M |a(t)|.$$

Thus, because of condition (C_2) , we can apply the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned} \lim_{\nu} \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ \int_{s_{n-1}}^\infty a(s) \Phi(x_\nu^* \langle g_0(s) \rangle, (D_r^{(1)} x_\nu)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x_\nu)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_2 ds_1 \\ = \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \\ \int_{s_{n-1}}^\infty a(s) \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)} x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_2 ds_1 \end{aligned}$$

for all $t \geq T$. So, for every $t \geq T$ we have the pointwise convergence

$$\lim_{\nu} (Sx_\nu)(t) = (Sx)(t).$$

It remains to prove that

$$(4) \quad || - \lim Sx_\nu = Sx.$$

To this end, we consider any subsequence (u_μ) of (Sx_ν) . Because of the relative compactness of SX , there exist a subsequence (v_λ) of (u_μ) and a $y \in E$ so that

$$|| - \lim v_\lambda = y.$$

Since $||$ -convergence implies the pointwise one to the same limit function, we must always have $y = Sx$, which proves (4).

Finally, by the Schauder theorem, there exists an $x \in X$ with $x = Sx$. Then, in view of condition (C_1) , we have

$$\begin{aligned} (D_r^{(n)} x)(t) &= (D_r^{(n)} w)(t) - a(t) \Phi(x^* \langle g_0(t) \rangle, (D_r^{(1)} x)^* \langle g_1(t) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(t) \rangle) \\ &= b(t) - a(t) \Phi(x^* \langle g_0(t) \rangle, (D_r^{(1)} x)^* \langle g_1(t) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(t) \rangle) \end{aligned}$$

for every $t \geq T$. Thus, the fixed point x of the mapping S is a solution on $[T, \infty)$

of the equation (E). Moreover, this solution is the required one, since

$$| [x(t) - w(t)] - L | \leq \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)} x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_2 ds_1 \rightarrow 0 \text{ as } t \rightarrow \infty$$

and for any i , $1 \leq i \leq n-1$,

$$| (D_r^{(i)} x)(t) - (D_r^{(i)} w)(t) | \leq \begin{cases} \int_t^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)} x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(s) \rangle) ds, & \text{if } i = n-1 \\ \leq \int_t^\infty \frac{1}{r_{i+1}(s_{i+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| \Phi(x^* \langle g_0(s) \rangle, (D_r^{(1)} x)^* \langle g_1(s) \rangle, \dots, (D_r^{(N)} x)^* \langle g_N(s) \rangle) ds ds_{n-1} \dots ds_{i+1}, & \text{if } i < n-1 \end{cases}$$

$\rightarrow 0$ as $t \rightarrow \infty$.

Corollary I below follows immediately from Theorem 1.

COROLLARY I. — Consider the differential equation (E) subject to the conditions (i), (ii), (C_2) and:

(C'_1) There exists an n -times continuously r -differentiable function w on $[t_0, \infty)$ with $D_r^{(n)} w = b$ and such that

$$\lim_{t \rightarrow \infty} (D_r^{(i)} w)(t) = 0 \quad (i = 0, 1, \dots, N).$$

Then for every real number L with $L \neq 0$ there exists a (bounded nonoscillatory) solution x of the equation (E) with

$$\lim_{t \rightarrow \infty} x(t) = L$$

and, if $N > 0$,

$$\lim_{t \rightarrow \infty} (D_r^{(i)} x)(t) = 0 \quad (i = 1, 2, \dots, N).$$

In particular, for $b = 0$ and $w = 0$, condition (C_1) is satisfied. Thus, from Theorem 1 we immediately obtain the following corollary concerning the differential equation

$$(E_0) \quad (D_r^{(n)} x)(t) + a(t) \Phi(x \langle g_0(t) \rangle, (D_r^{(1)} x) \langle g_1(t) \rangle, \dots, (D_r^{(N)} x) \langle g_N(t) \rangle) = 0.$$

COROLLARY II. — Consider the differential equation (E_0) subject to the conditions (i), (ii) and (C_2) . Then for every real number L with $L \neq 0$ there exists a (bounded nooscilla-

tory) solution x of the equation (E_0) with

$$\lim_{t \rightarrow \infty} x(t) = L \quad \text{and} \quad \lim_{t \rightarrow \infty} (D_r^{(i)} x)(t) = 0 \quad (i = 1, 2, \dots, n-1).$$

In our previous results (Theorem 1 and its corollaries) no restriction on the sign of the function a is assumed. Moreover, the only assumption on the functions r_i ($i = 1, 2, \dots, n-1$) is that they are positive and continuous on the interval $[t_0, \infty)$. Now, in order to obtain sufficient conditions for the oscillatory and asymptotic behavior of the bounded solutions of the differential equation (E_0) we shall restrict ourselves in the case where the function a is of constant sign on $[t_0, \infty)$ and the condition (R) below is satisfied:

(R) For every $i = 1, 2, \dots, n-1$

$$\int \frac{dt}{r_i(t)} = \infty.$$

We need the following lemma due to PHILOS [4, Lemmas 1 and 2].

LEMMA. — Suppose that condition (R) is satisfied. Let $\delta = \pm 1$ and h be a positive bounded and n -times r -differentiable function on the interval $[T, \infty)$, $T \geq t_0$, with

$$\delta(D_r^{(n)} h)(t) \leq 0 \quad \text{for every } t \geq T.$$

Then

$$\delta(-1)^{n+1+i} (D_r^{(i)} h)(t) \geq 0 \quad \text{for every } t \geq T \quad (i = 1, 2, \dots, n-1),$$

$$\lim_{t \rightarrow \infty} (D_r^{(i)} h)(t) = 0 \quad (i = 1, 2, \dots, n-1)$$

and, provided that $D_r^{(n)} h$ is continuous on $[T, \infty)$,

$$\int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |(D_r^{(n)} h)(s)| ds ds_{n-1} \dots ds_2 ds_1 < \infty.$$

THEOREM 2. — Consider the differential equation (E_0) subject to the conditions (i), (ii), (R) and:

(C_3) There exists an integer k with $0 \leq k \leq n-1$ and such that

$$\left\{ \begin{array}{l} \int^\infty |a(t)| dt = \infty, \quad \text{if } k = n-1 \\ \int^\infty \frac{1}{r_{k+1}(s_{k+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \dots ds_{k+1} = \infty, \quad \text{if } k < n-1. \end{array} \right.$$

If the function a is nonnegative (respectively, nonpositive) on the interval $[t_0, \infty)$, then every bounded solution x of the equation (E_0) for n even (resp. odd) is oscillatory while for n odd (resp. even) is either oscillatory or such that

$$\lim_{t \rightarrow \infty} (D_r^{(i)} x)(t) = 0 \quad \text{monotonically } (i = 0, 1, \dots, n-1).$$

PROOF. – Let x be a bounded nonoscillatory solution on an interval $[T_0, \infty)$, $T_0 \geq t_0$, of the equation (E_0) . Without loss of generality, we suppose that $x(t) \neq 0$ for every $t \geq T_0$. Furthermore, we assume that x is positive, since the substitution $z = -x$ transforms (E_0) into an equation of the same form satisfying the assumptions of the theorem.

By (i), we choose a $T \geq T_0$ so that

$$g_{ij}(t) \geq T_0 \quad \text{for every } t \geq T \quad (j = 1, 2, \dots, m_i; i = 0, 1, \dots, N).$$

Then, in view of (ii), from equation (E_0) we obtain

$$-\delta(D_r^{(n)} x)(t) = |a(t)| \Phi(x \langle g_0(t) \rangle, (D_r^{(1)} x) \langle g_1(t) \rangle, \dots, (D_r^{(N)} x) \langle g_N(t) \rangle) \geq 0$$

for all $t \geq T$, where

$$\delta = \begin{cases} +1, & \text{if } a \geq 0 \\ -1, & \text{if } a \leq 0. \end{cases}$$

Namely,

$$\delta(D_r^{(n)} x)(t) \leq 0 \quad \text{for every } t \geq T.$$

Thus, by the lemma, we have

$$(5) \quad \delta(-1)^{n+1+i} (D_r^{(i)} x)(t) \geq 0 \quad \text{for every } t \geq T \quad (i = 1, 2, \dots, n-1),$$

$$(6) \quad \lim_{t \rightarrow \infty} (D_r^{(i)} x)(t) = 0 \quad (i = 1, 2, \dots, n-1)$$

and

$$(7) \quad \int_{s_1}^{\infty} \frac{1}{r_1(s_1)} \int_{s_2}^{\infty} \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |(D_r^{(n)} x)(s)| ds ds_{n-1} \dots ds_2 ds_1 < \infty.$$

The functions $D_r^{(i)} x$ ($i = 0, 1, \dots, N$) are obviously bounded.

Now, it holds

$$(8) \quad \lim_{t \rightarrow \infty} x(t) = 0.$$

Indeed, in the opposite case for some positive constant K

$$x(t) \geq K \quad \text{for every } t \geq T_0.$$

So, because of (ii), the boundedness of the functions $D_r^{(i)}x$ ($i = 0, 1, \dots, N$) and the continuity of the function Φ , there exists a positive constant c such that for every $t \geq T$

$$\Phi(x\langle g_0(t) \rangle, (D_r^{(1)}x)\langle g_1(t) \rangle, \dots, (D_r^{(N)}x)\langle g_N(t) \rangle) \geq c.$$

Therefore,

$$|(D_r^{(n)}x)(t)| \geq c|a(t)| \quad \text{for all } t \geq T$$

and hence (7) gives

$$\int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty |a(s)| ds ds_{n-1} \dots ds_2 ds_1 < \infty,$$

which contradicts (C_2) .

Finally, by (5) and (8), we conclude that $\delta(-1)^n = -1$, namely that n is odd (resp. even) if $\delta = +1$ (resp. $\delta = -1$). Thus, (5), (6) and (8) prove the theorem.

COROLLARY III. — *Consider the differential equation (E_0) subject to the conditions (i), (ii) and (R). If the function a is of constant sign on $[t_0, \infty)$, then the condition (C_2) is a necessary and sufficient condition in order that the equation (E_0) have at least one (bounded nonoscillatory) solution x so that the $\lim_{t \rightarrow \infty} x(t)$ exists in $\mathbf{R} - \{0\}$.*

PROOF. — The necessity of (C_2) follows immediately from Theorem 3, while its sufficiency is contained in Corollary II.

3. — Applications.

We shall now clarify the importance of the results of Section 2 by applying them in the particular case where for some integer m , $1 \leq m \leq n-1$, we have

$$r_i = 1 \quad \text{for } i \neq n-m \quad \text{and} \quad r_{n-m} = r.$$

In this case the equation (E) takes the form

$$(\hat{E}) \quad [r(t)x^{(n-m)}(t)]^{(m)} + a(t)\Phi(x\langle g_0(t) \rangle, x'\langle g_1(t) \rangle, \dots, x^{(N)}\langle g_N(t) \rangle) = b(t),$$

if $N < n-m$, and the form

$$(\hat{E}) \quad [r(t)x^{(n-m)}(t)]^{(m)} + a(t)\Phi(x\langle g_0(t) \rangle, x'\langle g_1(t) \rangle, \dots, x^{(n-m-1)}\langle g_{n-m-1}(t) \rangle, \\ [rx^{(n-m)}]\langle g_{n-m}(t) \rangle, [rx^{(n-m)}]'\langle g_{n-m+1}(t) \rangle, \dots, [rx^{(n-m)}]^{(N-(n-m))}\langle g_N(t) \rangle) = b(t),$$

if $N \geq n - m$. Also, the condition (R) becomes:

$$(\hat{R}) \quad \int_{r(t)}^{\infty} \frac{dt}{r(t)} = \infty.$$

Let $(\hat{E}_0)$ denote the differential equation $(\hat{E})$ with $b = 0$. Corollaries 1, 2, 3, 4 and 5 below are now and follow from Theorem 1, Corollary I, Corollary II, Theorem 2 and Corollary III respectively.

COROLLARY 1. — Consider the differential equation $(\hat{E})$ subject to the conditions (i), (ii) and:

$(\hat{C}_1)$ There exists a function $w \in C^{n-m}([t_0, \infty))$ with $rw^{(n-m)} \in C^m([t_0, \infty))$ and $[rw^{(n-m)}]^{(m)} = b$ and such that $\hat{A} < \infty$, where

$$\hat{A} = \begin{cases} \max_{0 \leq i \leq N} \limsup_{t \rightarrow \infty} |w^{(i)}(t)|, & \text{if } N < n - m \\ \max \left\{ \max_{0 \leq i \leq n-m-1} \limsup_{t \rightarrow \infty} |w^{(i)}(t)|, \max_{0 \leq j \leq N-(n-m)} \limsup_{t \rightarrow \infty} |[r(t)w^{(n-m)}(t)]^{(j)}| \right\}, & \text{if } N \geq n - m. \end{cases}$$

$$(\hat{C}_2) \quad \int_{\frac{t^{n-1-m}}{r(t)}}^{\infty} \int_t^{\infty} (s-t)^{m-1} |a(s)| ds dt < \infty.$$

Then for every real number L with $|L| > 2\hat{A}$ there exists a (bounded nonoscillatory) solution x of the equation $(\hat{E})$ with

$$\lim_{t \rightarrow \infty} [x(t) - w(t)] = L$$

and

$$\begin{cases} \lim_{t \rightarrow \infty} [x^{(i)}(t) - w^{(i)}(t)] = 0 & (i = 1, 2, \dots, n-m-1), \text{ if } m < n-1 \\ \lim_{t \rightarrow \infty} ([r(t)x^{(n-m)}(t)]^{(j)} - [r(t)w^{(n-m)}(t)]^{(j)}) = 0 & (j = 0, 1, \dots, m-1). \end{cases}$$

COROLLARY 2. — Consider the differential equation $(\hat{E})$ subject to the conditions (i), (ii), $(\hat{C}_2)$ and:

$(\hat{C}'_1)$ There exists a function $w \in C^{n-m}([t_0, \infty))$ with $rw^{(n-m)} \in C^m([t_0, \infty))$ and $[rw^{(n-m)}]^{(m)} = b$ and such that:

$$\lim_{t \rightarrow \infty} w^{(i)}(t) = 0 \quad (i = 0, 1, \dots, N),$$

if $N < n - m$, and

$$\begin{cases} \lim_{t \rightarrow \infty} w^{(i)}(t) = 0 & (i = 0, 1, \dots, n - m - 1) \\ \lim_{t \rightarrow \infty} [r(t) w^{(n-m)}(t)]^{(j)} = 0 & (j = 0, 1, \dots, N - (n - m)) , \end{cases}$$

if $N \geq n - m$.

Then for every real number L with $L \neq 0$ there exists a (bounded nonoscillatory) solution x of the equation $(\hat{E})$ with

$$\lim_{t \rightarrow \infty} x(t) = L$$

and, provided that $N > 0$,

$$\lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \quad (i = 1, 2, \dots, N) ,$$

if $N < n - m$, and

$$\begin{cases} \lim_{t \rightarrow \infty} x^{(i)}(t) = 0 & (i = 1, 2, \dots, n - m - 1), \text{ when } m < n - 1 \\ \lim_{t \rightarrow \infty} [r(t) x^{(n-m)}(t)]^{(j)} = 0 & (j = 0, 1, \dots, N - (n - m)) , \end{cases}$$

if $N \geq n - m$.

COROLLARY 3. — Consider the differential equation $(\hat{E}_0)$ subject to the conditions (i), (ii) and $(\hat{C}_2)$. Then for every real number L with $L \neq 0$ there exists a (bounded nonoscillatory) solution x of the equation $(\hat{E}_0)$ with

$$\lim_{t \rightarrow \infty} x(t) = L$$

and

$$\begin{cases} \lim_{t \rightarrow \infty} x^{(i)}(t) = 0 & (i = 1, 2, \dots, n - m - 1), \text{ if } m < n - 1 \\ \lim_{t \rightarrow \infty} [r(t) x^{(n-m)}(t)]^{(j)} = 0 & (j = 0, 1, \dots, m - 1) . \end{cases}$$

COROLLARY 4. — Consider the differential equation $(\hat{E}_0)$ subject to the conditions (i), (ii), $(\hat{R})$ and:

$(\hat{C}_3)$ Either

$$\int_t^\infty t^{m-1} |a(t)| dt = \infty$$

or

$$\int_t^\infty \frac{t^{n-1-m}}{r(t)} \int_t^\infty (s-t)^{m-1} |a(s)| ds dt = \infty .$$

If the function a is nonnegative (resp. nonpositive) on $[t_0, \infty)$, then every bounded solution x of the equation $(\hat{E}_0)$ for n even (resp. odd) is oscillatory, while for n odd (resp. even) is either oscillatory or such that

$$\begin{cases} \lim_{t \rightarrow \infty} x^{(i)}(t) = 0 & \text{monotonically } (i = 0, 1, \dots, n-m-1) \\ \lim_{t \rightarrow \infty} [r(t) x^{(n-m)}(t)]^{(j)} = 0 & \text{monotonically } (j = 0, 1, \dots, m-1). \end{cases}$$

COROLLARY 5. — Consider the differential equation $(\hat{E}_0)$ subject to the conditions (i), (ii) and $(\hat{R})$. If the function a is of constant sign on $[t_0, \infty)$, then the condition $(\hat{C}_2)$ is a necessary and sufficient condition in order that the equation $(\hat{E}_0)$ have at least one (bounded nonoscillatory) solution x so that the $\lim_{t \rightarrow \infty} x(t)$ exists in $\mathbf{R} - \{0\}$.

It is easy to see that in the considered case the conditions (C_1) and (C'_1) become $(\hat{C}_1)$ and $(\hat{C}'_1)$ respectively. Furthermore, for any nonnegative integer μ and a nonnegative continuous function p on $[u, \infty)$ we have the formula

$$\int_u^\infty \int_v^\infty (s-v)^\mu p(s) ds = \int_u^\infty \frac{(s-u)^{\mu+1}}{\mu+1} p(s) ds.$$

By this formula, it is a matter of elementary calculus (cf. [4]) to verify that the conditions (C_2) and (C_3) follow from $(\hat{C}_2)$ and $(\hat{C}_3)$ respectively.

Before closing this paper, let us consider the usual case where $r_1 = r_2 = \dots = r_{n-1} = 1$. In this case the equation (E) takes the form

$$(\tilde{E}) \quad x^{(n)}(t) + a(t) \Phi(x \langle g_0(t) \rangle, x' \langle g_1(t) \rangle, \dots, x^{(N)} \langle g_N(t) \rangle) = b(t),$$

the condition (R) holds by itself and (cf. [4]) the conditions (C_2) and (C_3) become

$$(\tilde{C}_2) \quad \int_0^\infty t^{n-1} |a(t)| dt < \infty$$

and

$$(\tilde{C}_3) \quad \int_0^\infty t^{n-1} |a(t)| dt = \infty$$

respectively. Thus, the results of Section 2 can be easily formulated for the differential equation $(\tilde{E})$ so that to generalize some results of the book of STAIKOS [10].

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