A characteristic property of spheres.

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To Enrico Bompiani on his scientific Jubilee

Summary. - *We prove: Let S be a closed n-dimensional surface in an $(n+1)$-space of constant curvature $(n \geq 2)$; $k_1 \geq \dots \geq k_n$ denote its principle curvatures. Let $\varphi(\xi_1, \dots, \xi_n)$ be such that $\frac{\partial \varphi}{\partial \xi_i} > 0$. Then if $\varphi(k_1, \dots, k_n) = const$ on S and S is subject to some additional general conditions (those* (II$_0$) *or* (II) *n° 1), S is a sphere.*

1. **The Formulation of Theorem.** - Consider an n-dimensional, differentiable, oriented closed surface S in an $(n+1)$ space R which is either Euclidean, or Lobochewskian, or spherical. In the last case S is supposed to be inclosed in an open halfspace, i. e. in a hemisphere, and all further considerations pertain to such a hemisphere. We suppose that the space R is oriented and therfore the orientation of the surface determines the directions of its normals.

Let S be of class C^2 and let $k_1 \geq \dots \geq k_n$ denote its principal curvatures, at an arbitrary point $X \in S$.

Introduce two following conditions imposed on S.

(I) There exists such a function $\varphi(\xi_1, \dots, \xi_n)$ of class C^1 with the condition

$$\frac{\partial \varphi}{\partial \xi_i} > 0 \qquad (i = 1, \dots, n), \tag{1}$$

that $\varphi(k_1, \dots, k_n) = \text{const}$ on S.

(II$_0$) There exists in R a bounded domain G^0 with the smooth boundary S^0 and a smooth mapping h of $G^0 + S^0$ into R which maps S^0 onto S. (S^0 is oriented so that h transforms its orientation into that of S).

In the simplest case the surface S itself has no multiple points and therefore bounds a domain in R.

Instead of the condition (II$_0$) we impose on S the following more general condition.

(II) There exists in R such an n-dimensional, oriented, smooth closed surface S^0 that (*a*) S^0 bounds an open set G^0 so that $G^0 + S^0$ is compact and S^0 lies on the boundary of $G^0 + S^0$;

(It is not excluded that S^0 has multiple points and that some parts of S^0 do not belong to the boundary, of G^0; such parts of S^0 are, at least, twice covered).

(*b*) each part S_i^0 of S^0, which bounds a component G_i^0 of G^0 is oriented so that all normals to it are simultaniously directed either into G_i^0 or out of it;

(*c*) there exists a smooth mapping h of $G^0 + S^0$ into R which maps S^0 onto S with orientation;

We shall prove the following

THEOREM. - *The surface S of class C^2 subject to the conditions* (I), (II) *is a sphere.*

2. **Some remarks to the Theorem.** - (1) The simplest form of the function φ is $\varphi = \xi_1 + ... + \xi_n$ and then the condition (I) reduces to that of the constancy of mean curvature.

In general, the function φ can be non symmetrical and it suffice to consider it for $\xi_1 \geq ... \geq \xi_n$ in an n-interval a priori prescribed for the curvatures k_i of the surface S. In particular, if S is supposed to be of positive curvature (all $k_i > 0$), it is sufficient to suppose (1) only for $\xi_i > 0$. In this case φ can be, for instance, an elementary symmetric function.

(2) In the simplest particular case, when S has no multiple points and therefore bounds a domain, our Theorem was proved in [1]. If in addition to this simplest form of the condition (II) the condition (I) reduces to $k_1 + ... + k_n =$ const, our Theorem asserts that a domain bounded by a surface of constant mean curvature is a sphere. Thus the only possible form of an, even unstable, soap-bubble is a sphere.

Other particular cases of our Theorem can be found in the papers by H. HOPF and K. VOSS [7, 9]. We don't quote results got by diverse authors especially for convex surface. There takes place H. HOPF's result [6] which is not covered by our Theorem and which, in its essence, consists of the statement that in the 3-space a surface which is a regular image of a sphere and is subject to the condition (I) is a sphere.

(3) The condition (II) is not void: there is no difficulty in finding examples of closed surfaces which do not satisfy it.

It does not seem improbabile that any kind of the condition (II) is superfluous so that any closed surface subject to the only condition (I) is, necessarily, a sphere; but we have no proof for that.

(4) As we do not exclude the existance of the multiple points (selfintersection or selfcontacts) of the surface S as well as these of the surface S^0 in the condition (II), the above formulation of this condition is not entirely precize. We have to distinguish between a surface and the point-set covered by the surface, as well as between a point of the surface and a point of this set. (The surface can be defined by means of a mapping of a compact n-maifold M into R, and then its point is a pair: a point of M plus its image in R). The precize formulation of the condition (II) offers no difficulties and we omit it. As well we shall not explicitely distinguish in our exposition between the surface and the set covered by it, between the points of the surface and the points of this set.

(5) The condition (I) can be replaced by the following one

(I_1) If at two points $X', X'' \in S$ at least two curvatures are different, e. g. $k_i' > k_i''$, then there are the differences $k_i' - k_i''$, $k_j' - k_j''$ of opposite signs. And there exists such a constant $A > 0$ that for any pair of such points

(2) $$A > -\frac{\min(k_i' - k_i'')}{\max(k_i' - k_i'')} > \frac{1}{A}.$$

The connection between the conditions (I), (I_1) is established in [2].

(6) Our supposition that S is of class C^2 can be replaced by a weaker one: the upper curvatures of the normal sections of S are everywhere finite.

This implies that S is twice differentiable in a generalized sense almost everywhere and therefore has almost everywhere generalized principle curvatures k_i. (The generalized second differential meant here is defined in [6]). Accordingly, the condition (I) is to be replaced by the following one

(I_2) $\varphi(k_1, \dots, k_n) =$ const almost everywhere on S and φ is of class C^1 and such that

(3) $$\text{const} > \frac{\partial\varphi(\xi_1, \dots, \xi_u)}{\partial\xi_i} > \text{const} > 0 \qquad (i = 1, \dots, n),$$

at least on S, i. e. for $\xi_i = k_i$ $(i = 1, \dots, n)$.

If S is of ctass C^2 this condition is implied by (I).

If S is subject to the condition (II_0), it can be supposed to be not of class C^2 but having generalized SOBOLEFF's second derivatives summable with n-th power. Then k_i are to be understood in the above mentioned generalized sense and the condition (I) is to be replaced by (I_1) or (I_2).

3. **Basic Lemma.** - Let S', S'' be two oriented surfaces smooth up to their bounderies B', B''. We say that they touch each other from one side at a point X, if the following conditions are fulfilled.

(1) Either X is a common inner point of S' and S'', or it belongs to B' and B''.

(2) The normals to $S'+B'$, $S''+B''$ at the point X coincide.

(3) Introduce in a neighbourhood U of X the coordinates $x_1, \dots, x_n$, $x_{n+1}=z$ so that z-axis be directed along the common normal at X. Then in U S', S'' are represented by the equations

$$z = z'(x_1, \dots, x_n), \quad z = z''(z_1, \dots, z_n). \tag{4}$$

The condition demands that in U $z' \geq z''$ (or $z' \leq z''$).

LEMMA 1. - *Let the surfaces S', S'' with the bounderies B', B'' be subject to the following conditions.*

(1) *$S'+B'$, $S''+B''$ are of the class C^2 (in other words, S', S'' are of class C^2 up to their boundaries);*

(2) *B', B'' are of class C^2;*

(3) *S', S'' are subject to the condition (I) with the same value of φ, i. e. $\varphi(k_1', \dots, k_n') = \varphi(k_1'', \dots, k_n'') = const.$*

(4) *S', S'', touch each other from one side at a point X.*
Then S', S'' coincide in a neighbourhood of X.

This lemma is an immediate corollary of two following statements.

(A) Let the surfaces S', S'' or some their parts be represented by the equations (3). Denote

$$\Delta z = z''(x_1, \dots, x_n) - z'(x_1, \dots, x_n),$$

$$\Delta\varphi = \varphi(k_1'', \dots, k_u'') - \varphi(k_1', \dots, k_n').$$

The following theorem takes place (for the proof cf. [3]).
The diffecence $\Delta\varphi$ is representable in the form

$$\Delta\varphi = \Sigma\, a_{ik}\, \Delta z_{x_i x_k} + \Sigma\, b_i\, \Delta z_{x_i} + c\Delta z \tag{5}$$

and, if φ satisfies the inequalities (3), the expression on the right side has bounded coefficients and is elliptic, i. e. there exist such a positive constant A that

$$A\, \Sigma\, \xi_i^2 > \Sigma\, a_{ik}\, \xi_i\, \xi_k > \frac{1}{A}\, \Sigma\, \xi_i^2 .$$

(B) Let in a domain D with a boundary Γ of class C^2 there be given a linear ellidtic operator L like the right side of (5) and a function $z(x_1, \ldots, x_n)$ of class C^2 in the closed domain $D + \Gamma$. Suppose that everywhere in $D z \geq 0$, $L(z) \leq 0$ and there exists at least one point $X_0 \in D + \Gamma$ where z and all its derevatives vanish: $z = z_{x_1} = \ldots = z_{x_n} = 0$. Then $z = 0$ identically in D. (If $X_0 \in D$ the condition $z_{x_1}(X_0) = \ldots = z_{x_n}(X_0) = 0$ follows from $z \geq 0$ in D, $z(X_0) = 0$, but if $X_0 \in \Gamma$ this condition, obviously, is not superfluous).

This theorem is due to E. HOPF and GIRAUD and can be found in [8] though in somewhat different form.

The statements (A), (B) imply our Lemma 1.

REMARK. - The statement (B) remains valid if Γ is only smooth [4]. In accordance with this it is sufficient to suppose in Lemma 1 that the boundaries B', B'' of the surfaces S', S'' are smooth.

If the surfaces S', S'' are not supposed to be of class C^2 but subject to the weaker condition mentioned in n.° 2 (6), i. e. if their upper curvatures are everywhere finite, and B', B'' are supposed to be smooth, Lemma 1 remains valid. So it does if S', S'' have SOBOLEFF's second derevatives summable with n-th power, provided B', B'' are of class C^2.

These generalizations are based upon corresponding generalizations of the statement (B). Cf. [5].

4. The proof of the Theorem with the condition (II_0). - The presentation of the proof of our Theorem will be given in terms which pertain to the Euclidean or Lobachevskian space. In the case of the spherical space the proof is the same; few necessary changes in its presentation are so obvious that it would be superfluous to mention them.

In this and two following paragraphs we give the proof of our Theorem when the condition (II) is replaced by the stronger one (II_0). The proof reduces to that of following

LEMMA 2. - *The surface S subject to the conditions* (I), (II_0) *has a plane of symmetry of any direction, i. e. perpendicular to any given line. This, obviously, implies that S is a sphere.*

Let G^0 be the domain bounded by S^0 and h the smooth mapping defined over $G^0 + S^0$, which maps S^0 onto S. Let $G = h(G^0)$. Then G is open and bounded and its boundary is contained in S.

Take a line l and draw a supporting plane P_0 to $G + S$ perpendicular to l. Move this plane so that it starts to cross G, retaining its perpendicularity to l. Thus we have a variable plane P perpendicular to l, which cuts from S a part -a « hump » H which grows as P moves further.

Reflecting the hump H in the plane P we get the « reflected hump » F.

As the plane P, in its movement, reaches and passes a given point $X \in S$, there appears on F the corresponding point Y which moves along a line T_x. This line proceds from X into G, as far as the desplacement of the plane P from its initial position is sufficiently small. (T_x is a straight line parallel to l, if the space is Euclidean, and it is an equidistant, if the space is Lobachevskian).

Consider the total counter-image H^0 of the hump H: $H^0 = h^{-1}(H)$. Take a point $X^0 \in H^0$; then $h(X^0) = X \in H$. As the line T_x proceeds from X into G and the mapping h is locally homeomorphic, there exists in a neighbourhood of X^0 in G^0 an uniquely determined line $T^0_{X^0}$ - the counter-image of the line T_x. The mapping h being locally homeomorphic, each line $T^0_{X^0}$ can be prolonged in a unique manner. To the point $Y \in F$, which corresponds to X, there corresponds the point $Y^0 \in T^0_{X^0}$. All such points Y^0 corresponding to a definite position of the plane P form a set $F^0 \subset G^0$ which corresponds to H^0 as F corresponds to H. And evidentely; $F = h(F^0)$. (Although F^0 is not, in general, the total counter image of F).

When the plane P moves, the points $Y^0_{X^0}$ move along their trajectories $T^0_{X^0}$ and, correspondingly, the set F^0 varies in a determined manner. At each moment it consists of surfaces which lie in G^0 and have common boundaries with the corresponding parts of the set H^0. With the part $S^0 - H^0$ of S^0 it bounds a part of the set G^0, which becomes smaller as the plane P moves further.

But, sooner or later, the set F^0 must cease to exist, for, the displacement of the plane P being big enough, the reflected hump F leaves the set G and then the points of F lying beyond G have no counter-image sin G^0 at all.

We follow the change of the set F^0 till it exists, i. e. till to each point $X \in H$ there corresponds the point Y^0 on trajectory T^0_x. We fix the extreme plane P_1 which, with the initial plane P_0, bounds the set of planes P for which the set F^0 exists. For this plane one of two following situations takes place.

(1) The set F^0 ceases to exist when the plane P reaches the position P_1.

(2) The set F^0 still does exist, but it ceases to as soon as the plane P moves beyond P_1.

Further we consider these two cases separately.

5. **The first case.** - Let the first case take place. It means that at least one of the points Y^0 has reached the end of its trajectory at a point belonging to $S^0 - H^0$. In other words the limit F_1^0 of the sets F^0 corresponding to the planes $P \to P_1$ touches $S^0 - H_1^0$ at an inner point A^0 from inside of G^0.

Accordingly, in a neighbourhood of the point $A = h(A^0)$ the reflected hump F_1 touches the part $S - H_1$ of the surface S at an inner point from one side. Hence applying Lemma 1 we conclude that F_1 and $S - H_1$ coincide in a neighbourhood of A.

It means that F_1^0 and $S^0 - H_1^0$ coincide in a neighbourhood of A^0.

This consideration applies to any point of the boundary of such a neighbourhood, and so forth. Thus we conclude that the component of the reflected hump F_1 which contains the point A lies, as a whole, on the surface S and has no boundary except its common boundary with the corresponding component of the hump H_1.

Hence follows, at first, that these components of F_1 and H_1 exhaust the surface S. Secondly, F_1 and H_1 being symmetrical with respect to the plane P_1, this plane is the plane of symmetry of the surface S.

Thus our Lemma 2 is proved in the first of two above cases.

6. **The second case.** - Prove that the second of two above cases is impossible.

Take a point X^0 on the boundary B_1^0 of the set F_1^0 corresponding to the position P_1 of the plane P. The point $h(X^0) = X$ lies on the boundary B_1 of the reflected hump F_1, and therefore on the intersection $P_1 S$. Suppose that the tangent plane Q to S at X is not perpendicular to P_1.

The disposition of S, H_1, F_1 and P_1 in a neighbourhood of X is schematically repsesented in the Fig. 1. The h-image of a neighbourhood of the point X^0 in G^0 lies at the right side of S in the Fig. 1.

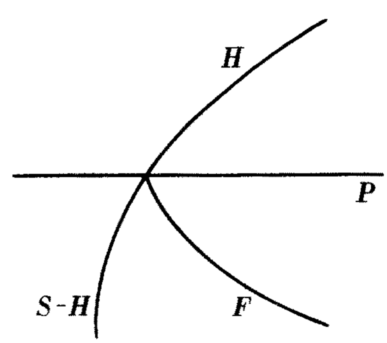

Fig. 1

From this picture one can easily conclude that a small displacement of the plane P beyond P_1 (downwards in the Fig. 1) cannot interfere with the existance of the set F^0 at least in a neighbourhood of the point X^0.

The surface S being smooth, we conclude that, as soon as the displacement of the plane P beyond P_1 is sufficiently small, the set F^0 will not cease to exist (in the large) if at no point of B_1 the tangent plane Q to S

is perpendicular to P_1. This contradicts the condition of the case under consideration and therefore there exists a point $X \in B_1$ where the tangent plane Q to S is perpendicular to P_1.

At such a point X the surface $F + B$ is tangent to $S - H$ and, in a neighbourhood of X, it lies at one side of S; namely, at that one which, owing to the mapping h, corresponds to the set G^0.

Thus the surfaces F and $S - H$ touch each other from one side at the boundary. (Their common boundary is regular because of its being the intersection of S and the plane P_1). Hence Lemma 1 applies and gives that F and $S - H$ coincide in a neighbourhood of X.

It means that the set F^0, in a neighbourhood of the point X_0, must lie on S^0. But according to the very definition of the set F^0 it lies in G^0. This contradiction proves that the case under consideration is impossible.

Thus Lemma 2 and with it our Theorem, with the condition (II_0) instead of (II), is proved.

REMARK. - If S is not of class C^2 but is subject to one of two weaker conditions mentioned in n° 2 (6), the proof of Lemma 2 is the same with the only difference that one has, instead of Lemma 1, to apply its suitable generalization quoted in a Remark at the end of n° 3.

7. **On the multiple points of a surface.** - A multiple point of a smooth surface can be of three kinds.

(1) A point of selfintersection where two pieces of the surface cross at an angle different from zero.

(2) A point X of selfcontact where two pieces of the surface have common tangent plane but do not coincide in any neighbourhood of X.

(3) A point of selfcoincidence in whose neighbourhood two pieces of the surface coincide.

A point can belong to two or all three of these classes, if more than two pieces of the surface meet at it.

LEMMA 3. - *The spherical representation of the set of poits of selfcontact of a surface of class C^2 has no inner points.*

The sperical representation being defined only in the Euclidian space, it is necessary to define it in the LOBACHEVSKIAN space.

Take in the space a point O. If T is a tangent plane to the surface, we draw through O a line perpendicular to T. Thus we get the spherical representation with respect to 0, which is meant in our Lemma, O being arbitrary.

THE PROOF OF LEMMA 3. - Let M be the set of the points of selfcontact of a given surface. The surface being of class C^2, one can easily verify that in a neighbourhood of a point $X \in M$ the spherical representation of M has no inner points. But a countable sum of sets without inner points have not them either [5]. Hence follows Lemma 3.

Now, let S^0 be the surface in the condition (II) of our Theorem. We distinguish two kinds of its poins of selfcoincidence, if there are any.

(1) The points of the essential selfcoincidence which are characterized by the following property. The pieces of the surface S^0, which coincide in a neighbourhood of such a point do not belong to the boundary of the open set G^0 bounded by S^0.

(2) The point of non essential selfcoincidence, where two coinciding pieces of the surface S^0 belong to the boundary of the open set G^0. We shall not consider such points as the multiple ones at all. Accordigly, the points of the first kind will be simply called these of selfcoincidence.

We consider a multiple point of the surface S subject to the condition (II) of our Theorem as an essential one, if and only if it is the h-image of a multiple point of the surface S^0, (the points of the non essential selfcoincidence of S^0 being excluded, in accordance with the above condition). In the same sense we shall understand the points of selfintersection, of selfcontact or of selfcoincidence.

LEMMA 4. - *Of the surface S satisfies the conditions* (I), (II) *of our Theorem the curvatures k_1, k_n at each point of selfcoincidence have opposit signs.*

PROOF. - Let X^0 S be a limit point of the points of selfcoincidence. Then it is a point of selfcontact and belongs to the boundary of the open set G^0 bounded by S^0. Owing to the condition (IIb) the normals to the part of S^0, which bounds a component of G^0, are all directed either into or out of it. Hence follows that in any neighbourhood of X^0 there exist points of S^0 with the normals which are almos opposite to each other.

Therefore the normals to the pieces of S^0, which coincide near X^0, are opposite. The mapping h transforms the orientation of S^0 into that of S. Hence at the point $X = h(X^0)$ the normals to the pieces of S are opposite and as these pieces coincide their principle curvatures have opposite signs: $k_1' = -k_n'', \dots, k_n' = -k_1''$.

Thus either among k_i — s there are those of opposite signs, or they are all zeros. Otherwise, the function φ in the condition (I) being monotonous the equality $\varphi(k_1', \dots, k_n') = \varphi(k_1'', \dots, k_n'')$ would be impossible.

But a closed surface always has points where all curvatures have the same sign; e. g. such are point where a sphere which encloses the surface touches it. The function $\varphi(k_1, \dots, k_n)$ in the condition (I) being monotonous it can not have the same value at a point where all curvatures have the same sign and at a point where they are all zeros. This contradicts the condition (I), and thus our Lemma is proved.

8. **The proof of the Theorem in the general case.** - We shall prove the following.

LEMMA 5. - *If the surface S is subject to the conditions of our Theorem, there exists such a cone that for any straight line $L \in K$ the surface S has a plane of symmetry perpendicular to L.*

Obviously, it implies our Theorem.

Let S be a surface subject to the conditions (I), (II) and let S^0, G^0, h have the meaning defined in the condition (II).

In accordance with Lemma 3, a point O being arbitrary fixed, we can take such a cone K with the vertex O that no line $L \in K$ is perpendicular to the tangent plane to S at a point of selfcontact. Take a line $L \in K$ and prove that S has a plane of symmetry perpendicular to L.

Draw a supporting plane P_0 to S perpendicular to L. It can not touch S at a multiple point. In fact, such a point $X \in SP_0$ obviously can not be one of selfintersection. Nor can it be a point of selfcontact because of the choise of the cone K. It can not be a point of selfcoincidence for, owing to Lemma 4, at such points there are curvatures of opposite signs.

In accordance with our definition of the multiple points of the surface S, as it is given in n.° 7, the set $M^0 = h^{-1}(SP_0)$ — the total counter image M^0 of the set P_0S contains no multiple points of S^0, and so does a neighbourhood of this set.

Hence the considerations of n.° 4 apply. We move the plane P from its initial position P_0 and cut off S the hump H. Reflecting it in P we get the reflected hump F, and we define the set $F^0 \subset G^0$ as it was done in n.° 4.

We move the plane P and follow the change of the sets F^0 and $H^0 = h^{-1}(H)$ till F^0 exists and the plane P does not meet multiple points of S. The last condition is equivalent to that one that neither H^0 nor its boundary contains a multiple point of S^0.

Let P_1 be the extreme plane which, with the plane P_0, bounds the set of the planes P for which the discribed situation takes place.

If P_1 does not contain multiple points of S, the situation is analogous to that considered in n.° 4. Then the considerations of nn.° 5, 6 prove that P_1 is the plane of symmetry of S. The a priori possible existance of multiple points on $S - H_1$ does not interfere with these considerations.

Thus we are left to suppose that the plane P_1 contains multiple points of S.

9. **The completion of the proof.** - Let the plane P_1 contain a multiple point X of the surface S. It can not be a point of selfcoincidence. Otherwise,

owing to Lemma 4, plane P_1 would cross S in a neighbourhood of X and on both sides of P_1 there would be multiple points. But this contradicts the definition of the plane P_1.

Suppose X is a point of selfcontact and let Q be the tangent plane at it. Owing to the choice of the cone K, which the line L belongs to, the plane P_1 can not coincide with Q. We prove that it is perpendicular to Q.

Suppose it crosses Q but is not perpendicular to Q. Let $X^0 \in S^0$ be the counter image of X, which is a point of self contact of the surface S^0, and let S_1^0, S_2^0 be two pieces of S^0, which touch each other at X^0. In a neighbourhood of X^0 they enclose a part U^0 of the set G^0.

Let $S_1 = h(S_1^0)$, $S_2 = h(S_2^0)$, $U = h(U^0)$. S_1, S_2 having at X the common tangent plane Q, the situation is like that schematically represented in the Fig. 2. It is not difficult to observe that in a neighbourhood of X the re-

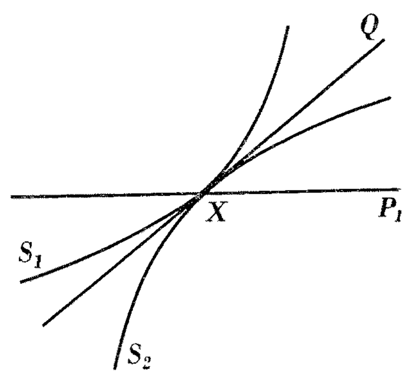

Fig. 2

flected hump F corresponding to a plane P sufficiently near to P_1 can not lie in U (if P is not perpendicular to Q). It means that there exists no counerimage F^0 of F. It contradicts the definition of the plane P_1. Thus the plane Q is perpendicular to P_1.

The plane Q being perpendicular to P_1, the situation is like that considered in n° 6. If H_1, H_2 are the parts of the hump H cut from the surfaces S_1, S_2 and F_1, F_2 are the corresponding parts of the reflected hump, F_1, F_2 thouch $S_1 - H_1$, $S_2 - H_2$ at the point from one side. Thus, owing to Lemma 1, F_1, F_2 coincide with $S_1 - H_1$, $S_2 - H_2$ in a neighbourbood of X and this coincidence spreads further up to multiple points and beyond the points of selfcontact because of the above consideratious.

Let, now, X be a point of selfintersection of the surface S. Let X^0 be the counterimage of X, which is a point of selfintersection of the surface S^0. Two pieces S_1^0, S_2^0 of S^0, which cross each other at X^0 bound in a neighbourhood of X^0 two parts U^0, V^0 of the set G^0. (It is not excluded that at X^0 there meet more than two pieces of the surface S^0, but we always can choose two of them so that the discribed situation takes place).

The h-himages S_1, S_2, U, V of S_1^0, S_2^0, U^0, V^0 are situated in a neighbourhood of X as it is schematically represented in the Fig. 3. The intersection of the planes Q_1, Q_2 tangent to S_1 and S_2 at X lies in the plane P_1. Otherwise on both sides of P_1 there would be the points of the intersection S_1S_2 what would contradict the definition of the plane P_1.

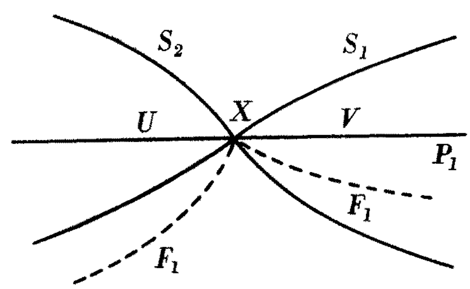

Fig. 3

The plane P_1 bisects the angle between the planes Q_1, Q_2. Otherwise one of two parts of the reflected hump F_1 would lie beyond the set $U + V$ (Observe Fig. 3). It would mean that the set F^0 ceased to exist before the variable plane P reaches P_1. But it contradicts the definition of the plane P_1.

As P_1 bisects the angle between S_1 and S_2 at X, either part of the refleted hump F_1 touches the corresponding part of $S - H_1$. These parts of F_1 lie in U and V respectively (for otherwise we would once more, have a contradiction with the definition of the plane P_1). Thus these parts of F_1 touch the corresponding parts of $S - H_1$ from one side (Observe the condition (IIb) about the normals).

Hence Lemma 1 applies and we see that in a neigbourhood of X the reflected hump F_1 coincides with $S - H_1$.

An obvious consideration using Lemma 1 shows that this coincidence spreads further up to multiple points and beyond them too, as it is obvious from above consideratious of the points of selfcontact and selfintersection. Therefore the coincidence of the reflected hump F_1 with $S - H_1$ spreads over the whole F_1. Hence P_1 is a plane of symmetry of the surface S.

Thus the proof of Lemma 5 and at the some time the proof of our Theorem is completed.

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