On subvarieties of a Picard variety.

Memoria di LEONARD ROTH (a Londra)

To Enrico Bompiani on his scientific Jubilee

Summary. - *See the following Introduction.*

INTRODUCTION

In all that follows we shall denote by V_r $(r \geq 1)$ an algebraic variety which is non singular (i. e. irreducible and free from multiple points) and which lies on a PICARD variety V_q $(q \geq r)$; we suppose always that V_q is in *normal form*, i. e. free from multiple points and exceptional manifolds. It is well known that the number $g_1(V_r)$ of linearly independent simple integrals of the first kind attached to V_r cannot exceed q; the case where $g_1(V_r) < q$ is discussed in n. 2

We shall denote by W_r a non singular algebraic variety of *superficial irregularity* q, i. e. such that $g_1(W_r) = q$. With these notations we sketch the background to the present work.

The earliest result in this field dates back to 1900 when — according to ENRIQUES ([14], p. 368) — CASTELNUOVO proved the theorem: *if the surface* W_2 *has geometric genus* $P_g(W_2)$ *zero, then it contains an irrational pencil of curves.* This theorem was later extended and made precise by SEVERI [25], who showed that, if $P_g(W_r) = 0$ $(r \geq 2)$, then W_r contains a *congruence* (system of subvarieties of index unity) of superficial irregularity q.

A second result, likewise due to CASTELNUOVO [5], asserts that V_q *cannot contain any rational curves.* More generally, SEVERI [25] showed that, for any V_r of V_q, $P_g(V_r) > 0$.

In the third place, CASTELNUOVO [4] proved that, if W_2 contains no irrational pencil, then $P_g(W_2) > 2(q-2)$. Subsequently COMESSATTI [7] extended this theorem to any variety W_r (see n. 9). The demonstrations of these results are analytical and ingenious, but, for $r \geq 3$, are somewhat complicated (1); however, in 1950 d'ORGEVAL [17] gave a simple geometrical proof of the CASTELNUOVO inequality which extends immediately to the case $r \geq 3$ ([22]).

Now it is an interesting fact that all these results form part of a general theory which is based on the study of the subvarieties of V_q. In the

(1) In fact COMESATTI gives a detailed proof only for the case $r = 3$.

present paper three aspects of this theory are considered:

(*i*) An improvement of the above theorems, which is effected by introducing the notion of *pseudo-Abelian* variety, first treated by Dantoni [11].

(*ii*) An extension of the results, which deal only with canonical systems of highest dimension $r-1$, to those of any dimension.

(*iii*) Applications to the study of the varieties $W_{r'}$.

For the general theory of the Picard variety which we require here, we refer the reader to the posthumous book by Conforto [9]. For more particular results the monograph [22] may be consulted. Other applications of the present methods will be found in the note [21].

I

1. **General properties of V_q. Continuous systems, pseudo-Abelian varieties.** We begin by recalling the main facts concerning V_q which will be needed here. In the first place we note that V_q admits a parametric representation by means of theta functions of q independent complex variables u_i $(i=1, 2, \dots, q)$ which may be taken as universal coordinates on V_q. Denoting by u the vector whose components are u_i, we can find $n+1$ theta functions $\theta_0(u)$, $\theta_1(u), \dots, \theta_n(u)$ of the same order such that a projective model of V_q in space S_n is represented by

$$x_0 : x_1 : \dots : x_n = \theta_0(u) : \theta_1(u) : \dots : \theta_n(u). \tag{1}$$

A first consequence of this result is that V_q *admits a group G_q of automorphisms which is continuous, commutative and completely transitive over V_q*; this group is represented by the equations

$$u_i' = u_i + c_i \qquad (i = 1, 2, \dots, q), \tag{2}$$

where the c_i are arbitrary constants; for any given set of values c_i the equations define *a transformation of the first kind* on V_q. It is important to note that the group G_q serves to *characterise* V_q.

A second consequence is the so-called Appell-Humbert theorem: *any hypersurface V_{q-1} of V_q can be specified by the parametric equations* (1) *together with a single equation* $\theta(u)=0$. From this we can calculate the freedom of the complete linear system $|V_{q-1}|$ defined by V_{q-1}; we find that, excepting the case where V_{q-1} belongs to an irrational (elliptic) pencil, the system has positive freedom, and is without base points. Hence, by Bertini's theorem, *the general member of the system is non singular.*

Applying the transformations of G_q to any non singular V_{q-1} we obtain a continuous system $\{V_{q-1}\}$ of birationally equivalent hypersurfaces, the general member of which is likewise non singular. Moreover the characteristic (linear) system associated with any complete linear system $|V_{q-1}|$ of $\{V_{q-1}\}$ is incomplete.

Similarly, the transformations of G_q, applied to any non singular variety V_r $(1 \leq r \leq q-2)$, generate a continuous system $\{V_r\}$ of birationnally equivalent varieties, the general member of which is non singular. In general this system is ∞^q; it will be of lower dimension if, and only if, V_r admits a continuous group of automorphisms contained in G_q. The obvious case is that in which V_r is a Picard variety; the system $\{V_r\}$ is then ∞^{q-r}. Supposing instead that the group in question has dimension t $(1 \leq t \leq r-1)$, its trajectories on V_r form a congruence $\{V_t\}$ of birationally equivalent Picard varieties; in this case V_r is a *pseudo-Abelian variety of type t.* We may then prove ([11, 19, 20]) that V_r contains a second congruence $\{V_{r-t}\}$ whose members are birationally equivalent.

It should be noted that a pseudo-Abelian variety V_r which lies on V_q is somewhat particular; in the general case, the congruence $\{V_t\}$ of trajectories contains certain aggregates of trajectories $V_{t,s}$ $(s \geq 2)$ such that the multiples $sV_{t,s}$ are algebraically equivalent to the generic V_t ([19, 20]). When V_r lies on V_q, however, there are no such varieties $V_{t,s}$.

The case where V_r is Picardian may be included in this scheme by describing V_r as a pseudo-Abelian variety of type r. In conclusion, the complete continuous system $\{V_r\}$ associated with a given variety V_r is in general ∞^q; it is ∞^{q-t} if, and only if, V_r is pseudo-Abelian of type $(1 \leq t \leq r)$. In all cases, in view of the complete transitivity of the group G_q, we see that the system $\{V_r\}$ invades V_q and is free from base points.

2. **Special Picard varieties: the characteristic property.** It is a well known result (in the theory of reducible integrals) that, if V_q has general moduli, it contains no Picard subvarieties. If, however, for particular values of the moduli, V_q contains one such variety V_p $(1 \leq p \leq q-1)$, then it contains a congruence $\{V_p\}$; this congruence is Picardian, and its members are transforms of one another nnder G_q. In fact $\{V_p\}$ is a continuous system of the kind described in n. 1. In this case V_q contains a second Picard congruence $\{V_{q-p}\}$ of Picard varieties V_{q-p}, analogous to the first; and the members of both these congruences are non singular. Also neither congruence possesses singular points, through which pass an infinity of members.

In what follows we shall require the converse of this last result; *If V_q contains a congruence $\{V_r\}$, without base points, of (non singular) varieties V_r, then V_r and $\{V_r\}$ are both Picardian.* (Here, and in the sequel, it is understood that the case $r = q-1$ of an irrational pencil is included).

The proof is a simple consequence of the following theorem due to Castelnuovo [5]: if the variety V_r has superficial irregularity p $(0 < p < q)$, then either V_r is Picardian — in which case $p = r$ — or else V_r lies on a Picard subvariety V_p of V_q, and has the same superficial irregularity as V_p.

We first observe that the congruence $\{V_r\} = \Gamma$, say, is superficially irregular, since otherwise any complete linear system $|V_{q-1}|$ of hypersurfaces belonging to Γ would have a complete characteristic system. It follows that a certain number of the integrals u_i attached to V_q — $q-p$, say, where $0 < p < q$ — must be constant along each variety V_r, and that V_r has irregularity p. Hence, by Castelnuovo's theorem, either V_r is Picardian (in which case $r = p$) or else V_r lies on a Picard variety V_p which itself is a member of a congruence free from base points. Now through every point of V_q there passes just one variety V_p; hence those varieties V_r which pass through points of any such variety V_p must be contained in it. Thus V_p carries a congruence, without base points, which, by a previous remark, must be irregular. And the members V_r of this congruence have irregularity p: which is impossible. Hence only the first alternative can hold: both V_r and Γ are therefore Picardian

We note the following corollary: *if a continuous system* $\{V_r\}$ *is compounded of a congruence, then* V_r *is either Picardian or pseudo-Abelian.*

For, on each variety V_r we have a congruence of Picard varieties V^t $(1 \leq t \leq r-1)$ which are the trajectories of a continuous ∞^t group of automorphisms.

3. **On canonical varieties.** Let V_d be any non singular variety of any dimension $d \geq 1$, regular or irregular: then we may define on V_d a set of d invariant varieties $X_h(V_d)$ $(h = 0, 1, \dots, d-1)$ which may be effective or virtual, and which are called the canonical varieties of V_d. Their definition is by induction, as follows.

First, suppose that S is a non singular hypersurface of V_d which can vary in a linear pencil of general character (2); then denoting by δ the Jacobian set of the pencil, we define $X_0(V_d)$, by induction on d, by means of the rational equivalence

$$X_0(V_d) \equiv \delta - 2X_0(S) - X_0(S^2). \tag{3}$$

The aggregate $\{X_0(V_d)\}$ is termed the *Severi series* of V_d.

Next, supposing that S is *any* non singular hypersurface of V_d, we define

(2) This means that the base variety of the pencil is non singular (save when $d = 2$), that all members of the pencil are irreducible, and that the only members which possess multiple points are nodal.

$X_h(V_d)$, for $h \geq 1$, by induction on d and h, by means of the rational equivalences

(4) $$X_h(V_d) \equiv A_h(S) - X_h(S) \qquad (h = 1, 2, \ldots, d-1),$$

where $A_h(S)$ is an *adjoint variety*, with the property

(5) $$A_h(S) \cdot S \equiv X_{h-1}(S) \ (\text{on } S) \qquad (h = 1, 2, \ldots, d-1).$$

This definition obviously rests on a theorem (TODD [31, 33]). The elimination of $A_h(S)$ between (4) and (5) leads to the so-called adjunction law.

In (4) the variety $X_{d-1}(S)$ is simply S itself. Note also that it is customary to write $A_{d-1}(S) = S'$; the hypersurface $X_{d-1}(V_d) = S' - S$ varies in a linear system, the canonical system of classical algebraic geometry.

For other definitions and results concerning the canonical varieties, we refer to the work [23-4] of B. SEGRE, the monograph [3] by BALDASSARRI, and the critical review [34] by TODD.

It should be observed that the above equivalences are established by ascending induction: in the applications that follow the induction we employ is descending; the reason is that initially it is the variety $X_{d-1}(S) = S$, of highest dimension, that is given, while all the canonical varieties are determined from this.

Suppose now that V_d has superficial irregularity $q \geq d$: then in certain cases we can give a transcendental definition of $X_h(V_d)$, due to ÉGER [13], which generalises a method of SEVERI'S for the case $d = 2$. Taking any $h + 1$ linearly independent integrals u_i of the first kind on V_d, we consider the Jacobian locus $J(u_1, u_2, \ldots, u_{h+1})$; in the case where this is an h-dimensional variety, with simple components only, we may show it to be identical with $X_h(V_d)$ — which is thus effective (though possibly null).

For a self-contained exposition by transcendental methods, one actually requires $q > d$; if, however, the existence and properties of $X_h(V_d)$ have first been established geometrically, the present method can be applied provided that $q \geq d$.

We now wish to make such an application to the PICARD variety V_q: indentifying V_d with V_q, we may prove by the transcendental method that *all the varieties $X_h(V_q)$ are the null varieties of the corresponding rational equivalences on V_q*: we write $X_h(V_q) = 0$ $(h = 0, 1, \ldots, q-1)$. SEVERI has conjectured that this property is characteristic of V_q, but the question is still open.

4. **The canonical varieties of complete intersections.** We next apply the above results to the subvarieties of V_q. Consider a continuous system $\{V_{q-1}\}$ of hypersurfaces on V_q; we denote the complete intersections V_{q-1}^2, $V_{q-2}^3, \ldots,$

by $V_{q-2}, V_{q-3}, \ldots$. With this notation we have

THEOREM I. - *The canonical varieties of the sequence* (V_{q-k}) *are given by the equivalences*

$$X_h(V_{q-k}) \equiv \binom{q-h-1}{k-1} V_{q-1}^{q-h} \qquad (1 \leq k \leq q-1;\ 0 \leq h \leq q-k). \tag{6}$$

First, let $k=1$; then from (4) and (5), using the fact that $X_h(V_q)=0$, we obtain

$$\begin{aligned} &A_h(V_{q-1}) \equiv X^h(V_{q-1}) \qquad (\text{on } V_q) \\ &A_h(V_{q-1}) \cdot V_{q-1} \equiv X_{h-1}(V_{q-1}) \qquad (\text{on } V_{q-1}) \end{aligned} \qquad (h = q-1,\ q-2, \ldots, 1).$$

From these relations we deduce the equivalences

$$X_h(V_{q-1}) \equiv V_{q-1}^{q-h} \qquad (h = q-1,\ q-2, \ldots, 0).$$

Next, inserting these results in (4) and (5), we have

$$\begin{aligned} &V_{q-1}^{q-h} \equiv A_h(V_{q-2}) - X_h(V_{q-2}) \qquad (\text{on } V_{q-1}) \\ &A_h(V_{q-2}) \cdot V_{q-2} \equiv X_{h-1}(V_{q-2} \qquad (\text{on } V_{q-2}). \end{aligned}$$

Hence $X_h(V_{q-2}) \equiv (q-h-1)V_{q-1}^{q-h} \qquad (h = q-2,\ q-3, \ldots, 0)$.

The general result now follows by induction; in fact the algorithm represented by the adjunction law of n. 3 follows the law of formation of PASCAL'S arithmetical triangle.

5. **On the varieties** $X_h(V_r)$. Given any non singular variety V_r of V_q, we associate with it a sequence (V_{q-k}) $(k = 1, 2, \ldots, q-r)$ constructed as follows. First, let V_{q-1} be any hypersurface chosen generically from the continuous system $\{V_r\}$ to which V_r belongs; next, we choose for V_{q-2} a member of the *characteristic system* on V_{q-1}; then we take V_{q-3} to be a member of the characteristic system on V_{q-2}; and so on. We thus obtain a succession V_{q-1}, $V_{q-2}, \ldots, V_{r+1}$, of varieties which are non singular and which are all generated by varieties V_r. Finally, after these we have a variety $\overline{V}_r$, say, which is in general reducible, consisting of a certain number of varieties V_r.

We now prove

THEOREM II. - *The canonical varieties* $X_h(V_r)$ $(h = 0, 1, \ldots, r-1)$ *are all effective (possibly null).*

Consider V_r as a hypersurface on the variety V_{r+1} just defined. By Theorem I, $X_h(V_{r+1}) \geq 0$ (all $h \geq 0$); hence, by the equivalences (4), and with a descending induction on h, the required result is established.

COR. 1. - $P_g(V_r) > 0$ (this result is due to CASTELNUOVO and SEVERI).

COR. 2. - *V_r is free from exceptional manifolds.* For any such manifold is a locus of rational curves; and we have just seen that V_q cannot contain such curves.

6. **The inequalities for $P_g(V_r)$.** More expressive inequalities than the above Corollary 1 can be found as follows. In the first place we prove

THEOREM III. *If V_r is not pseudo-Abelian, then $P_g(V_r) > r(r - r)$; and if further $q > r + 1$, then $P_g(V_r) \geq (r + 1)(q - r)$.*

By Theorem I, the variety $\bar{V}'_r$ adjoint to $\bar{V}_r$, namely

$$\bar{V}'_r = X_{r-1}(\bar{V}_r) + \bar{V}_r \text{ (on } V_{r+1}),$$

is given by the equivalence

$$\bar{V}'_r \equiv (q - r)\bar{V}_r \text{ (on } V_{r+1}). \tag{7}$$

It follows from this that *the canonical system* $| X_{r-1}(V_r) |$ *is cut on V_r by varieties of the characteristic system, each counted $(q - r)$ times;* thus, in order to find a lower bound for $P_g(V_r)$, we have to estimate the dimension of this characteristic system.

Suppose first that $r = q - 1$; then since, by hypothesis, V_{q-1} belongs to a continuous system $\{V_{q-1}\}$ of ∞^q linear systems, the characteristic system has freedom $q - 1$ at least.

Next, assuming that $r < q - 1$, we follow the method given by Enriques ([14], p. 449) in the case $r = 2$. Consider a hypersurface V_{q-1} defined as in n. 5; any such V_{q-1} is generated by ∞^{q-r-1} V_r's, and belongs to a complete linear system $| V_{q-1} |$ of freedom ρ, say, contained in a continuous system $\{V_{q-1}\}$; thus there are $\infty^{q+\rho}$ V_{q-1}'s in $\{V_{q-1}\}$. Denoting by x the freedom of those V_{q-1}'s which contain a given V_r, we thus have

$$x \leq \{(q - r - 1) + (q + \rho)\} - q,$$

whence

$$x \leq q + \rho - r - 1.$$

Since the characteristic system cut on V_{q-1} has freedom $q + \rho - 1$, the characteristic system cut on V_r has freedom at least

$$q + \rho - 1 - (x - 1) \geq r + 1,$$

whence the result. The inequality $P_g(V) > r(q - r)$ was established by CASTELNUOVO and COMESSATTI on the hypothesis — unduly restrictive — that V_r contains no irregular congruence whatever. The above geometric proof of their result was first given by d'ORGEVAL [17] for the case $r = 2$; the obvious extension was noted in [22].

COR. - *If $P_g(V_r) < r(q - r)$, then V_r is either pseudo-Abelian or Picardian.*

In the case where V_r is pseudo-Abelian we have

THEOREM IV. - *If V_r is pseudo-Abelian of type t $(1 \leq t \leq r - 1)$, then $P_g(V_r) > (r - t)(q - r)$.*

Here the equivalence (7) again holds, but the variety V_r and all its transforms under the group G_q (n. 1) belong to a PICARD congruence $\{V_t\}$. The above calculation has therefore to be modified. Or we can reason as follows. Regarding the congruence $\{V_t\}$ as a PICARD variety whose points are the V_t's, we have on it a continuous system of varieties V_{r-t} which are the images of the V_r's. By the previous result, the characteristic system on such a V_{r-t} has freedom $r - t$ at least; whence the result (3).

COR. 1. - *For any V_r on V_q, the canonical system $|X_{r-1}(V_r)|$ is free from base points.*

This follows from (7), which holds in all cases.

COR. 2. - *A necessary and sufficient condition that V_r should be pseudo-Abelian is that $|X_{r-1}(V_r)|$ should be compounded of a congruence.*

The necessity is well known ([19-20]); the congruence in question is that of the trajectories. The sufficiency is a consequence of (7) and the theorem of n. 2 (more precisely, the corollary)

COR. 3. - *A necessay and sufficient condition that V_r should be a Picard variety is that $P_g(V_r) = 1$.*

The necessity follows from the last theorem of n. 3.

(3) If V_r can be regarded as pseudo-Abelian in more than one way, we choose the least possible value of t.

With regard to the sufficiency, by the corollary to Theorem III, V_r must be either pseudo-Abelian or Picardian, and the former possibility is excluded by Theorem IV.

7. Characterisation of pseudo-Abelian varieties. It is an important property of pseudo-Abelian varieties that the canonical varieties of any dimension less than that of the trajectories are all null ([19-20]). On a PICARD variety V_q this result can be inverted: we have

THEOREM V. - *A necessary and sufficient condition that V_r should be pseudo-Abelian or Picardian is that, for some value of h $(0 \leq h \leq r-1)$, $X_h(V_r) = 0$.*

(i) Suppose first that $r = q - 1$, so that, by Theorem I,

$$X_h(V_{q-1}) \equiv V_{q-1}^{q-h} \quad (0 \leq h \leq q-1).$$

If, for some value of h, $X_h(V_{q-1}) = 0$, we deduce that $V_{q-1}^{q-h} = 0$, and hence that $V_{q-1}^{q-h+1} = V_{q-1}^{q-h+2} = \ldots = 0$. Thus the continuous system $\{V_{q-1}\}$, which is effectively free from base points, is such that its characteristic manifolds of every dimension less than $h + 1$ are all null. Hence the system belongs to a congruence, which proves the result (n. 2).

(ii) Next, let $r < q - 1$; and consider the succession (V_{q-k}) defined in n. 5. On V_{r+1} we have, by (4) and (5),

$$\left.\begin{aligned} A_h(V_r) &\equiv X_h(V_{r+1}) + X_h(V_r) \\ A_h(V_r) \cdot V_r &\equiv X_{h-1}(V_r) \end{aligned}\right\} (h = 1, 2, \ldots, r-1).$$

Suppose that $X_{h-1}(V_r) = 0$; then since all the varieties which appear in the above equivalences are effective, it follows that

$$X_h(V_r) \cdot V_r = 0, \quad X_h(_{r+1}) \cdot V_r = 0 \text{ (on } V_{r+1}).$$

Now since V_r is variable in an ∞^1 system on V_{r+1}, the latter result shows that either $X_h(V_{r+1})$ belongs to a congruence of which V_r is a member or is compounded, or else that $X_h(V_{r+1}) = 0$.

In the first case, considering the members of the sequence (V_{q-k}) in turn, we arrive at the conclusion that the system $\{V_{q-1}\}$ is compounded of this same congruence; whence the result.

In the second case, we have, exactly as before,

$$X_{h+1}(V_{r+2}) \cdot V_{r+1} = 0 \text{ (on } V_{r+2}).$$

Again there are two possibilities : either $X_{h+1}(V_{r+2})$ belongs to a congruence of which V_{r+1} is a member or is compounded, or else $X_{h+1}(V_{r+2}) = 0$.

Continuing in this way, either we obtain a congruence on V_q of which $\{V_r\}$ is compounded, or else we conclude that

$$X_{q-r+h+2}(V_{q-1}) = 0 \ (1 \leq h \leq r-1).$$

And in this latter case the result follows from (i).

Recalling the details of the proof of Theorem II, we now have

Cor. 1. - *Except when V_r is pseudo-Abelian or Picardian, the varieties $X_h(V_r)$ are all effective, h-dimensional and of positive order. In particular, the generic element of the Severi series $\{X_0(V_r)\}$ consists of a positive number of points.*

In n. 2 we defined a special variety V_q as one containing a Picard subvariety. By virtue of the previous theorem we can say that: *In order that V_q should be special, it is necessary and sufficient that V_q should contain a variety V_r such that, for some value of $h \geq 0$, $X_h(V_r) = 0$.* For such a V_r is either Picardian or pseudo-Abelian, and in the latter case the trajectories of V_r are Picard varieties.

This result may also be expressed as follows:

Cor. 2. - *The existence on V_q of a single variety V_r such that, for some value of $h \geq 0$, $X_h(V_r) = 0$, implies the existence of an invariant subgroup in the group G_q; and conversely.*

It is interesting to consider the above theorem in the light of what we may call the generalised Severi conjecture (cf. n. 3); according to this, any algebraic variety U_r such that $X_h(U_r) = 0$ $(0 \leq h \leq t-1)$, while $X_t(U_r) > 0$, is pseudo-Abelian of type t. Evidently a far stronger result holds when U_r lies on V_q. In that case U_r is of course superficially irregular. A first step towards proving the conjecture would be to establish this property in the general case.

II

8. **The Picard-Severi variety.** Consider a varieti W_r of superficial irregulation $q > 0$; and let u_i $(i = 1, 2, \ldots, q)$ be the corresponding simple integrals of the first kind attached to W_r. These define a period matrix ω which is a Riemann matrix; denoting by V_q the Picard variety, in normal form, belonging to ω, we say that V_q is the *Picard-Severi variety* (or second Picardian) associated with W_r.

Assuming u_i to be the universal coordinates on V_q, we suppose V_q to be represented by the parametric equations (1); then denoting by x a general point of W_r, we obtain a mapping of W_r on V_q by means of the equations

$$u_i(x) = u_i \quad (i = 1, 2, \ldots, q). \tag{8}$$

We first enquire under what conditions this yields a **proper model** of W_r, i. e. an irreducible r-dimensional variety V_r, in simple or multiple correspondence with W_r. The answer, due to SEVERI [26-7] is as follows: *the proper model V_r of W_r on V_q exists if, and only if, W_r contains no congruence of superficial irregularity q.* When $r > q$, such a congruence always exists, as we shall now see. In fact, to construct the model V_r we have to consider the relations (modulo the periods)

$$u_i(x) \equiv u_i(y) \quad (i = 1, 2, \ldots, q), \tag{9}$$

where x, y are points of W_r, the former chosen generically and the latter to be determined. If (9) admit one and only one solution, namely $y = x$, we obtain a model V_r which is birationally equivalent to W_r. If instead (9) admit $\nu(>1)$ solutions, we have a V_r which is the image of an involution I_ν on W_r, which is called the *fundamental involution* on W_r, and then V_r is a ν-fold model of W_r. In every other case the relations (9) represent a congruence, of superficial irregularity q, of algebraic varieties along which the integrals u_i all remain constant in this case the proper model is non-existent.

Supposing now that the model V_r exists, we still do not know whether it is necessarily non singular. At present the question can be answered (in the affirmative) only in certain special cases. In all that follows we shall make the assumption that, whenever W_r admits a proper model, there is also a proper model which is free from multiple points (But see n. 12).

9. **First applications.** The first applications of the mapping of W_r on V_r are immediate consequences of the preceding theorems, together with the remark that, whether the model V_r is simple or multiple, we always have $P_g(W_r) \geq P_g(V_r)$.

To begin with, it follows from Theorem III that if W_r admits a model V_r which is not pseudo-Abelian, then in all cases $P_g(W_r) > r(q-r)$; if also $q > r+1$, $P_g(W_r) > (r+1)(q-r)$. Whence

THEOREM VI. - *If $P_g(W_r) \leq r(q-r)$ $(q \geq r)$, or if $P_g(W_r) \leq (r+1)(q-r)$ $(q > r+1)$, then either W_r contains a congruence of (superficial) irregularity q, or else V_r is pseudo-Abelian; and in the latter case W_r — in correspondence with V_r — contains two congruences of complementary irregularities.*

Cor. - *If $P_g(W_r) = 0$, then W_r contains a congruence of irregularity q.* As already stated, this result is due to Castelnuovo and Severi.

Theorem VI is an improved form of the inequality due to Castelnuovo and Comessatti. We may add that Comessatti [8], using an ingenious idea in the differential geometry of line systems, has determined all the surfaces for which $P_g(W_2) \leq 2(q-2)$; such surfaces reduce to a few types. Comessatti's results have been confirmed by Nollet [16]. The analogous problem for threefolds has been attacked by Rosenblatt [18], employing the same methods, but only partial results have been obtained.

In the original version of Theorem VI, the sole inference made from the inequality was to the effect that W_r must contain an irregular congruence. It is interesting to compare this form of the result with one obtainable from the theory of differential forms ([29]) If we assume that W_r contains no irregular congruence whatever, we know that none of the differential forms such as $du_1 du_2 \dots du_r$ can vanish identically on W_r. Hence the number of differential forms of the first kind and of degree r attached to W_r is at least $\binom{q}{r}$: whence the theorem ([29]): *If $P_g(W_r) < \binom{q}{r}$, then W_r contains an irregular congruence.* We may remark that the case $r = 2$ of this theorem was proved by Castelnuovo [6] many years before (4) the general result was published by Severi. Castelnuovo's proof was different in character.

Using Theorem IV in the same way as Theorem III, we have

Theorem VII. - *If W_r is pseudo-Abelian of type t $(1 \leq t \leq r-1)$, and if $P_g(W_r) \leq (r-t)(q-r)$, then W_r contains a congruence of irregularity q.*

From Cor. 3 to the same theorem we deduce

Theorem VIII. - *If $P_g(W_r) = 1$, either W_r contains a congruence of irregularity q or else — in the case $q = r$ — W_r can be mapped, simply or multiply, on a Picard variety V_r.*

The case $r = 3$ of this result was given by Comessatti [7].

Cor - *If $X_{r-1}(W_r) = 0$, either W_r contains a congruence of irregularity q or else is a Picard variety.*

For supposing that V_r exists, in the case $\nu = 1$ there is nothing to prove. If however $\nu > 1$, the fact that $X_{r-1}(W_r)$ is null implies that in the mapping

(4) Castelnuovo's work remained unpublished for a long time.

on V_r there can be no branch elements. Hence, as ENRIQUES ([14], p. 360) has shown, W_r admits the continuous group of automorphisms which characterises a PICARD variety (n. 1).

10. **Characterisation of W_r by means of subvarieties.** An obvious way of ensuring that W_r must contain a congruence of irregularity q is to impose on W_r a condition which will render the proper model V_r nonexistent. One such condition (Theor. VI. cor.) is $P_g(W_r) = 0$. We shall now consider conditions of a different kind.

Suppose that W_r contains a subvariety W_s $(1 \leq s \leq r-1)$ which does not admit a proper model on V_q; this will be the case if, for example, $P_g(W_s) = 0$, or again if $g_1(W_s) = 0$. Then there are two possibilities: either V_r does not exist or else W_s must correspond, in the mapping, to some variety of dimension less then s. In the latter event this means that W_s must either be, or be on, an exceptional or a fundamental manifold, according as $\nu = 1$ or $\nu > 1$. Since the number of such manifolds is necessarily finite, we have

THEOREM IX. - *If W_r contains a system (continuous or discontinuous) of varieties W_s which invade W_r and for which either $P_g(W_s) = 0$, or $g_1(W_s) = 0$, then W_r carries a congruence of irregularity q.*

Consider the particular case in which $s = r-1$, $g_1(W_{r-1}) = 0$. Here, by hypothesis, W_{r-1} cannot carry a superficially irregular congruence of varieties or involution of points, so the system $\{W_{r-1}\}$ to which W_{r-1} belongs must be contained in the congruence — necessarily unique — of irregularity q on W_r. Hence

THEOREM X. - *If W_r contains an ∞^1 system (continuous or discontinuous) of superficially regular hypersurfaces, then this system must belong to the unique congruence of irregularity q on W_r.*

In the case where this congruence is a *pencil*, the system $\{W_{r-1}\}$ in question consists of members of that pencil, so that if the system is discontinuous *a priori*, it is actually continuous in fact. A classical instance of this result is that of a surface W_2 which carries a system of rational curves; here all the curves in question are contained in a pencil of genus q, and W_2 is birationally equivalent to a scroll.

Consider next the case where $P_g(W_s) = 1$; if W_s admits a proper model V_s, this is a simple or multiple PICARD variety, in which case $g_1(W_s) = s$ (Theorem VIII). It follows that, if W_r likewise possesses a proper model V_r, this most be a simple or multiple pseudo-Abelian or PICARD variety, for V_s

is a member of a congruence of Picard varieties on V_r. Hence

Theorem XI. - *If W_r contains a system (continuous or discontinuous) of varieties W_s which invade W_r and for which $P_g(W_s)=1$, then either W_r carries a congruence of irregularity q or else the proper model V_r is pseudo-Abelian or Picardian.*

An important instance of this result is afforded by a pseudo-Abelian variety W_r of type t; if this admits a model V_r, then the congruence of trajectories W_t on W_r is mapped by a congruence $\{V_t\}$ of Picard varieties on V_r, which is likewise pseudo-Abelian. If when $\nu > 1$, there are branch elements in the representation, these must consist exclusively of varieties belonging to the congruence $\{V_t\}$; for in that case, and in that case only, the generic V_t does not meet the branch manifold, and so corresponds to a set of ν Picard varieties on W_r.

11. **Further applications.** We must now examine the nature of the correspondence between the varieties V_r and W_r. Suppose first that $\nu=1$: this means that the congruences (9) in general admit just one solution $y=x$. Thus to a point of W_r there always corresponds just one point of V_r; but to a point of V_r there may correspond a variety of W_r. Now, by Theor. II, Cor. 2, V_r is free from exceptional manifolds. Thus, *If a variety W_r possesses a simple model V_r on its Picardian V_q, its exceptional manifolds can all be removed by birational transformation.* In other words, if the relations (9) in general admit just one solution, we can find a birational transform of W_r on which the relations always admit just one solution.

In the case $\nu > 1$, it is still true that V_r is free from exceptional manifolds. But we do not know whether, for $r > 2$, the fundamental elements (if any) in the correspondence between V_r and W_r can be removed by suitable birational transformation of W_r. It may be that such a transformation cannot always be found.

The remaining preliminary remarks concern the transcendental definition of the canonical varieties $X_h(W_r)$ which we have already mentioned in n. 3. Consider first the case $h=0$; here the essential results for surfaces are due to De Franchis [12] and Severi [28]. As Severi has stated elsewhere ([30]), they extend to any variety W_r $(r > 2)$.

Let u be any simple integral of the first kind attached to W_r. Then, if the Jacobian set $J(u)$ of the pencil is infinite, the set in question consists of one or more algebraic varieties, along each of which u is constant. If instead $J(u)$ consists of a finite set of points, then we have $J(u)=X_0(W_r)$; this last result was established independently for $r > 2$ by Todd [32] and Éger [13]. It may be deduced that *a necessary and sufficient condition that $J(u)$ be infi-*

nite for the general integral u of W_r is that W_r should contain a congruence of irregularity q. It follows that, if W_r does not contain such a congruence, the generic set $X_0(W_r)$ is effective and consists of a finite number of points.

We thus see that, in the (simple or multiple) correspondence between W_r and the model V_r, whenever the latter exists, a set $X_0(V_r)$ transforms in general into an effective finite set $X_0(W_r)$. Hence, *if, for a variety W_r, the set $X_0(W_r)$ is in general virtual or infinite, W_r must carry a congruence of irregularity q.* This proposition, which is analogous to Theor. VI, Cor., generalises one of Dantoni's results for surfaces ([10]). However, in the case $r = 2$, one can be more precise; in fact the only surfaces — regular or irregular — which possess this property are irrational scrollar (of genus greater than unity).

In the case $h = r - 1$, the definition of $X_{r-1}(W_r)$ by means of a JACOBIAN was likewise first given by SEVERI for surfaces. This generalises at once to any variety V_r of V_q; if $u_1, u_2, \ldots, u_r$ are linearly independent integrals attached to V_r, the Jacobian $J(u_1, u_2, \ldots, u_r)$ is in general a hypersurface on V_r, and in that case it is identical with a variety $X_{r-1}(V_r)$. The exceptional case is that in which V_r carries an irregular congruence: we cannot then employ in the definition any integral which is constant along members of the congruence.

Instead of the Jacobian locus, we may introduce the concept of differential form (n. 9), following an idea due to KÄHLER [15] Consider, on V_r, the outer product $du_1\, du_2 \ldots du_r$; unless this vanishes identically, it is a differential form of the first kind and of degree r; there are in all $P_g(V_r)$ linearly independent differential forms of this type. In the mapping of V_r on W_r, the form in question corresponds to an analogous form

$$\frac{\partial(u_1, u_2, \ldots, u_r)}{\partial(x_1, x_2, \ldots, x_r)}\, dx_1 dx_2 \ldots dx_r,$$

where $x_1, x_2, \ldots, x_r$ are local coordinates on W_r. We use this result in the next section.

12. Coincidences in I_ν. Assuming now that $\nu > 1$, we take $x_1, x_2, \ldots, x_r$ for local coordinates on W_r, and $u_1, u_2, \ldots, u_r$ for local coordinates on V_r; the remaining integrals u_i ($i = r + 1, r + 2, \ldots, q$ will then be holomorphic functions of $u_1, u_2, \ldots, u_r$.

A coincidence in the involution I_ν (and, correspondingly, a branch point on V_r) can occur when and only when the determinant

$$j \equiv \partial(u_1, u_2, \ldots, u_r)/\partial(x_1, x_2, \ldots, x_r) \tag{10}$$

vanishes (n. 8). When this condition is satisfied, it is clear that every other such determinant obtained by choosing any r different variables from the q integrals u_i will likewise vanish. As in the case $r=2$ (ANDREOTTI [1-2]), we may show that *coincidences in I_ν cannot be isolated.* We now prove

THEOREM XII. - *Coincidences in I_ν can occur only when V_r is pseudo-Abelian or Picardian (in which case $q=r$); and in all such cases the branch locus on V_r belongs to the congruence of trajectories.*

(In the case where V_r is Picardian it is to be understood that the «trajectories» in question are those of some subgroup of G_q (n. 1)).

(*i*) First let $q=r$, in which case V_r is a PICARD variety. If V_r contained no irregular congruence whatever, there could be no branching, since in that case the Jacobian j defined by (10) would give rise to the canonical hypersurface $X_{r-1}(V_r)$, which we know to be null (n. 3). It follows that V_r must contain at least one congruence represented by the equations $u_i=c_i$ $(i=1, 2, \ldots, r-t)$; and then the branch locus belongs to a congruence $\{V_t\}$ of PICARD varieties.

(ii) Supposing that $q>r$, we may also assume that $P_g(V_r)>r$. For we have $r(q-r)>r$, so that if $P_g(V_s)\leq r$, V_r would be pseudo-Abelian (Theor. III, Cor.) Thus the complete system $|K| \equiv |X_{r-1}(V_r)|$ has dimension r at least.

We may also assume that the system $|K|$ does not belong to a congruence, since, if it did, it would again follow that V_r is pseudo-Abelian (Theor. IV, Cor. 2).

Suppose, if possible, that $|K|$ does not belong to a congruence. We may then select from $|K|$ at least $r+1$ linearly independent members $K_1, K_2, \ldots, K_{r+1}$, given by Jacobians j whose arguments are chosen from the set (u_i); for otherwise it would follow from Comessatti's work [7] that $|K|$ must belong to an irregular congruence on V_r.

By what has been said before, if there exists a branch locus on V_r, this must be common to all the hypersurfaces $K_1, K_2, \ldots, K_{r+1}$. Now this locus is identical with the Jacobian variety $J(K)$ of the set (K_i), provided the Jacobian effectively exists — as, on our hypotheses, it certainly does. But, from the classical linear equivalence ([30])

$$J(K)-(r+1)K \equiv K,$$

it follows that $J(K)=(r+2)K$ is a pluricanonical variety, in contradiction to the previous result that the branch locus must be contained in $|K|$.

Hence $|K|$ must be compounded of a congruence $\{V_t\}$, say $(1 \leq t \leq \leq r-1)$, the members of which are PICARD varieties (n. 5); thus V_r is pseudo-Abelian of type t. And we have shown incidentally that the branch locus belongs to $\{V_t\}$. This locus may be reducible and also pure or impure.

The above theorem generalises Andreotti's results [1 - 2] for the case $r = 2$. In that case, however, it is possible to go on to assert that the surface W_2 is elliptic, i. e pseudo-Abelian of type 1. For $r > 2$, the immediate conclusion is less precise: we observe that, since the generic element V_t of the congruence $\{V_t\}$ does not meet the branch locus, it must correspond to a set of ν varieties W_t on W_r, each of which is birationally equivalent to V_t. It follows that W_r *likewise contains a congruence of birationally equivalent Picard varieties* W_t; there is of course a complementary congruence corresponding to that carried by V_r. But in the mapping of W_t on V_t there are in general exceptional features — in other words, W_t is not in normal form. At any rate, we see from the correspondence between W_r and V_r that W_r admits a continuous group of ∞^t automorphisms the trajectories of which are the varieties W_t. But in the general case the transformations of the group are not completely transitive over W_t. If, as seems plausible, the exceptional elements in the correspondence could be removed, it would then follow that W_r is pseudo-Abelian (of type t).

One important feature of this mapping should be noted. Consider a variety $V_{t,s}$ of the congruence $\{V_t\}$ which is an s-fold component ($s \geq 2$) of the branch locus, or which lies on such a component. To it there corresponds on W_r a member $W_{t,s}$ of the congruence of « quasi-trajectories » such that the multiple $sW_{t,s}$ is algebraically equivalent to the generic W_t of $\{W_t\}$. The varieties $W_{t,s}$, for a given value of s, may be isolated or they may generate a certain number of manifolds belonging to the congruence $\{W_t\}$. If in particular W_r is pseudo-Abelian, then we must expect such submultiple varieties to occur in the congruence of trajectories. When, however, W_r lies on a PICARD variety these submultiples are always absent (c. f. n. 1).

It is worth remarking that *whenever a variety W_r admits a proper model V_r (simple or multiple) which is pseudo-Abelian, then the hypothesis of n. 8 is superfluous.* In particular, then if the involution I_ν possesses coincidences, there *always* exists a non singular model V_r of W_r.

13. **The case $\nu > 1$.** We have seen that, when $\nu > 1$, and there are branch elements in the correspondence, then V_r must be pseudo-Abelian. The question now arises: what can be said concerning the case $\nu > 1$ when branching is absent? Here we have a formula, established by ANDREOTTI [2] for surfaces, but of general validity. Considering the fundamental groups of V_r and W_r, he shows that the SEVERI divisors of the two varieties are connected by the relation $\sigma(V_r) = \nu\sigma(W_r)$. It thus follows that $\sigma(W_r) < \sigma(V_r)$. Whence

THEOREM XIII. - *If W_r admits a proper model V_r, then this model can be multiple if, and only if, one of the following alternatives holds:*

(i) *V_r is pseudo-Abelian;*

(ii) *The divisor of W_r is less than the divisor of V_r.*

In all other cases V_r and W_r are birationally equivalent. Condition (ii) is obviously satisfied whenever V_r is free from torsion.

14. Some unsolved problems. In conclusion we draw attention to some of the questions raised and left unanswered in the present work.

(i) We have used throughout II the working hypothesis that, whenever the variety W_r admits a proper model V_r, then it admits a non singular model. This assumption is needed even in the case $\nu = 1$; for although we have seen that in that case the variety V_r is certainly free from exceptional manifolds, it is not necessarily free from multiple points.

(ii) In the case $\nu = 1$, we know that any exceptional elements in the correspondence between V_r and W_r can be removed. When, however, $\nu > 1$, we do not know whether the elimination can always be effected. When V_r happens to be pseudo-Abelian this seems likely; if so, we should then be able to conclude that W_r is pseudo-Abelian if, and only if, V_r is pseudo-Abelian. But even so, no light is shed on the remaining cases, i. e. all those in which the correspondence has no branch elements.

(iii) Given a variety W_r which carries no congruence of irregularity q, we may wish to know in what circumstances it will possess a simple model V_r. Theorem XIII answers this question, but the hypotheses entail some knowledge of the model itself. It would be interesting to obtain a statement involving only properties of W_r.

(iv) All our results concerning W_r in relation to V_r require the hypothesis that W_r carries no congruence of irregularity q. When this condition is unfulfilled, we have a mapping of W_r on a Picardian of dimension less than r, and the various question we have discussed cannot (it seems) be broached with the present methods. Other problems, such as classification, cannot be attempted either. The question of finding some alternative approach is important, since various significant classes of irregular varieties, e. g. the pseudo-Abelian varieties of geometric genus zero, fall within this category.

(v) In n. 3 we have mentioned SEVERI's conjecture concerning V_q. It is natural to ask whether our methods would suffice to establish a weaker form of this conjecture: *any variety W_r for which $g_1(W_r) = q > 0$, $X_h(W_r) = 0$*

(all h) is necessarily Picardian. By Theorem VIII, any such variety W_r is either Picardian or else contains a congruence of irregularity q; we have therefore to eliminate the latter possibility. However, in order to do so, it appears that we must use the original SEVERI conjecture as applied to varieties of any dimension less than r!

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