Differential equations with fixed critical points (*).

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Summary. - *See section 1.*

1. This paper is the second part of a group of studies concerning differential equations with fixed critical points. We shall use the methods, notations, and terminologies employed in the first part [denoted Part I] where second order equations where considered. In particular, K with or without a subscript will denote a constant, not always the same.

The object of this paper is: to determine all the equations of the form

$$\dddot{y} = P_1(y)\ddot{y} + P_2(y)\dot{y}^2 + P_2(y)\dot{y} + P_4(y) \tag{1.1}$$

where $P_n(y)$ is a polynomial in y of degree n with analytic coefficients in x, whose general integral has no parametric critical points.

A few classes of stable equations of order four are considered (see Part III).

Part II. - Equations of the third order.

I. The theorems of stability.

2. In this article, we introduce two theorems which are fundamental in our investigation.

Let x be a complex independent variable and ε a complex parameter. Let $y = (y_1, \dots, y_p)$ be a set of functions satisfying the system of ordinary

(*) The research reported in this document has been sponsored by the Office of Scientific Research, OAR through the European Office, Aerospace Research, USAF.

differential equations

$$\frac{dy}{dx} = A(x;\ y;\ \varepsilon) \tag{2.1}$$

where $A = (A_1, \ldots, A_n)$ is a given vector-function of x, y, ε, holomorphic within a certain domain containing the point $x = x_0$, $y = y^0 = (y_1^0, \ldots, y_n^0)$, $\varepsilon = 0$.

General theorem of stability. - *If the general solution of the differential system* (2.1) *is single-valued in x for all values of ε in a neigborhood $|\varepsilon| \leq \varepsilon_0$ of $\varepsilon = 0$, except possibly $\varepsilon = 0$, then it will also be single-valued in x for $\varepsilon = 0$. Moreover, if one writes*

$$y(x) = y(x;\ \varepsilon) = v(x) + \sum_{p=1}^{\infty} u_p(x)\,\varepsilon^p, \tag{2.2}$$

then the coefficients $v(x)$, $u_p(x)$ of (2.2) *are also single-valued.*

For fuller information about this theorem, we refer to Part I, § 1.

Now, in this and the next paragraphs, let c_i, $(i = 1, \ldots, p)$, denote constants, $k \geq 0$ an integer, $P(x;\ z)$ a polynomial in z of degree $\leq k - 2$, and $H_i(x;\ z;\ u)$, $(i = 1, 2, \ldots)$, a polynomial in u, $\dot{u}$, ..., $u^{(p-1)}$. Further, we suppose that $h(x;\ z)$, $H_i(x;\ z;\ u)$ are holomorphic functions of z at $z = 0$ and that $P(x;\ z)$, $h(x;\ z)$, $H_i(x;\ z;\ u)$ are analytic functions of x in a given domain D.

We consider the differential system

$$\left\{\begin{aligned} &\dot{z} = 1 + zP(x;\ z) + z^k u, \\ &z^r \frac{d^r u}{dx^r} + c_1 z^{r-1} \frac{d^{r-1} u}{dx^{r-1}} + \ldots + c_{r-1} z \frac{du}{dx} + c_r u \\ &= h(x;\ z) + zH_1(x;\ z;\ u) \end{aligned}\right. \tag{2.3}$$

concerning which the following fundamental theorem holds:

THEOREM I. - *In order that the differential system* (2.3) *be stable, it is necessary*

i. *that the roots of the indicial equation*

$$\begin{aligned} &\Theta(\Theta - 1) \ldots (\Theta - r + 1) + c_1\Theta(\Theta - 1) \ldots (\Theta - r + 2) + \ldots \\ &\ldots + c_{r-1}\Theta + c_r = 0 \end{aligned} \tag{24}$$

be distinct integers, positive, negative or zero;

ii. *if one of these roots is zero, that* $h(x;\,0) \equiv 0$.

Proof. - Set $x = a + \varepsilon t$, $z = \varepsilon v$; $a \in D$ is a constant. Substitution in (2.3) yields

$$\frac{dv}{dt} = 1 + \varepsilon v P(a + \varepsilon t;\; \varepsilon v) + \varepsilon^k v^k u,$$

$$v^r \frac{d^r u}{dt^r} + c_1 v^{r-1} \frac{d^{r-1} u}{dt^{r-1}} + \dots + c_{r-1} v \frac{du}{dt} + c_r u$$

$$= h(a + \varepsilon t;\; \varepsilon v) + \varepsilon v H_1(a + \varepsilon t;\; \varepsilon v;\; u).$$

When $\varepsilon = 0$, one finds the reduced system

$$(2.5)\qquad \begin{cases} \dfrac{dv}{dt} = 1, \\[2ex] v^r \dfrac{d^r u}{dt} + c_1 v^{r-1} \dfrac{d^{r-1} u}{dt^{r-1}} + \dots + c_{r-1}\, v \dfrac{du}{dt} + c_r u = h(a;\; 0). \end{cases}$$

Therefore, if b denotes a constant, one obtains

$$v(t) = t - b,$$

$$(2.6)\qquad (t-b)^r \frac{d^r u}{dt^r} + c_1 (t-b)^{r-1} \frac{d^{r-1} u}{dt^{r-1}} + \dots + c_{r-1}(t-b) \frac{du}{dt} + c_r u$$

$$= h(a;\; 0).$$

The Euler equation (2.6) may be transformed into a linear equation with constant coefficients by means of a substitution $t - b \to e^t$. This leads to the indicial equation (2.4).

Suppose that Θ_1 is a simple root of (2.4); then equation (2.6) has a solution of the form t^{Θ_1}.

Suppose that Θ_1 is a multiple (say j-tuple) root of (2.4); then equation (2.6) has j distinct solutions of the form

$$t^{\Theta_1},\; t^{\Theta_1} \lg t,\; \dots,\; t^{\Theta_1} (\lg t)^{j-1}.$$

In order that equation (2.6) be stable, it is thus necessary that the roots of the indicial equation (2.4) be distinct integers.

Suppose now that one of the roots of (2.4) is zero. Then, $c_r = 0$ and

$$C \equiv (-1)^{r-1}(r-1)! + (-1)^{r-2}c_1(r-2)! + \dots + c_{r-1} \neq 0.$$

Equation (2.6) has the particular solution

$$\frac{1}{C} h(a;\ 0)\ lg(t-b).$$

Therefore, in order that $u(t)$ be stable, it is necessary that $h(a;\ 0) = 0$; because $a \in D$ is arbitrary, this condition becomes $h(x;\ 0) = 0$.

3. For differential equations of the second order, theorem I takes the simpler form:

In order that the differential system

$$\text{(3.1)} \qquad \begin{cases} \dot{z} = 1 + zP(x;\ z) + z^k u, \\ z\dot{u} = pu + h(x;\ z) + zH_1(x;\ z;\ u) \end{cases}$$

be stable, it is necessary

i. *that the constant p be an integer, positive, negative or zero;*

ii. *if $p = 0$, that $h(x;\ 0) \equiv 0$.*

This theorem was used throughout our previous paper (Part I).

Now consider, in particular, the case where the differential system is

$$\text{(3.2)} \qquad \begin{cases} \dot{z} = 1 + zu, \\ z^2\ddot{u} + c_1 z\dot{u} + c_2 u = h(x;\ z) + zH_1(x;\ z;\ u). \end{cases}$$

Denote by p and q, the roots of the indicial equation

$$\text{(3.3)} \qquad \Theta(\Theta - 1) + c_1\Theta + c_2 = 0;$$

p and q are integers and $q > p$. Suppose $p > 0$ and set

$$\text{(3.4)} \qquad \begin{cases} u = P_{p-1}(x;\ z) + z^p v, \\ P_{p-1}(x;\ z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_{p-1} z^{p-1}; \end{cases}$$

v is an unknown function and $P_{p-1}(x;\ z)$ is a polynomial in z of degree $p-1$ whose coefficients will be determined later on. We shall write P instead of $P_{p-1}(x;\ z)$ when the meaning is clear.

Substitution in (3.2) gives

$$\text{(3.5)} \qquad \dot{z} = 1 + zP + z^{p+1} v$$

so that from (3.4), it follows that

$$\dot{u} = \dot{P} + z^p \dot{v} + p z^{p-1} v + z^p H_2(x;\ z;\ v),$$

$$\ddot{u} = \ddot{P} + z^p \ddot{v} + 2p z^{p-1} \dot{v} + p(p-1) z^{p-2} v$$

$$+ p(2p-1) P z^{p-1} v + z^p H_3(x;\ z;\ v;\ \dot{v})$$

where $H_2(x;\ z;\ v)$ and $H_3(x;\ z;\ v;\ \dot{v})$ have the same properties as $H_1(x;\ z;\ v)$. Therefore, because of (3.3), the second equation (3.2) takes the form

$$\text{(3.6)} \qquad \begin{aligned} z^p[z^2 \ddot{v} + (c_1 + 2p) z \dot{v}] &= -z^2 \ddot{P} - c_1 z \dot{P} - c_2 P + h(x;\ z) \\ &+ z H_1(x,\ z;\ P + z^p v) + z^{p+1} H_4(x;\ z;\ v). \end{aligned}$$

Now, use (3.5) and write the right member of (3.6) in the form

$$\text{(3.7)} \qquad A_0 + A_1 z + \ldots + A_p z^p + z^{p+1} H_5(x;\ z;\ v);$$

the A's are analytic functions of x and $H_5(x;\ z;\ v)$ has the same properties as $H_1(x;\ z;\ u)$.

Suppose that we determine the p coefficients α_i by setting

$$\text{(3.8)} \qquad A_0 = A_1 = \ldots = A_{p-1} = 0;$$

then, the differential system (3.2) becomes

$$(3.9)\qquad \begin{cases} \dot{z} = 1 + zP_{p-1}(x;\, z) + z^{p+1}v, \\ z^2\ddot{v} + (c_1 + 2p)\, z\dot{v} = A_p(x) + zH_5(x;\, z;\, v). \end{cases}$$

Then the condition for stability, $A_p(x) \equiv 0$, follows from theorem I.

It is clear that when $A_p(x) \equiv 0$, the system (3.9) has a unique holomorphic solution $z(x)$, $v(x)$ such that $z(x_0) = 0$, $v(x_0) = v_0$, where v_0 is an arbitrary constant.

The differential system (3.9) has thus the form

$$(3.10)\qquad \begin{cases} \dot{z} = 1 + zP_{p-1}(x;\, z) + z^{p+1}v, \\ z\ddot{v} + (c_1 + 2p)\,\dot{v} = H_5(x;\, z;\, v); \end{cases}$$

the related indicial equation is

$$\Theta(\Theta - 1) + (c_1 + 2p)\,\Theta = 0$$

and has the roots $\Theta = 0$, $\Theta = q - p$.

We apply again the same method to the differential system (3.10). To do this, we set

$$(3.11)\qquad \begin{cases} v = Q_{q-p-1}(x;\, z) + z^{q-p}\, w, \\ Q_{q-p-1}(x;\, z) \equiv \alpha_p + \alpha_{p+1} z + \dots + \alpha_{q-1} z^{q-p-1}; \end{cases}$$

w is an unknown function and $Q_{q-p-1}(x;\, z)$ is a polynomial in z of degree $q - p - 1$ whose coefficients will be determined as follows.

Substitution of v into (3.10) gives rise to a polynomial

$$B_0 + B_1 z + \dots + B_{q-p} z^{q-p}$$

analogous to $A_0 + \dots + A_p z^p$. Note that α_p remains arbitrary and may be assumed to be a constant parameter; then $\alpha_{p+1}, \dots, \alpha_{q-1}$ are determined by setting

$$B_0 = B_1 = \dots = B_{q-p-1} = 0$$

and the condition for stability is

$$B_{q-p}(x) \equiv 0.$$

Instead of making the substitutions (3.4) and (3.11) successively, one may also set

$$u = P_q(x;\ z) + z^q v$$

where

$$P_q(x;\ z) \equiv P_{p-1}(x;\ z) + z^p Q_{q-p-1}(x;\ z)$$

or

$$P_q(x;\ z) = \alpha_0 + \ldots + \alpha_{q-1} z^{q-1}.$$

The coefficients $\alpha_0, \ldots, \alpha_{q-1}$ are determined as before; α_p remains arbitrary and will be assumed to be a constant parameter.

We shall content ourselves with these general lines of the argument; it is hoped that subsequent applications of the method will make these operations transparent.

Finally, we note that the method applies also when the second equation of the differential system (3.2) [or (2.3)] is of order r greater than 2. In that case, $r-1$ coefficients α_p remain arbitrary and may be assumed to be constant parameters. This method will be used below to determine stable differential equations of the fourth order (see Part III).

4. - We shall conclude this section by giving formulas which will be used often in what follows.

Set

$$u = P(x;\ z) + z^5 v,$$

$$P = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4,$$

$$\dot{z} = 1 + uz = 1 + zP + z^6 v$$

and *assume* α *to be a constant.*

In addition, set

$$P^2 = s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4 + O(z^5),$$

$$P^3 = t_0 + t_1 z + t_2 z^2 + t_3 z^3 + O(z^4),$$

$$\dot{P} = p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + O(z^5),$$

$$\ddot{P} = q_0 + q_1 z + q_2 z^2 + q_3 z^3 + O(z^4)$$

$$P\dot{P} = r_0 + r_1 z + r_2 z^2 + r_3 z^3 + O(z^4),$$

where

$$s_0 = \alpha^2, \qquad t_0 = \alpha^3, \qquad r_0 = \alpha\beta,$$

$$s_1 = 2\alpha\beta, \qquad t_1 = 3\alpha^2\beta, \qquad r_1 = \beta^2 + \alpha p_1,$$

$$s_2 = \beta^2 + 2\alpha\gamma, \qquad t_2 = 3\alpha\beta^2 + 3\alpha^2\gamma, \qquad r_2 = \beta\gamma + \beta p_1 + \alpha p_2,$$

$$s_3 = 2\beta\gamma + 2\alpha\delta, \qquad t_3 = \beta^3 + 6\alpha\beta\gamma + 3\alpha^2\delta, \quad r_3 = \beta\delta + \gamma p_1 + \beta p_2 + \alpha p$$

$$s_4 = \gamma^2 + 2\beta\delta + 2\alpha\varepsilon,$$

and

$$p_0 = \beta, \qquad q_0 = \dot{\beta} + p_1,$$

$$p_1 = \dot{\beta} + 2\gamma + \alpha\beta, \qquad q_1 = \dot{p}_1 + 2p_2 + \alpha p_1,$$

$$p_2 = \dot{\gamma} + 3\delta + 2\alpha\gamma + \beta^2, \qquad q_2 = \dot{p}_2 + 3p_3 + 2\alpha p_2 + \beta p_1,$$

$$p_3 = \dot{\delta} + 4\varepsilon + 3\alpha\delta + 3\beta\gamma, \qquad q_3 = \dot{p}_3 + 4p_4 + 3\alpha p_3 + 2\beta p_2 + \gamma p_1,$$

$$p_4 = \dot{\varepsilon} + 4\alpha\varepsilon + 4\beta\delta + 2\gamma^2.$$

Moreover, one also has

$$P^4 = \alpha^4 + 4\alpha^3\beta z + (6\alpha^2\beta^2 + 4\alpha^3\gamma) z^2 + O(z^3),$$

$$\dddot{P} = (\dot{q}_0 + q_1) + (\dot{q}_1 + 2q_2 + \alpha q_1) z + (q_2 + 3\dot{q}_3 + 2q_2\alpha + \beta q_1) z^2 + O(z^3),$$

$$P\ddot{P} = \alpha q_0 + (\beta q_0 + \alpha q_1) z + (\gamma q_0 + \beta q_1 + \alpha q_2) z^2 + O(z^3),$$

$$\dot{P}^2 = \beta^2 + 2\beta p_1 z + (p_1^2 + 2\beta p_2) z^2 + O(z^3),$$

$$\dot{P}P^2 = p_0 s_0 + (p_1 s_0 + p_0 s_1) z + (p_2 s_0 + p_1 s_1 + p_0 s_2) z^2 + O(z^3).$$

It is often most convenient to rewrite the preceding table for the case $\alpha = 0$ and for the case α replaced by 2α. This is left to the reader.

II. **Equations of the third order.**

5. - In order to be more specific, equation (1.1) is rewritten as follows

$$\dddot{y} = ay\ddot{y} + b\dot{y}^2 + cy^2\dot{y} + dy^4 + F(x, y) \tag{5.1}$$

where

$$F(x, y) = a_1\ddot{y} + c_1y\dot{y} + c_0\dot{y} + d_3y^3 + d_2y^2 + d_1y + d_0; \tag{5.2}$$

a, b, c, d, with or without subscripts, are analytic functions of x in a certain domain D.

The reduced equation corresponding to (5.1) is easily determined by setting $x = x_0 + \varepsilon t$, where x_0 is a point in D and $\varepsilon \neq 0$ a parameter; one finds $\dddot{y} = 0$.

The only value of y for which CAUCHY'S general existence theorem does not apply to equation (5.1) is $y = \infty$. To determine necessary conditions for the absence of parametric critical points for equation (5.1), suppose that in a neighborhood of $x = x_0$, $y(x)$ takes the form

$$y(x) = \frac{s(x)}{(x - x_0)^r} \tag{5.3}$$

where $r > 0$ and $s(x) \neq 0$; $s(x)$ is a holomorphic function of x.

Substitute $y(x)$ given by (5.3) into (5.1) and note that

$$\dot{y}(x) = -\frac{rs(x_0)}{(x - x_0)^{r+1}}[1 + O(x - x_0)],$$

$$\ddot{y}(x) = \frac{r(r + 1)s(x_0)}{(x - x_0)^{r+2}}[1 + O(x - x_0)],$$

$$\dddot{y}(x) = -\frac{r(r + 1)(r + 2)s(x_0)}{(x - x_0)^{r+3}}[1 + O(x - x_0)].$$

i. First, suppose that at least one of $a(x_0)$, $b(x_0)$, $c(x_0)$, $d(x_0)$ is not zero; then, the dominant terms arise from $\dddot{y}$, $y\ddot{y}$, $\dot{y}^2$, $y^2\dot{y}$, y^4 and are respectively proportional to

$$(x - x_0)^{-r-3}, \quad (x - x_0)^{-2r-2}, \quad (x - x_0)^{-3r-1}, \quad (x - x_0)^{-4r}.$$

Therefore, to obtain an identity at least two of the numbers $r + 3$, $2r + 2$, $3r + 1$, $4r$ must be equal; this gives $r = 1$.

In addition, $s(x_0)$ must satisfy the equation

$$6 + (2a + b)s - cs^2 + ds^3 = 0 \tag{5.4}$$

so that $y(x)$ has at most three sets of parametric poles.

Moreover, the general integral of the equation

$$\dddot{y} = ay\ddot{y} + b\dot{y}^2 + cy^2\dot{y} + dy^4 \tag{5.5}$$

where a, b, c, d are constant, must be one-valued.

These conditions will determine the possible values of a, b, c, d; the solution of this problem will be given below.

ii. Second, suppose $d(x_0) = c(x_0) = 0$, $2a(x_0) + b(x_0) = 0$.

Substitution of $y(x)$ given by (5.3) into (5.1) shows that the general integral of the equation

$$\dddot{y} = a(y\ddot{y} - 2\dot{y}^2) \tag{5.6}$$

has no pole at $x = x_0$. If $y(x)$ given by (5.6) has no parametric critical points (algebraic or not algebraic), it must be an entire function of x; this is known to be impossible [see Chazy [2, *b*] and Valiron [4, *a*]]. Therefore, $a = 0$ and accordingly $b = 0$.

[In (5.6), one may set $y \rightarrow \alpha y$, $a\alpha = 1$; equation (5.6) becomes

$$\dddot{y} = y\ddot{y} - 2\dot{y}^2.] \tag{5.7}$$

iii. Third, suppose $a = b = c = d = 0$. The dominant terms arise from $\dddot{y}$, $y\dot{y}$, y^3. Therefore, to obtain an identity at least two of the numbers $r + 3$, $2r + 1$, $3r$ must be equal.

When $c_1 \neq 0$, this gives $r + 3 = 2r + 1$ or $r = 2$; because $r + 3 < 3r$, one has also $d_3(x_0) = 0$ or $d_3(x) \equiv 0$ (because x_0 is arbitrary in D).

Hence, the stable equations are of the form

$$\dddot{y} = a_1 \ddot{y} + (c_1 y + c_0)\dot{y} + d_2 y^2 + d_1 y + d_0; \tag{5.8}$$

the general integral of (5.8) has only one set of parametric poles of the second order; $s(x_0)$ is given by

$$s(x_0)\, c_1(x_0) = 12. \tag{5.9}$$

The reduced equation corresponding to (5.8) is

$$\dddot{y} = c_1 y \dot{y} \tag{5.10}$$

and must be stable; c_1 is a constant, not zero.

By setting $y \to \alpha y$, one may suppose $c_1 = 12$. The equation (5.10) is then

$$\dddot{y} = 12\, y \dot{y}. \tag{5.11}$$

and is stable. In fact, the integral of (5.11) satisfies $\ddot{y} = 6y^2 + K$, where K is a constant; therefore $y(x)$ is 0, x^{-2} or $\wp(x;\, 0,\, K)$.

iv. Suppose $a = b = c = d = c_1 = 0$. The dominant terms arise from $\dddot{y}$, $\ddot{y}$, y^2 and $r = 3$.

The reduced equation is

$$\dddot{y} = d_2 y^2 \tag{5.12}$$

and $s(x_0)$ is determined by $s(x_0)\, d_2(x_0) = -60$.

This equation (5.12) is not stable except when $d_2 = 0$. To prove this, assume $d_2 = -60$ (by setting $y \to \alpha y$). The equation

$$\dddot{y} + 60\, y^2 = 0$$

is satisfied by $y = x^{-3}$. Then, set $y = x^{-3} + \varepsilon z$, $z(x)$ is determined by

$$\dddot{z} + 120 \frac{z}{x^3} + 60 \varepsilon z^2 = 0.$$

According to the general theorem of stability, [see Part I, art. 2], $z(x; \varepsilon)$ given by

$$z(x; \varepsilon) = z_0(x) + \varepsilon z_1(x) + \dots$$

must be single-valued together with $z_0(x)$, $z_1(x)$, ...

In particular, $z_0(x)$ is determined by

$$\dddot{z}_0 + 120 \frac{z_0}{x^3} = 0; \tag{5.13}$$

the related indicial equation is

$$r(r-1)(r-2) + 120 = (r+4)(r^2 - 7r + 30) = 0$$

and has two complex roots. Therefore, when $d_2 \neq 0$, $z_0(x)$ and also $z(x)$ are multiple-valued functions of x.

The stable equations corresponding to (5.12) where $d_2 = 0$ are thus of the form

$$\dddot{y} = a_0 \ddot{y} + c_0 \dot{y} + d_1 y + d_0$$

and are linear.

6. - Our purpose is now to determine a, b, c, d in order that the equation [cf. (5.5)]

$$\dddot{y} = ay\ddot{y} + b\dot{y}^2 + cy^2\dot{y} + dy^4 \tag{6.1}$$

be stable; a, b, c, d are constant.

To do this, set

$$y = \frac{s}{z}, \quad \dot{z} = 1 + uz \tag{6.2}$$

where s, a constant, satisfies the equation

$$ds^3 - cs^2 + (2a + b)s + 6 = 0. \tag{6.3}$$

Note also that

$$(6.4)\qquad \begin{cases} z^2\dot{y} = -s(1+uz), \\ z^2\ddot{y} = s\left(-z\dot{u} + \dfrac{2}{z} + 3u + u^2 z\right), \\ z^3\dddot{y} = -s\left(z^2\ddot{u} + \dfrac{6}{z} + 12u - 4z\dot{u} + 7u^2 z - 3u\dot{u}z^2 + u^3 z^2\right). \end{cases}$$

Substitution of $y(x)$ into (6.1) gives

$$(6.5)\qquad z^2\ddot{u} - z\dot{u}(4+as) + u(12+3as+2bs-cs^2) + \\ + uz^2(u^2 - 3\dot{u}) + zu^2(7+as+bs) = 0.$$

According to theorem I, a necessary condition for this equation (6.5) to be stab.e is that the solutions of the indicial equation

$$(6.6)\qquad \Theta(\Theta-1) - (4+as)\Theta + 12 + (3a+2b)s - cs^2 = 0$$

be integers.

For convenience, we set $\Theta = \chi - 1$, so that equation (6.6) becomes

$$(6.7)\qquad \chi^2 - (7+as)\chi + 18 + 2(2a+b)s - cs^2 = 0.$$

Hence, we now have to solve the following problem: *to determine a, b, c, d so that the solutions of* (6.7) *where s satisfies* (6 3), *will be integers.*

We consider the following cases:

i. $c = d = 0$, a or b is not zero;

ii. $d = 0$, $c \neq 0$;

iii. $d \neq 0$.

7. - $c = d = 0$; a or b is not zero. Then, s is given by

$$(7.1)\qquad (2a+b)s + 6 = 0$$

and accordingly, (6.7) becomes

$$(7.2)\qquad \chi^2 - (7+as)\chi + 6 = 0.$$

In order that the solutions χ_1 and χ_2 of (7.2) be integers, one must have

$$\chi_1 \chi_2 = 6, \tag{7.3}$$

$$\chi_1 + \chi_2 = 7 + as. \tag{7.4}$$

These conditions will determine the possible values of a and b.

Corresponding to each solution of (7.3), there is an a given by (7.4) and a b given by (7.1). The possible solutions of (7.3) are given in the following table together with the corresponding values of as and bs.

χ_1	χ_2	as	bs	
1	6	0	-6	$a = 0$
2	3	-2	-2	$a = b$
-1	-6	-14	22	$11a + 7b = 0$
-2	-3	-12	18	$3a + 2b = 0$

The related differential equations are

$$\dddot{y} = b\dot{y}^2, \qquad \Theta = 0,\ 5; \tag{7.5}$$

$$\dddot{y} = a(y\ddot{y} + \dot{y}^2), \quad \Theta = 1,\ 2; \tag{7.6}$$

$$\dddot{y} = \frac{a}{7}(7y\ddot{y} - 11\dot{y}^2), \qquad \Theta = -2,\ -7; \tag{7.7}$$

$$\dddot{y} = \frac{a}{2}(2y\ddot{y} - 3\dot{y}^2), \qquad \Theta = -3,\ -4. \tag{7.8}$$

We must now investigate the stability of these equations. For convenience, note that by setting $y \to \alpha y$, one may suppose $b = -6$ in (7.5), $a = 2$ in (7.6), $a = 7$ in (7.7) and $a = 2$ in (7.8).

To verify that equation (7.5) or

$$\dddot{y} + 6\dot{y}^2 = 0 \tag{7.9}$$

is stable, set $\dot{y} = z$ where z is given by

$$\ddot{z} + 6z^2 = 0; \tag{7.10}$$

the solutions of (7.10) are 0, $-\frac{1}{x^2}$ or $\wp(x)$ and therefore, $z(x)$ and $y(x)$ are stable

To prove that equation (7.6) or

$$\dddot{y} = 2y\ddot{y} + 2\dot{y}^2 \tag{7.11}$$

is stable, observe that one obtains (7.11) by differentiating

$$\dot{y} = y^2 + Kx + K_1 \tag{7.12}$$

with respect to x; K and K_1 are arbitrary constants. This RICCATI equation is stable and so is equation (7.11).

The equation (7.8) or

$$\dddot{y} = 2y\ddot{y} - 3\dot{y}^2 \tag{7.13}$$

is stable as was proved by CHAZY. Its general solution has a movable singular line and is defined only in one region of the x-plane. However, because $\Theta = -3$ or -4, no additional condition for stability can be obtained by our method; equation (7.13) will not be considered in this paper.

Now, we shall see that equation (7.7) or

$$\dddot{y} = 7y\ddot{y} - 11\dot{y}^2 \tag{7.14}$$

is not stable. This equation is equivalent to the differential system

$$\begin{cases} \dot{z} = 1 + uz \\ z^2\ddot{u} + 10z\dot{u} + 14u = 0. \end{cases} \tag{7.15}$$

By setting

$$x = \varepsilon t, \qquad z = \varepsilon Z,$$

where t and Z are new variables, one obtains

$$\frac{dZ}{dt} = 1 + \varepsilon Z u,$$

$$Z^2 \frac{d^2u}{dt^2} + 10 Z \frac{du}{dt} + 14u = 0.$$

Now set

$$Z = Z_0 + \varepsilon Z_1 + \dots,$$

$$u = u_0 + \varepsilon u_1 + \dots$$

and determine z_0, z_1, u_0 by

$$\frac{dZ_0}{dt} = 1, \quad \frac{dZ_1}{dt} = Z_0 u_0,$$

$$Z_0^2 \frac{d^2u_0}{dt^2} + 10 Z_0 \frac{du_0}{dt} + 14u_0 = 0;$$

one has

$$Z_0 = t,$$

$$u_0 = c_1 t^{-2} + c_2 t^{-7}$$

where c_1 and c_2 are arbitrary constants. Then consider $u_0 = t^{-2}$ so that Z_1 is given by

$$\frac{dZ_1}{dt} = \frac{1}{t};$$

therefore $Z_1 = \lg t$ and $Z_1(x)$ is not stable. Accordingly, $y(x)$ given by $y = \frac{s}{z}$ is not stable.

8. - $d = 0$, $c \neq 0$. Denote by s_1, s_2 the solutions of equation (6.3), i.e.

$$cs^2 - (2a + b)s - 6 = 0 \tag{8.1}$$

and note that $s_1 \neq s_2$ (otherwise, one has only one family of parametric poles).

One has

$$c = -\frac{6}{s_1 s_2}, \quad 2a + b = -6\left(\frac{1}{s_1} + \frac{1}{s_2}\right). \tag{8.2}$$

By setting

$$6\,P(s) = 18 + 2s\,(2a + b) - cs^2, \tag{8.3}$$

one finds

$$P(s_1) = 1 - \frac{s_1}{s_2}, \qquad P(s_2) = 1 - \frac{s_2}{s_1}. \tag{8.4}$$

It then follows that the solutions of the equations [cf. eq. (6.7)]

$$\chi^2 - (7 + as_1)\,\chi + 6\,P(s_1) = 0, \tag{8.5}$$

$$\lambda^2 - (7 + as_2)\,\lambda + 6\,P(s_2) = 0 \tag{8.6}$$

must be integers. Therefore, $6\,P(s_1) = p$ and $6\,P(s_2) = q$ are also integers.

Then, it follows from (8.4) that

$$6\,(p + q) = pq \tag{8.7}$$

or because $p \cdot q \neq 0$ (otherwise $s_1 = s_2$),

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{6}. \tag{8.8}$$

Our next problem is to find all the integral solutions of (8.8).

Suppose one integral solution (p, q) of (8.7) is known. Substitution in (8.4) gives $\frac{s_1}{s_2}$.

Now, the solutions of equations (8.5) and (8.6), namely

$$\chi^2 - (7 + as_1)\,\chi + p = 0, \tag{8.9}$$

$$\lambda^2 - (7 + as_2)\,\lambda + q = 0 \tag{8.10}$$

must be integers. Suppose that the integral solutions of these equations are known. One may determine s_1 and s_2 and consequently $\frac{s_1}{s_2}$. The desired solutions are those which correspond to the same value $\frac{s_1}{s_2}$ given by (8.4). An example will clarify the process.

9. - Integral solutions of (8.8).

a. If $p = 6$, then $q = \infty$ and $s_2 = \infty$ or $c = 0$ i.e. case i. [see § 7].

b. If $p = q$, then $p = q = 12$.

Now we consider two cases according as to whether $p > 0$, $q > 0$ or $p > 0$, $q < 0$.

c. $p > 0$, $q > 0$; we may suppose $p < q$ and therefore $p < 12 < q$. Because $q = \frac{6p}{p-6}$, one has $p > 6$ so that the integral solutions are

$$(p, q) = (7,42), \quad (8,24), \quad (9,18), \quad (10,15).$$

d. $p > 0$, $q < 0$. Set $q = -r$, $r > 0$; then (8.8) becomes $\frac{1}{p} - \frac{1}{r} = \frac{1}{6}$ and consequently $r > p$. Because $r = \frac{6p}{6-p}$, one has $p < 6$ so that the integral solutions are

$$(p, q) = (2, -3), \quad (3, -6), \quad (4, -12), \quad (5, -30).$$

10. - We proceed further by considering equations (8.9) and (8.10). To bring out the chief features of the method, we shall use the case $(p, q) = (3,6)$ as an illustration.

From (8.4), one obtains $2s_1 = s_2$. The equations (8.9-10) are

$$\chi^2 - (7 + as_1)\chi + 3 = 0,$$

$$\lambda^2 - (7 + as_2)\lambda - 6 = 0;$$

we give the integral solutions (χ_1, χ_2) and (λ_1, λ_2) respectively of these equations, together with the corresponding values of as_1 and as_2:

$$(\chi_1, \chi_2, as_1) = (1, 3, -3), \quad (-1, -3, -11);$$

$$(\lambda_1, \lambda_2, as_2) = (1, -6, -12), \quad (2, -3, -8), \quad (-1, 6, -2), \quad (-2, 3, -6).$$

Because $2s_1 = s_2$, the only solution of our problem is

$$as_1 = -3, \quad as_2 = -6.$$

It then follows

$$a(s_1 + s_2) = -9, \quad a^2 s_1 s_2 = 18$$

and from (8.2),

$$3c = -a^2, \quad b = a.$$

Therefore, the stable equation of this type is

(10.1) $$\dddot{y} = ay\ddot{y} + a\dot{y}^2 - \frac{a^2}{3} y^2 \dot{y}.$$

11. - No solution arises from

$$(p, q) = (2, -3), \quad (7,42), \quad (10,15)$$

because the corresponding relation between s_1 and s_2 is not satisfied.

We now give the results for the other cases.

$$\underline{p = 3, \quad q = -6.}$$

$$\chi_1 = 1, \quad \chi_2 = 3, \quad \lambda_1 = -2, \quad \lambda_2 = 3;$$

$$2s_1 = s_2, \quad as_1 = -3, \quad as_2 = -6;$$

$$3c + a^2 = 0, \quad b = a.$$

(11.1) Equation: $\dddot{y} = ay\ddot{y} + a\dot{y}^2 - \frac{a^2}{3} y^2 \dot{y}.$

$$\underline{p = 4, \ q = -12.}$$

$$\chi_1 = 1, \quad \chi_2 = 4, \quad \lambda_1 = -3, \quad \lambda_2 = 4;$$

$$3s_1 = s_2, \ as_1 = -2, \quad as_2 = -6;$$

$$2c + a^2 = 0, \quad b = 2a.$$

(11.2) Equation: $\dddot{y} = ay\ddot{y} + 2a\dot{y}^2 - \frac{a^2}{2} y^2\dot{y}.$

$$\underline{p = 5, \ q = -30.}$$

$$\chi_1 = 1, \quad \chi_2 = 5, \quad \lambda_1 = -5, \quad \lambda_2 = 6;$$

$$6s_1 = s_2, \quad as_1 = -1, \quad as_2 = -6;$$

$$c + a^2 = 0, \quad b = 5a.$$

(11.3) Equation: $\dddot{y} = ay\ddot{y} + 5a\dot{y}^2 - a^2y^2\dot{y}.$

$$\underline{p = 8, \quad q = 24.}$$

$$\chi_1 = 2, \quad \chi_2 = 4, \quad \lambda_1 = 4, \quad \lambda_2 = 6;$$

$$3s_1 + s_2 = 0, \quad as_1 = -1, \quad as_2 = 3;$$

$$c = 2a^2, \quad b = 2a.$$

(11.4) Equation: $\dddot{y} = ay\ddot{y} + 2a\dot{y}^2 + 2a^2y^2\dot{y}.$

$$\underline{p = 12, \quad q = 12.}$$

$$\chi_1 = 3, \quad \chi_2 = 4, \quad \lambda_1 = 3, \quad \lambda_2 = 4;$$

$$s_1 + s_2 = 0, \quad as_1 = 0, \quad as_2 = 0;$$

$$a = b = 0, \quad cs_1^2 = 6.$$

(11.5) Equation: $\dddot{y} = cy^2\dot{y}.$

$$\underline{p = 9, \quad q = 18.}$$

$$\chi_1 = 3, \quad \chi_2 = 3, \quad \lambda_1 = 3, \quad \lambda_2 = 6;$$

$$2s_1 + s_2 = 0, \quad as_1 = -1, \quad as_2 = 2;$$

$$c = 3a^2, \quad b = a.$$

(11.6) Equation: $\dddot{y} = ay\ddot{y} + a\dot{y}^2 + 3a^2y^2\dot{y}$ (not stable).

As will be shown in the next articles, equations (10.1) and (11.1–5) are stable; equation (11.6) is not stable.

12. **Equation** (10.1) – A transformation $y \to \alpha y$, $a\alpha = 3$ brings equation (10.1) to

$$\dddot{y} = 3y\ddot{y} + 3\dot{y}^2 - 3y^2\dot{y} \tag{12.1}$$

which is obtained on differentiating

$$\ddot{y} = 3y\dot{y} - y^3 + K \tag{12.2}$$

with respect to x. This equation (12.2) of the second order is of type I; $P1$, $p = 0$, 3(see Part I, eq. (20.4)] and is stable; then equation (10.1) is also stable.

To integrate (12.2), set $y = -\frac{\dot{v}}{v}$ and find $\dddot{v} + Kv = 0$.

Equation (11.2). – A transformation $y \to \alpha y$, $a\alpha = 2$ brings equation (11.2) to

$$\dddot{y} = 2y\dot{y} + 4\dot{y}^2 - 2y^2\dot{y}. \tag{12.3}$$

Now, multiply both members of (12.3) by $2y$ and observe that

$$2y\dddot{y} = \frac{d}{dx}(2y\ddot{y} - \dot{y}^2); \tag{12.4}$$

y is thus determined by

$$\ddot{y} = \frac{\dot{y}^2}{2y} + 2y\dot{y} - \frac{y^3}{2} + \frac{K}{y} \tag{12.5}$$

which is a stable equation of the second order, of type III and class $E.16$.

To integrate equation (12.5) set

$$\dot{y} = y^2 + h + 2vy \tag{12.6}$$

where h is given by $h^2 + 2K = 0$; therefore

$$\dot{v} + v^2 = \frac{h}{2}. \tag{12.7}$$

On setting $v = \frac{\dot u}{u}$, $y = -\frac{\dot z}{z}$, one sees that equation (12.5) is equivalent to the differential system

$$\ddot u = \frac{h}{2} u, \quad \ddot z - 2\frac{\dot u}{u}\dot z + hz = 0. \tag{12.8}$$

Equation (11.4). - A transformation $y \to \alpha y$, $a\alpha = 1$ brings equation (11.4) to

$$\dddot y = y\ddot y + 2\dot y^2 + 2y^2\dot y. \tag{12.9}$$

On multiplying both members by $2y$ and taking (12.4) into account, one sees that $y(x)$ satisfies

$$\ddot y = \frac{\dot y^2}{2y} + y\dot y + \frac{y^3}{2} + \frac{K}{y} \tag{12.10}$$

which is a stable equation of the second order, of type III and class $E.6$.

To integrate (12.10), set

$$12w = \dot y - y^2 \tag{12.11}$$

so that

$$12\dot w = 72\,\frac{w^2}{y} + \frac{K}{y}$$

or

$$y = \frac{72w^2 + K}{12\dot w}. \tag{12.12}$$

Elimination of y between (12.11) and (12.12) gives

$$\ddot w = -\,6w^2 + \frac{K}{12};$$

w is thus an elliptic function.

Equation (11.5). - A transformation $y \to \alpha y$, $\alpha^2 c = 6$ brings equation (11.5) to

$$\dddot y = 6y^2\dot y.$$

Therefore $y(x)$ satisfies

$$\ddot{y} = 2y^3 + K$$

and is stable; $y(x)$ is an elliptic function.

Equation (11.6). - A transformation $y \to \alpha y$, $a\alpha = -1$ brings equation (11.6) to

$$\dddot{y} = -y\ddot{y} - \dot{y}^2 + 3y^2\dot{y}.$$

Therefore $y(x)$ satisfies

$$\dddot{y} = -y\dot{y} + y^3 + K$$

which is an equation of the second order, of type I and class vi, namely

$$\ddot{y} = -y\dot{y} + y^3 - 12Vy + 12\dot{V}$$

where V is given by

$$\ddot{V} = 6V^2 + K_1 \quad \text{or} \quad \ddot{V} = 6V^2 + x.$$

Therefore, V cannot be an arbitrary constant and consequently, $y(x)$ is not stable except when $K = 0$.

13· - It remains to consider equation (11.3). A transformation $y \to \alpha y$, $a\alpha = 1$ brings this equation to

(13.1) $$\dddot{y} = y\ddot{y} + 5\dot{y}^2 - y^2\dot{y}.$$

Set

(13.2) $$P = \ddot{y} - y\dot{y} - y^3$$

and write (13.1) in the form

(13.3) $$\dot{P} = 4\dot{y}(\dot{y} - y^2).$$

Multiply (13.3) by P and note that

$$\frac{d}{dx}(\dot{y} - y^2)^2(2\dot{y} + y^2) = 6\dot{y}(\dot{y} - y^2)P;$$

then (13.3) becomes

$$P^2 = \frac{4}{3}(\dot{y} - y^2)^2(2\dot{y} + y^2) + K$$

and y defined by (13.1) satisfies

$$(13.4) \qquad (\ddot{y} - y\dot{y} - y^3)^2 = \frac{4}{3}(\dot{y} - y^2)^2(2\dot{y} + y^2) + K.$$

This equation (13.4) is of the second degree in the higher derivative $\ddot{y}$; this type of equation will be the object of another part of these studies.

However equation (13.4) is stable; the proof is given below for $K = 0$.

Note that equation (13.4) when $K = 0$ is the reduced equation obtained on setting $x = x_0 + \varepsilon t$ and $\varepsilon y = u$.

For convenience, we put $b = \sqrt{\frac{2}{3}}$ and write

$$(13.5) \qquad \ddot{y} - y\dot{y} - y^3 = 2b(\dot{y} - y^2)\sqrt{\dot{y} + \frac{1}{2}y^2}$$

instead of (13.4).

Now set

$$(13.6) \qquad \dot{y} + \frac{1}{2}y^2 = z^2y^2$$

and note that

$$(13.7) \qquad \ddot{y} - y\dot{y} - y^3 = 2zy^2[\dot{z} + zyQ]$$

where

$$(13.8) \qquad Q = z^2 - \frac{1}{b^2}.$$

Therefore, equation (13.5) is equivalent to the differential system

$$(13.9) \qquad \dot{y} = y^2\left(z^2 - \frac{1}{2}\right),$$

$$(13.10) \qquad \dot{z} = -Qy(z - b).$$

The function z satisfies a differential equation which is obtained on eliminating y between (13.9) and (13.10).

To proceed further, we consider $\frac{\dot{y}}{y}$ given by (13.9) and take the logarithmic derivative of (13.10); one obtains

$$\frac{\dot{z}}{\dot{z}^2}=\frac{2z^2-2zb-1}{Q(z-b)}=\frac{2\left(z+\frac{1}{b}-b\right)}{\left(z+\frac{1}{b}\right)(z-b)}$$

or

$$\frac{\ddot{z}}{\dot{z}^2}=\frac{6}{5}\cdot\frac{1}{z-b}+\frac{4}{5}\cdot\frac{1}{z+\frac{1}{b}}.$$

Therefore, z is a solution of

$$\dot{z}^5=K^5(z-b)^6\left(z+\frac{1}{b}\right)^4. \tag{13.11}$$

The explicit form of $z(x)$ may be obtained by setting

$$t^5=\frac{z+\frac{1}{b}}{z-b},\quad z=\frac{\frac{1}{b}+bt^5}{t^5-1};$$

then

$$t=-\left(b+\frac{1}{b}\right)\frac{K}{5},$$

$$t=-\frac{K}{5}\left(b+\frac{1}{b}\right)x+K_1$$

and $z(x)$ is a rational function of x.

14. $d\neq 0$. - Denote by s_1, s_2, s_3 the solutions of equation (5.4) i.e.

$$ds^3-cs^2+(2a+b)s+6=0 \tag{14.1}$$

and note that $s_1 \neq s_2 \neq s_3$. One has

$$(14.2)\qquad \begin{cases} d = -\dfrac{6}{s_1 s_2 s_3}, \\[2ex] c = -\dfrac{6}{s_1 s_2 s_3}(s_1 + s_2 + s_3), \\[2ex] 2a + b = -\dfrac{6}{s_1 s_2 s_3}(s_1 s_2 + s_2 s_3 + s_3 s_1). \end{cases}$$

By setting again

$$(14.3)\qquad 6P(s) = 18 + 2(2a + b)s - cs^2,$$

one finds

$$(14.4)\qquad \begin{cases} P(s_1) = \left(1 - \dfrac{s_1}{s_2}\right)\left(1 - \dfrac{s_1}{s_3}\right), \\[2ex] P(s_2) = \left(1 - \dfrac{s_2}{s_1}\right)\left(1 - \dfrac{s_2}{s_3}\right), \\[2ex] P(s_3) = \left(1 - \dfrac{s_3}{s_1}\right)\left(1 - \dfrac{s_3}{s_2}\right). \end{cases}$$

According to the general theory, the solutions of the equations

$$(14.5)\qquad \chi^2 - (7 + as_1)\chi + 6P(s_1) = 0,$$

$$(14.6)\qquad \lambda^2 - (7 + as_2)\lambda + 6P(s_2) = 0,$$

$$(14.7)\qquad \mu^2 - (7 + as_3)\mu + 6P(s_3) = 0$$

must be integers. Therefore,

$$(14.8)\qquad 6P(s_1) = p, \quad 6P(s_2) = q, \quad 6P(s_3) = r$$

are also integers.

These integers p, q, r satisfy a relation which can be found as follows. Set

$$(14.9)\qquad \alpha = s_2 - s_3, \quad \beta = s_3 - s_1, \quad \gamma = s_1 - s_2$$

so that

$$(14.10) \qquad p = -\frac{6\beta\gamma}{s_2 s_3}, \quad q = -\frac{6\gamma\alpha}{s_1 s_3}, \quad r = -\frac{6\alpha\beta}{s_1 s_2},$$

$$pq + qr + rp = 36\,\frac{\alpha\beta\gamma}{s_1 s_2 s_3}\left(\frac{\alpha}{s_1} + \frac{\beta}{s_2} + \frac{\gamma}{s_3}\right).$$

Then

$$\frac{\alpha}{s_1} + \frac{\beta}{s_2} + \frac{\gamma}{s_3} = \frac{1}{s_1 s_2 s_3}(s_2 s_3 \alpha + s_3 s_1 \beta + s_1 s_2 \gamma)$$

$$= -\frac{\alpha\beta\gamma}{s_1 s_2 s_3}$$

and finally

$$(14.11) \qquad 6(pq + qr + rp) = pqr$$

or because $pqr \neq 0$,

$$(14.12) \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{6}.$$

Our next problem is to determine all the integral solutions of the DIOPHANTINE equation (14.12). For convenience, we denote by χ_1, χ_2; λ_1, λ_2; μ_1, μ_2 the solutions of the equations (14.5,6,7) respectively. Therefore

$$(14.13) \qquad \chi_1 + \chi_2 = -7 - as_1, \; \lambda_1 + \lambda_2 = -7 - as_2, \;\; \mu_1 + \mu_2 = -7 - as_3.$$

If $a \neq 0$, then as_1, as_2, as_3 are real.

We have to consider two cases according as a is or is not zero.

15. $a = 0$. - Then equation (14.5) becomes

$$\chi^2 - 7\chi + p = 0;$$

thus

$$\chi_1 + \chi_2 = 7, \quad \chi_1 \chi_2 = p$$

where p is a positive integer. Because

$$(\chi_1 - \chi_2)^2 = (\chi_1 + \chi_2)^2 - 4\chi_1\chi_2 = 49 - 4p$$

must be a positive square, one has $4p = 49 - h^2 > 0$ where h, an integer, is such that $0 \leq h < 7$; therefore, p must be 6, 10 or 12.

i. Suppose $p = 6$; then $\frac{1}{q} + \frac{1}{r} = 0$ [see (14.12)]. Therefore $q > 0$ and because of (14.6), $q = 6$, 10 or 12 and $r = -6$, -10 or -12. However, because of (14.7), $49 - 4r$ must be a square, which is impossible.

ii. Suppose $p = 10$. Then $\frac{1}{q} + \frac{1}{r} = \frac{1}{15}$; therefore q is positive and is equal to 10 or 12.

When $q = 10$, then $r = -30$; when $q = 12$, then $r = -60$. Thus, we have the two solutions

$$(p,\ q,\ r) = (10,\ 10, -30), \quad (10,\ 12, -60). \tag{15.1}$$

iii. Suppose $p = 12$. The same method gives again the solutions (15.1).

16. $a \neq 0$. - From (14.12), it follows that one of p, q, r is a positive integer. Moreover from (14.10), one obtains

$$6pqr = -\ 6^4 \frac{\alpha^2\beta^2\gamma^2}{s_1^2 s_2^2 s_3^2}$$

and because as_1, as_2, as_3 are real, one may write

$$pqr = -6N^2 \tag{16.1}$$

where N is a positive integer. Therefore, one of p, q, r is a negative integer.

By setting $r = -t$, equation (14.12) becomes

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{t} + \frac{1}{6} \tag{16.2}$$

where p, q and t are positive integers. Note also [cf. (16.1)] that

$$pqt = 6N^2 \tag{16.3}$$

and that

$$t(p + q) = pq + N^2. \tag{16.4}$$

Because $(p+q)^2 \geq 4pq$, one sees on using (16.4) that

$$(p+q)^2 - 4t(p+q) - 4N^2 \geq 0$$

and thus that $4t^2 - 4N^2 \leq 0$; therefore $t \leq N$.

i. Suppose $t = N$; then $pq = 6N$, $p + q = 6 + N$ and

(16.5) $$p = 6, \quad q = N = t.$$

ii. Suppose $t < N$. Then $pq > 6N$ and because of (16.4),

$$p + q > 6 + N.$$

Moreover, one easily sees that $p \neq q$. [If $p = q$, one has on using (16.4),

$$p^2 - 2pt + N^2 = 0$$

and p would be a complex number].

Now, we suppose $p < q$; then from (16.2), one obtains

(16.6) $$p < 12.$$

Note that $p = 6$ yields (16.5) [then $q = t$, cf. (16.2)].

Suppose $p < 6$. From (16.2), one obtains

(16.7) $$q = \frac{6pt}{6p - (6-p)t},$$

and from (16.3),

(16.8) $$p^2t^2 = N^2[6p - (6-p)t].$$

Therefore, $6p - (6-p)t$ must be a positive square; this limits the possible values of t.

When $6 < p < 12$, one employs

$$t = \frac{6pq}{6p - (p-6)q},$$

$$p^2q^2 = N^2[6p - (p-6)q]$$

instead of (16.7–8).

The results are given in the following table, together with the corresponding values of N. The values corresponding to $q = 6$ are omitted.

p	q	t	N
6	N	N	N
4	60	10	20
5	29.30	29	29.5
5	13.15	13.2	13.5
5	7.10	7.3	7.5
7	41	41.42	41.7
7	19.2	21.19	7.19
7	11.3	11.14	7.11
8	11.2	11.24	8.11
8	8.2	8.6	8.4
9	15	6.15	3.15
10	14	15.14	5.14
11	13	66.13	11.13

Except for $p = 6$, $q = t = N$ [see section 19 below], no stable equation arises from the values of p, q, t given in this table.

[One sees that at least one of the corresponding Θ_i's is not an integer].

17. - Now we consider the case $p = 10$, $q = 10$, $r = -30$ and proceed to determine s_1, s_2, s_3.

From (14.10), one obtains

$$p \frac{\alpha}{s_1} = q \frac{\beta}{s_2} = r \frac{\gamma}{s_3} = k$$

or

$$p(s_2 - s_3) = ks_1, \quad q(s_3 - s_1) = ks_2, \quad r(s_1 - s_2) = ks_3$$

where k is a constant to be determined.

Actually, one has

$$- ks_1 + 10s_2 - 10s_3 = 0,$$

$$- 10s_1 - ks_2 + 10s_3 = 0,$$

$$- 30s_1 + 30s_2 - ks_3 = 0,$$

from which follows $k = \pm 10\sqrt{5}$.

When $k = 10\sqrt{5}$, one obtains

$$\frac{s_1}{1 - \sqrt{5}} = \frac{s_2}{1 + \sqrt{5}} = \frac{s_3}{6} = \rho$$

and thus by using (14.2)

$$b = \frac{2}{\rho}, \quad c = \frac{2}{\rho^2}, \quad d = \frac{1}{4\rho^3}.$$

The solutions of equations (14.5-7) are $\chi_1 = 2$, $\chi_2 = 5$ and $\mu_1 = 10$, $\mu_2 = -3$ respectively.

Now set $a\rho = 1$ and write the corresponding differential equation

$$\dddot{y} = 2a\dot{y}^2 + 2a^2y^2\dot{y} + \frac{a^3}{4} y^4. \tag{17.1}$$

The solution of this equation possesses three sets of simple parametric poles, namely

$$as_1 = 1 - \sqrt{5}, \quad \Theta = 1,\ 4;$$

$$as_2 = 1 + \sqrt{5}, \quad \Theta = 1,\ 4;$$

$$as_3 = 6 \quad , \quad \Theta = 9,\ -4.$$

That this eqnation is or is not stable remains to be investigated.

A simple transformation enables one to suppose $a = 6$ and to bring equation (17.1) to

$$\dddot{y} = 12\dot{y}^2 + 72y^2\dot{y} + 54y^4$$

or

$$\dddot{y} + 6\dot{y}^2 = 18(\dot{y} + y^2)(\dot{y} + 3y^2).$$

When $u = 3y$, this equation takes the form

$$\dddot{u} = 4\dot{u}^2 + 8u^2\dot{u} + 2u^4$$

or

$$\dddot{u} = 4(\dot{u} + u^2)^2 - 2u^4.$$

[When $k = -10\sqrt{5}$, one obtains again equation (17.1)].

18. - Suppose $p = 10$, $q = 12$, $r = -60$. The same method yields $k = \pm 20\sqrt{3}$. Assume $k = 20\sqrt{3}$; then

$$s_1 = \rho(\sqrt{3} - 1), \quad s_2 = -\rho\sqrt{3}, \quad s_2 = \rho(-6 + \sqrt{3})$$

and

$$b = -\frac{3(7 + 3\sqrt{3})}{11\rho}, \quad c = \frac{40 + 14\sqrt{3}}{11\rho^2}, \quad d = -\frac{7 + 3\sqrt{3}}{11\rho^3}.$$

By setting $a\rho = 1$, one obtains the differential equation

$$\ddot{y} = -\frac{3}{11}(7 + 3\sqrt{3})\, a\dot{y}^2 + \frac{40 + 14\sqrt{3}}{11}\, a^2 y^2 \dot{y} - \frac{7 + 3\sqrt{3}}{11}\, a^3 y^4. \tag{18.1}$$

The solution of this equation has three sets of simple parametric poles, namely

$$as_1 = \sqrt{3} - 1, \quad \Theta = 1,\ 4;$$

$$as_2 = -\sqrt{3}\ , \quad \Theta = 2,\ 3;$$

$$as_3 = -6 + \sqrt{3}, \quad \Theta = 11,\ -6.$$

A simple transformation enables one to assume $a = -\sqrt{3}$ and hence to bring equation (18.1) to

$$\dddot{y} = 6y^2\dot{y} + \frac{3}{11}(9 + 7\sqrt{3})(\dot{y} + y^2)^2.$$

That this equation is or is not stable remains to be shown.

19. Suppose $a \neq 0$ and $p = 6$, $q = -r = N$, N a positive integer. Then,

$$6(s_2 - s_3) = ks_1, \quad N(s_3 - s_1) = ks_2, \quad -N(s_1 - s_2) = ks_3$$

and $k = \pm N$.

For $k = N$, one obtains $s_1 = 0$, $s_2 = s_3$; this is not a solution of our problem because all the s_i must be different from zero.

For $k = -N$, one has

$$s_1 = 12\rho, \quad s_2 = (6 - N)\rho, \quad s_3 = (6 + N)\rho$$

where ρ is a parameter. Therefore

$$d = -\frac{1}{2(36 - N^2)\rho^3}, \quad c = -\frac{12}{(36 - N^2)\rho^2},$$

$$2a + b = -\frac{180 - N^2}{2(36 - N^2)\rho}.$$

Now, consider equation (14.5) with $s_1 = 12\rho$: one finds the two admissible solutions

$$\chi_1 = 2, \qquad \chi_2 = 3, \qquad 6a\rho = -1;$$

$$\chi_1 = -2, \quad \chi_2 = -3, \quad a\rho = -1.$$

i. Suppose $6a\rho = -1$; then

$$as_2 = -\frac{6 - N}{6}, \quad as_3 = -\frac{6 + N}{6}$$

and the solutions of equations (14.6-7) are

$$\lambda_1 = 6, \quad \lambda_2 = \frac{N}{6}; \quad [\text{cf. } (14.6)];$$

$$\mu_1 = 6, \quad \mu_2 = -\frac{N}{6}. \quad [\text{cf. } (14.7)].$$

Therefore, one must have $N = 6k$, $k > 1$ an integer. Consequently,

$$b = \frac{13 - k^2}{1 - k^2}\, a, \quad c = -\frac{12a^2}{1 - k^2}, \quad d = \frac{3a^3}{1 - k^2}$$

and the desired equation is

$$\dddot{y} = ay\ddot{y} + \frac{13 - k^2}{1 - k^2}\, a\dot{y}^2 - \frac{12a^2}{1 - k^2}\, y^2\dot{y} + \frac{3a^3}{1 - k^2}\, y^4. \tag{19.1}$$

This equation has three sets of simple parametric poles, namely

$$as_1 = -2, \qquad \Theta = 1, \ 2;$$

$$as_2 = k - 1, \qquad \Theta = 5, \ k - 1;$$

$$as_3 = -(k + 1), \quad \Theta = 5, \ -k - 1.$$

A simple transformation enables one to set $a = \frac{1 - k^2}{2}$ and to bring equation (19.1) to

$$\dddot{y} = \frac{1 - k^2}{2}\, y\ddot{y} + \left(\frac{1 - k^2}{2} + 6\right)\dot{y}^2 - 3(1 - k^2)\, y^2\dot{y} + \frac{3}{8}(1 - k^2)^2 y^4$$

or to

$$\frac{d^2}{dx^2}(\dot{y} - Ay^2) = 6(\dot{y} - Ay^2)^2$$

if

$$1 - k^2 = 4A.$$

To prove that this equation is stable, we only have to observe that it is equivalent to the differential system

$$\dot{y} = \frac{1-k^2}{4} y^2 + u, \quad \ddot{u} = 6u^2.$$

ii. Suppose $a\rho = -1$; then

$$as_2 = -(6-N), \quad (N \neq 6), \quad as_3 = -(6+N)$$

and the solutions of equations (14.6–7) are

$$\lambda_1 = 1, \quad \lambda_2 = N; \qquad [\text{cf. } (14.6)];$$

$$\mu_1 = 1, \quad \mu_2 = -N. \quad [\text{cf. } (14.7)].$$

Therefore, one has

$$b = \frac{36+3N^2}{2(36-N^2)} a, \quad c = -\frac{12a^2}{36-N^2}, \quad d = \frac{a^3}{2(36-N^2)};$$

consequently, the desired equation is

$$\dddot{y} = ay\ddot{y} + \frac{36+3N^2}{2(36-N^2)} a\dot{y}^2 - \frac{12a^2}{36-N^2} y^2\dot{y} + \frac{a^3}{2(36-N^2)} y^4. \tag{19.2}$$

This equation has three sets of simple parametric poles, namely

$$as_1 = -12, \qquad \Theta = -3, \ -4;$$

$$as_2 = N-6, \qquad \Theta = 0, \ N-1;$$

$$as_3 = -(N+6), \quad \Theta = 0, \ -N-1;$$

we recall that $N \neq 6$ is a positive integer greater than 1.

A simple transformation enables one to suppose $a = 2$ and to bring equation (29.2) to

$$\dddot{y} = 2y\ddot{y} - 3\dot{y}^2 + \frac{4}{36 - N^2}(6\dot{y} - y^2)^2. \tag{19.3}$$

[Note that when $N = \infty$, equation (19.3) becomes $\dddot{y} = 2y\ddot{y} - 3\dot{y}^2$; one sees easily that this equation is equivalent to the differential system

$$\dot{y} = \frac{y^2}{6} + u, \quad uy^2 - 3\dot{u}y + 6u^2 + \frac{9}{5}\ddot{u} = 0\Big].$$

20. We sum up the results obtained above in the following table where the relevant equations are listed together with the related values of s and Θ.

A. One set of double parametric poles.

Class I.

$$\dddot{y} = c_1 y\dot{y} + F(x, y), \tag{20.1}$$

$$F(x, y) = a_1\ddot{y} + c_0\dot{y} + d_2 y^2 + d_1 y + d_0;$$

$$sc_1 = 12, \qquad \Theta = 3, \ 5.$$

B. One set of simple parametric poles.

Class II.

[In the remainder of this paragraph, $F(x, y)$ is given by (5.2)].

$$\dddot{y} = b\dot{y}^2 + F(x, y); \tag{20.2}$$

$$bs = 6, \quad \Theta = 0, \ 5.$$

Class III.

$$\dddot{y} = a(y\ddot{y} + \dot{y}^2) + F(x, y); \tag{20.3}$$

$$as = -2, \qquad \Theta = 1, \ 2.$$

C. Two sets of simple parametric poles.

Class IV.

$$\dddot{y} = ay\ddot{y} + a\dot{y}^2 - \frac{a^2}{3} y^2 \dot{y} + F(x, y); \tag{20.4}$$

$$as_1 = -3, \qquad \Theta = 0,\ 2;$$

$$as_2 = -6, \qquad \Theta = -3,\ 2.$$

Class V.

$$\dddot{y} = ay\ddot{y} + 2a\dot{y}^2 - \frac{a^2}{2} y^2 \dot{y} + F(x, y); \tag{20.5}$$

$$as_1 = -2, \qquad \Theta = 0,\ 3;$$

$$as_2 = -6, \qquad \Theta = -4,\ 3.$$

Class VI.

$$\dddot{y} = ay\ddot{y} + 5a\dot{y}^2 - a^2 y^2 \dot{y} + F(x, y); \tag{20.6}$$

$$as_1 = -1, \qquad \Theta = 0,\ 4;$$

$$as_2 = -6, \qquad \Theta = -6,\ 5.$$

Class VII.

$$\dddot{y} = ay\ddot{y} + 2a\dot{y}^2 + 2a^2 y^2 \dot{y} + F(x, y); \tag{20.7}$$

$$as_1 = -1, \qquad \Theta = 1,\ 3;$$

$$as_2 = 3, \qquad \Theta = 3,\ 5.$$

Class VIII.

$$\dddot{y} = cy^2\dot{y} + F(x, y); \tag{20.8}$$

$$cs_1^2 = 6, \qquad \Theta = 2,\ 3.$$

D. Three sets of simple parametric poles.

Class IX.

$$\dddot{y} = 2a\dot{y}^2 + 2a^2y^2\dot{y} + \frac{a^3}{4}y^4 + F(x, y); \tag{20.9}$$

$$as_1 = 1 - \sqrt{5}, \quad \Theta = 1, \ 4;$$

$$as_2 = 1 + \sqrt{5}, \quad \Theta = 1, \ 4;$$

$$as_3 = 6 \qquad , \quad \Theta = -4, \ 9.$$

Class X.

$$\dddot{y} = -\frac{3}{11}(7 + 3\sqrt{3})a\dot{y}^2 + \frac{40 + 14\sqrt{3}}{11}a^2y^2\dot{y} - \frac{7 + 3\sqrt{3}}{11}a^3y^4 + F(x, y); \tag{20.10}$$

$$as_1 = \sqrt{3} - 1, \qquad \Theta = 1, \ 4;$$

$$as_2 = -\sqrt{3}, \qquad \Theta = 2, \ 3;$$

$$as_3 = -6 + \sqrt{3}, \quad \Theta = -6, \ 11.$$

Class XI.

$$\dddot{y} = ay\ddot{y} + \frac{13 - k^2}{1 - k^2}a\dot{y}^2 - \frac{12a^2}{1 - k^2}y^2\dot{y} + \frac{3a^3}{1 - k^2}y^4 + F(x, y); \tag{20.11}$$

$$as_1 = -2, \qquad \Theta = 1, \ 2;$$

$$as_2 = k - 1, \qquad \Theta = 5, \ k - 1;$$

$$as_3 = -k - 1, \ \Theta = 5, \ -k - 1.$$

Class XII.

$$\dddot{y} = ay\ddot{y} + \frac{36 + 3N^2}{2(36 - N^2)}a\dot{y}^2 - \frac{12a^2}{36 - N^2}y^2\dot{y} + \frac{a^3}{2(36 - N^2)}y^4 + F(x, y); \tag{20.12}$$

$$as_1 = -12, \qquad \Theta = -3, \ -4;$$

$$as_2 = N - 6, \qquad \Theta = 0, \ N - 1;$$

$$as_3 = -N - 6, \qquad \Theta = 0, \ -N - 1.$$

If $N = \infty$, this equation reduces to

$$\dddot{y} = \frac{a}{2}(2y\ddot{y} - 3\dot{y}^2) \tag{20.13}$$

and has only one set of simple parametric poles such that

$$as = -12, \quad \Theta = -3, \ -4.$$

We shall proceed further with the equations of class I-VIII, leaving the remaining equations for another paper.

21. - To obtain canonical forms for the stable equations, it is often most convenient to use a transformation $T(\lambda, \mu, \varphi)$, namely

$$y(x) = \lambda(x)u + \mu(x), \quad t = \varphi(x) \tag{21.1}$$

which does not alter the main features of the equations considered [$\lambda(x)$, $\mu(x)$, $\varphi(x)$ are analytic functions of x; see Part I, § 18].

We note for future use the following formulas where primes denote differentiations with respect to t, i.e. $u' = \frac{du}{dt}$, $u'' = \frac{d^2u}{dt^2}$, $u''' = \frac{d^3u}{dt^3}$;

$$\left\{\begin{aligned} \dot{y} &= \lambda\dot{\varphi}u' + \dot{\lambda}u + \dot{\mu}, \\ \ddot{y} &= \lambda\dot{\varphi}^2u'' + (2\Lambda + \Phi)\lambda\dot{\varphi}u' + \ddot{\lambda}u + \ddot{\mu}, \\ \dddot{y} &= \lambda\dot{\varphi}^3u''' + 3(\Lambda + \Phi)\lambda\dot{\varphi}^2u'' + M\lambda\dot{\varphi}u' + \dddot{\lambda}u + \dddot{\mu} \end{aligned}\right. \tag{21.2}$$

where

$$\Lambda = \frac{\dot{\lambda}}{\lambda}, \quad \Phi = \frac{\ddot{\varphi}}{\dot{\varphi}}, \quad M = 3\frac{\ddot{\lambda}}{\lambda} + 3\Lambda\Phi + \frac{\dddot{\varphi}}{\dot{\varphi}}. \tag{21.3}$$

According to (5.2), set

$$F(x, y) = a_1\ddot{y} + c_1y\dot{y} + c_0\dot{y} + d_3y^3 + d_2y^2 + d_1y + d_0; \tag{21.4}$$

we note that

$$(21.5)\qquad \begin{cases} F(x,\ y) = a_1\lambda\dot{\varphi}\ u'' + c_1\lambda^2\dot{\varphi}uu' \\ + [a_1(2\Lambda + \Phi) + c_1\mu + c_0]\lambda\dot{\varphi}u' \\ + d_3\lambda^3u^3 + (\Lambda c_1 + 3d_3\mu + d_2)\lambda^2u^2 \\ + [a_1\ddot{\lambda} + c_1\ddot{\mu}\lambda + c_1\dot{\lambda}\mu + c_0\dot{\lambda} + 3d_3\lambda\mu^2 + 2d_2\lambda\mu + d_1\lambda]u \\ + F(x,\ \mu). \end{cases}$$

III. **Equations of class** I.

22. The stable equations of this class are of the form

$$(22.1)\qquad \dddot{y} = ay\dot{y} + F(x,\ y)$$

where

$$(22.2)\qquad F(x,\ y) = a_1\ddot{y} + c_0\ddot{y} + d_2y^2 + d_1y + d_0.$$

Set

$$(22.3)\qquad y = \frac{t}{z^2},\quad \dot{z} = 1 + zu$$

and suppose that t is a constant (this is obtained by a transformation T); we have

$$(22.4)\qquad \begin{cases} z^2\dot{y} = -2t\left(\dfrac{1}{z} + u\right), \\ z^3\ddot{y} = 2t\left[\dfrac{3}{z} - z\dot{u} + 5u + 2zu^2\right], \\ z^4\dddot{y} = 2t\left[-\dfrac{12}{z} - z^2\ddot{u} + 7z\dot{u} - 27u + 6z^2u\dot{u} - 19zu^2 - 4z^2u^3\right]. \end{cases}$$

Recall that $ta = 12$ and $\Theta = 3,\ 5$.

On using a transformation T, we may suppose that

$$a = 12, \quad 6a_1 + d_2 = 0, \quad c_0 = 0;$$

one has only to determine λ, μ, φ by

$$\frac{\dot{a}}{a} + \Lambda - 2\Phi = 0,$$

$$\Lambda + 3\Phi = a_1 + 2\,\frac{d_2}{a},$$

$$a\mu = M - a_1(2\Lambda + \Phi) - c_0 .$$

Equation (32.1) may be written

$$\dddot{y} = 12y\dot{y} + a_1(\ddot{y} - 6y^2) + d_1 y + d_0 \tag{22.5}$$

and $t = 1$.

Substitution of $y(x)$ given by (22.3) into (22.5) shows that this equation is equivalent to the differential system

$$\dot{z} = 1 + uz, \tag{22.6}$$

$$z^2\ddot{u} - z\dot{u}(7 + a_1 z) + 5u(3 + a_1 z) \tag{22.7}$$

$$= -\frac{d_1}{2}\,z^2 - \frac{d_0}{2}\,z^4 + 6u\dot{u}z^2 - zu^2(19 + 2a_1 z) - 4z^2u^3.$$

Because $\Theta = 3,\ 5,$ one sets [see § 3]

$$u = P + z^5 v, \tag{22.8}$$

$$P = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 \tag{22.9}$$

where δ remains arbitrary and may be assumed to be a constant.

We note immediately that $\alpha = \beta = 0$ which simplifies our problem.

Now substitute u given by (22.8) into (22.7); one has

$$
(22.10) \qquad \left\{
\begin{aligned}
&\dot{u} = z^5\dot{v} + 5z^4 v + \dot{P} + O(z^5),\\
&\ddot{u} = z^5\ddot{v} + 10z^4\dot{v} + 20z^3 v + \ddot{P} + O(z^4),\\
&u^2 = P^2 + O(z^5), \quad u\dot{u} = P\dot{P} + O(z^5),\\
&u^3 = P^3 + O(z^5)
\end{aligned}
\right.
$$

so that equation (22.7) becomes

$$
(22.11) \qquad z^7\ddot{v} + 3z^6\dot{v} + A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + A_5 z^5 + O(z^6) = 0
$$

where A_0, A_1, A_2, A_3, A_4, A_5 are determined by

$$
\begin{aligned}
&z^2\ddot{P} - z\dot{P}(7 + a_1 z) + 5(3 + a_1 z)P\\
&+ \frac{d_1}{2} z^2 + \frac{d_0}{2} z^4 - 6z^2 P\dot{P} + z(19 + 2a_1 z)P^2 + 4z^2 P^3\\
&= A_0 + A_1 z^2 + A_2 z^2 + A_3 z^3 + A_4 z^4 + A_5 z^5 + O(z^6).
\end{aligned}
$$

Define α, β, γ, δ, ε by setting

$$
(22.12) \qquad A_0 = 0,\ A_1 = 0,\ A_2 = 0,\ A_3 = 0,\ A_4 = 0.
$$

Equation (22.11) yields

$$
(22.13) \qquad z^2\ddot{v} + 3z\dot{v} = A_5 + O(z).
$$

For equation (22.13) to be stable, it is necessary that

$$
(22.14) \qquad A_5 = 0.
$$

From $A_0 = 0$, $A_1 = 0$, $A_2 = 0$, one deduces immediately

$$
\alpha = 0, \quad \beta = 0,
$$

$$
(22.15) \qquad 6\gamma + d_1 = 0.
$$

Then $A_3 = 0$ gives

$$\dot{\gamma} = a_1 \gamma; \tag{22.16}$$

therefore δ remains arbitrary. Further, ε is determined by $A_4 = 0$; taking into account (22.16), one obtains

$$\varepsilon = \dot{a}_1 \gamma + 2a_1 \delta + \frac{d_0}{2}. \tag{22.17}$$

It remains to consider $A_5 = 0$ or

$$\dot{\varepsilon} + a_1 \varepsilon + 3\gamma^2 = 0. \tag{22.18}$$

Now we proceed to determine a_1, d_0, d_1 from (22.15-18).

First, from (22.16-18), one deduces

$$\gamma (\ddot{a}_1 + 2a_1 \dot{a}_1) + 2\delta(\dot{a}_1 + a_1^2) + \frac{\dot{d}_0}{2} + \frac{d_0}{2} a_1 + 3\gamma^2 = 0$$

or because δ is an arbitrary parameter,

$$\dot{a}_1 + a_1^2 = 0, \tag{22.19}$$

$$\dot{d}_0 + d_0 a_1 + 6\gamma^2 = 0. \tag{22.20}$$

Two cases are considered according as to whether a_1 is or is not zero.

i. $a_1 = 0$. From (22.16), (22.20) and (22.15), one deduces

$$\gamma = K, \quad d_1 = -6K, \quad d_0 = -6K^2 x + K_1 \tag{22.21}$$

where K, K_1 are as usual arbitrary constants. Therefore, the desired stable equation is

$$\dddot{y} = 12y\dot{y} - 6Ky - 6K^2 x + K_1. \tag{22.22}$$

By introducing x in place of $x - \frac{K_1}{6K^2}$, one may assume $K_1 = 0$. Then a transformation $x \to \alpha_1 x$, $y \to \beta_1 y$ where $\alpha_1^2 \beta_1 = 1$ and $K\alpha_1^3 = -1$ reduces equation (22.22) to the canonical form

$$\dddot{y} = 12y\dot{y} + 6y - 6x. \tag{22.23}$$

Set $y = u$ in (22.23) and integrate with respect to x; one obtains

$$\dddot{u} - 6\dot{u}^2 = 6u - 3x^2 + 6K$$

or by replacing $u + K$ by u,

$$\dddot{u} - 6\dot{u}^2 = 6u - 3x^2 \tag{22.24}$$

which is an equation of class II (see below eq. (27.3)).

ii. $a_1 \neq 0$. Then from (22.19), one has

$$a_1 = \frac{1}{x+K};$$

by replacing $x + K$ by x, one may assume $K = 0$. Then, from (22.16)

$$\gamma = Kx$$

and from (22.20),

$$d_0 = \frac{K_1}{x} - \frac{3}{2}K^2x^3.$$

The desired stable equation is thus

$$\dddot{y} = 12y\dot{y} + \frac{1}{x}(\ddot{y} - 6y^2) - 4Kxy + \frac{K_1}{x} - \frac{3}{2}K^2x^3. \tag{22.25}$$

The transformation $x \to \alpha_1 x$, $y \to \beta_1 y$ where $\alpha_1^2\beta_1 = 1$, $K\alpha_1^4 = -2$ reduces (22.25) to the canonical form

$$\dddot{y} = 12y\dot{y} + \frac{1}{x}(\ddot{y} - 6y^2 - K) + 12xy - 6x^3 \tag{22.26}$$

where K is again an arbitrary constant.

Moreover, it is clear that equation (22.26) is equivalent to the differential system

$$(22.27)\qquad \begin{cases} \ddot{y} = 6y^2 + K + z, \\ x\dot{z} = z + 12x^2y - 6x^4. \end{cases}$$

IV. Equations of class II.

23. - In the following, we are concerned with stable equations whose integrals have only simple parametric poles.

Assuming s to be a constant (which is obtained by a transformation T), we set

$$(23.1)\qquad y = \frac{s}{z}, \qquad \dot{z} = 1 + uz$$

and note the useful relations

$$(23.2)\qquad \begin{cases} z^2\dot{y} = -s(1 + uz), \\ z^2\ddot{y} = s\left(\frac{2}{z} - z\dot{u} + 3u + zu^2\right), \\ z^3\dddot{y} = -s\left[\frac{6}{z} + z^2\ddot{u} - 4z\dot{u} + 12u + 7zu^2 - z^2(3u\dot{u} - u^3)\right], \\ z^3\dot{y}^2 = s^2\left[\frac{1}{z} + 2u + zu^2\right], \\ z^3y\dot{y} = -s^2(1 + uz). \end{cases}$$

24. - Consider the equation of class II

$$(24.1)\qquad \dddot{y} = b\dot{y}^2 + F(x, y)$$

for which $bs = -6$, $\Theta = 0$, 5 [see § 20] and determine $T(\lambda, \mu, \varphi)$ in order that

$$b = -6, \quad a_1 = 0, \quad d_2 = c_0.$$

This is achieved by setting

$$\Lambda - \Phi + \frac{\dot{b}}{b} = 0, \qquad 3(\Lambda + \Phi) - a_1 = 0,$$

$$-M + 2b\dot{\mu} + (2\Lambda + \Phi)a_1 + c_1\mu + c_0 + \frac{6}{b}[b^2\Lambda^2 + c_1\Lambda + 3d_3\mu + d_2] = 0$$

so that μ is determined by a linear differential equation.

[The choice of $b = -6$ is obvious for then $s = 1$. Assume the coefficients of $F(x, y)$ to be arbitrary and apply our method. The first condition for stability is $2a_1 - c_1 + d_3 = 0$. Further, the condition $d_2 = c_0$ simplifies the problem].

With these simplifications, equation (24.1) takes the form

(24.2) $$\dddot{y} + 6\dot{y}^2 = c_1 y\dot{y} + c_0(\dot{y} + y^2) + d_3 y^3 + d_1 y + d_0$$

and its associated equation in u is

(24.3) $$z^2\ddot{u} - 4z\dot{u} + 7zu^2 - z^2(3u\dot{u} - u^3) - 6u^2z$$
$$- c_1(1 + uz) - c_0 z^2 u + d_3 + d_1 z^2 + d_0 z^3 = 0.$$

Because $\Theta = 0,\ 5,$ there follows the condition for stability

(24.4) $$c_1 = d_3.$$

Equation (24.3) becomes

(24.5) $$z\ddot{u} - 4\dot{u} - c_1 u + d_1 z + d_0 z^2 - c_0 z u + u^2 - z(3u\dot{u} - u^3) = 0.$$

According to the general theory, one sets

(24.6) $$u = P + z^5 v$$

where

$$P = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4;$$

α remains arbitrary and will be assumed to be a constant parameter.

On using the formulas given above, this substitution brings equation (24.5) to

$$z^6\ddot{v} + 6z^5\dot{v} + A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + O(z^5) = 0$$

where the A's are determined by

$$z\ddot{P} - 4\dot{P} - c_1 P + d_1 z + d_0 z^2 - c_0 z P + P^2 - 3zP\dot{P} + zP^3$$

$$= A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + O(z^5).$$

Now, we determine β, γ, δ, ε by setting

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0;$$

then, the condition for stability is $A_4 = 0$.

25. - We give the results.

From $A_0 = 0$, one obtains

(25.1) $$4\beta = \alpha^2 - c_1\alpha.$$

From $A_1 = 0$ and (25.1), one deduces

(25.2) $$6\gamma = -2\dot{\beta} + c_1(\alpha^2 - \beta) - c_0\alpha + d_1 .$$

From $A_2 = 0$ and (25.1-2), one has

(25.3) $$6\delta = (\dot{\beta} + \alpha\beta)^{\cdot} - c_1\alpha^3 + 2c_1\alpha\beta - c_1\gamma + c_0(\alpha^2 - \beta) - \alpha d_1 + d_0;$$

then $A_3 = 0$ yields

(25.4) $$4\varepsilon = (\dot{\gamma} + 2\delta + \alpha\gamma)^{\cdot} - 4\alpha\delta - 8\beta\gamma + \alpha^2\gamma - c_1\delta - c_0\gamma.$$

Finally, the condition for stability as given by $A_4 = 0$ can be written

(25.5) $$(\dot{\delta} + 4\varepsilon + 3\alpha\delta + 2\beta\gamma)^{\cdot} - 4\beta\delta + 2\alpha\beta\gamma - 3\gamma^2 + 2\alpha\varepsilon + 3\alpha^2\delta$$

$$- \gamma\dot{\beta} - c_1\varepsilon - c_0\delta = 0.$$

It is clear that β, γ, δ, ε are polynomials in α and so is also the first member of (25.5). This condition for stability shows that a certain polynomial in α is identically zero; because α is arbitrary, the coefficients of this polynomial must be zero. Thus, one obtains conditions which determine the coefficients of the stable equations of class II.

26. To simplify the problem, we replace α, c_0, c_1, d_1, d_0, by 2α, $6c_0$, $6c_1$, $6d_1$, $6d_0$ respectively and obtain

$$(26.1)\quad \begin{cases} \beta = \alpha^2 - 3c_1\alpha, \\ \gamma = \dot{c}_1\alpha + 3c_1\alpha^2 + 3c_1^2\alpha - 2c_0\alpha + d_1, \\ \delta = \frac{1}{6}(\dot{\beta} + 2\alpha\beta)^{\cdot} - 8c_1\alpha^3 + 4c_1\alpha\beta - c_1\gamma + c_0(4\alpha^2 - \beta) \\ \quad - 2\alpha d_1 + d_0, \\ 4\varepsilon = (\dot{\gamma} + 2\delta + 2\alpha\gamma)^{\cdot} - 8\alpha\delta - 8\beta\gamma + 4\alpha^2\gamma - 6c_1\delta - 6c_0\gamma, \end{cases}$$

$$(26.2)\quad (\dot{\delta} + 4\varepsilon + 6\alpha\delta + 2\beta\gamma)^{\cdot} - 4\beta\delta + 4\alpha\beta\gamma - 3\gamma^2 + 4\alpha\varepsilon + 12\alpha^2\delta$$
$$- \gamma\dot{\beta} - 6c_1\varepsilon - 6c_0\delta = 0.$$

To further simplify the notations, we set

$$(26.3)\quad \begin{cases} \beta = \alpha^2 + \beta_1\alpha, \\ \gamma = \gamma_2\alpha^2 + \gamma_1\alpha + \gamma_0, \\ \delta = \delta_3\alpha^3 + \delta_2\alpha^2 + \delta_1\alpha + \delta_0, \\ 4\varepsilon = \varepsilon_4\alpha^4 + \varepsilon_3\alpha^3 + \varepsilon_2\alpha^2 + \varepsilon_1\alpha + \varepsilon_0 \end{cases}$$

where

$$(26.4)\quad \begin{cases} \beta_1 = -3c_1, \\ \gamma_2 = 3c_1, \quad \gamma_1 = \dot{c}_1 + 3c_1^2 - 2c_0, \\ \delta_3 = -4c_1, \quad \delta_2 = -(\dot{c}_1 + 15c_1^2 - 3c_0), \\ \varepsilon_4 = 20c_1, \quad \varepsilon_3 = 2\dot{c}_1 + 204c_1^2 - 16c_0. \end{cases}$$

The left member of (26.2) is a polynomial in α of degree 5: however the coefficient of α^5 is zero and that of α^4 is $-45\,c_1^2$. Because α is an arbitrary parameter, one obtains

$$c_1 = 0 \tag{26.5}$$

and therefore [see (24.4)],

$$d_3 = 0. \tag{26.6}$$

These conditions greatly simplify β, γ, δ, ε; one obtains

$$\beta = \alpha^2, \quad \gamma = -2c_0\alpha + d_1,$$

$$\delta = 3c_0\alpha^2 - 2d_1\,\alpha + d_0,$$

$$4\varepsilon = -16c_0\alpha^3 + (2\dot{c}_0 + 12d_1)\alpha^2 + (12c_0^2 - 8d_0 - 2\ddot{c}_0 - 2\dot{d}_1)\alpha$$

$$+ \ddot{d}_1 + 2\dot{d}_0 - 6c_0d_1.$$

The left member of (26.2) is now a polynomial in α of the second degree; one has

$$(3\ddot{c}_0 - 18c_0^2)\alpha^2 + [12c_0^2 - 2\ddot{c}_0)^{\cdot} - 3(\ddot{d}_1 - 6c_0d_1)]\,\alpha$$

$$+ 3\ddot{d}_0 - 6c_0d_0 - 3d_1^2 + (\ddot{d}_1 - 6c_0d_1)^{\cdot} = 0.$$

Therefore

$$\ddot{c}_0 = 6c_0^2, \tag{26.7}$$

$$\ddot{d}_1 = 6c_0d_1, \tag{26.8}$$

$$\ddot{d}_0 - 2c_0d_0 = d_1^2. \tag{26.9}$$

27. - Note that $F(x, y)$ assumes the form

$$F(x, y) = 6c_0(\dot{y} + y^2) + 6d_1y + 6d_0. \tag{27.1}$$

From (26.7), one deduces

$$\dot{c}_0^2 = 4c_0^3 - g_3$$

where g_3 is an arbitrary constant. Therefore, one has to consider three cases according to the values of g_3, namely

i. $g_3 = 0, \quad c_0 = 0;$

ii. $g_3 = 0, \quad c_0 = x^{-2};$

iii. $g_3 \neq 0, \quad c_0 = \wp(x;\ 0,\ g_3).$

i. $c_0 = 0$. Then $\ddot{d}_1 = 0$ and

$$d_1 = K_1 x + K_2.$$

We consider again three cases, namely

a. $K_1 = K_2 = 0,$

b. $K_1 = 0, \quad K_2 \neq 0,$

c. $K_1 \neq 0, \quad K_2 = 0,$

$\left[K_1 \neq 0,\ K_2 \neq 0\right.$ is reduced to case *c.* by replacing x by $\left. x - \dfrac{K_2}{K_1}\right]$.

a. From (26.9), one has $\ddot{d}_0 = 0$ and

$$d_0 = K_3 x + K_4.$$

The stable equation of this class is thus

$$\dddot{y} + 6\dot{y}^2 = K_3 x + K_4$$

and is easily reduced to the two canonical forms

$$\text{(27.2)} \qquad \left\{ \begin{aligned} &\dddot{y} + 6\dot{y}^2 = K_4, \\ &\dddot{y} + 6\dot{y}^2 = -x. \end{aligned} \right.$$

By setting $\dot{y} = z$, these equations become respectively

$$\ddot{z} + 6z^2 = K_4,$$

$$\ddot{z} + 6z^2 = -x$$

i.e. well known stable equations of the second order.

b. From (26.9), one obtains $\ddot{d}_0 = K_2$ and

$$d_0 = \frac{K_2}{2} x^2 + K_3 x + K_4 .$$

The stable equation of this class is of the form

$$\dddot{y} + 6\dot{y}^2 = K_2 y + \frac{K_2}{2} x^2 + K_3 x + K_4 ;$$

it may be reduced to the canonical form

$$\dddot{y} + 6\dot{y}^2 = 6y + 3x^2 + K_3 x + K_4 . \tag{27.3}$$

[see eq. (22.24)].

c. From (26.9), one deduces $\ddot{d}_0 = K_1 x$ and

$$d_0 = \frac{K_1^2}{12} x^4 + K_3 x + K_4 .$$

The stable equation of this class takes the form

$$\dddot{y} + 6\dot{y}^2 = K_1 x y + \frac{K_1^2}{12} x^4 + K_3 x + K_4$$

and may be reduced to the canonical form

$$\dddot{y} + 6\dot{y}^2 = 12 x y + 12 x^4 + K_3 x + K_4 . \tag{27.4}$$

ii. $c_0 = x^{-2}$. Then d_1 satisfies

$$\ddot{d}_1 - \frac{6}{x^2} d_1 = 0$$

and thus

$$d_1 = \frac{K_1}{x^2} + K_2 x^3 .$$

Three cases are again considered, namely

a. $K_1 = K_2 = 0$,

b. $K_1 = 0, \quad K_2 \neq 0$,

c. $K_1 \neq 0, \quad K_2 = 0$;

then d_0 is determined by

$$\ddot{d}_0 - \frac{2}{x^2} d_0 = d_1^2. \tag{27.5}$$

[Note that the solution of the homogenous equation is

$$d_0 = \frac{K_3}{x} + K_4 x^2 \Big]. \tag{27.6}$$

a. Then $d_1 = 0$ and d_0 is given by (27.6); the stable equation of this class is

$$\dddot{y} + 6\dot{y}^2 = \frac{6}{x^2}(\dot{y} + y^2) + \frac{K_3}{x} + K_4 x^2. \tag{27.7}$$

This equation is equivalent to the differential system

$$\begin{cases} \dot{y} = - y^2 + u, \\ 4uy^2 + 2y\dot{u} - \ddot{u} - 4u^2 + \dfrac{6}{x^2} u + \dfrac{K_3}{x} + K_4 x^2 = 0. \end{cases} \tag{27.8}$$

b. One has

$$d_1 = K_2 x^3;$$

then d_0 is determined by

$$\ddot{d}_0 - \frac{2}{x^2} d_0 = K_2^2 x^6$$

so that

$$d_0 = \frac{K_2^2}{54}\, x^8 + K_4 x^2 + \frac{K_3}{x}.$$

The stable equation of this class is of the form

$$\dddot{y} + 6\dot{y}^2 = \frac{6}{x^2}(\dot{y} + y^2) + K_2 x^3 y + \frac{K_2^2}{54}\, x^8 + K_4 x^2 + \frac{K_3}{x} \tag{27.9}$$

and may be reduced to the canonical form

$$\dddot{y} + 6\dot{y}^2 = \frac{6}{x^2}(\dot{y} + y^2) + 18 x^3 y + 6 x^8 + K_4 x^2 + \frac{K_3}{x}.$$

This last equation is equivalent to the differential system

$$\begin{cases} \dot{y} = -y^2 + u, \\ 4y^2u + 2y(\dot{u} + 9x^3) - \ddot{u} - 4u^2 + \dfrac{6u}{x^2} + 6x^3 + K_4x^2 + \dfrac{K_3}{x} = 0. \end{cases}$$

c. One has

$$d_1 = \frac{K_1}{x^2},$$

$$\ddot{d}_0 - \frac{2}{x^2}d_0 = \frac{K_1^2}{x^4}$$

and

$$d_0 = \frac{K_1^2}{4x^2} + K_4x^2 + \frac{K_3}{x}.$$

The stable equation of this class is

(27.10) $$\dddot{y} + 6\dot{y}^2 = \frac{6}{x^2}(\dot{y} + y^2) + \frac{K_1y}{x^2} + \frac{K_1^2}{4x^2} + K_4x^2 + \frac{K_3}{x}$$

and may be reduced to the canonical form

$$\dddot{y} + 6\dot{y}^2 = \frac{6}{x^2}(\dot{y} + y^2) + \frac{2y}{x} + \frac{1}{x^2} + K_4x^2 + \frac{K_3}{x}.$$

iii. $c_0 = \wp(x;\ 0,\ g_3)$; then d_1 and d_0 are given respectively by

(27.11) $$\ddot{d}_1 - 6\wp(x;\ 0,\ g_3)\,d_1 = 0,$$

(27.12) $$\ddot{d}_0 - 2\wp(x;\ 0,\ g_3)\,d_0 = d_1^2.$$

These equations are of the type (LAME's equation)

$$\ddot{w} - [n(n+1)\wp + h]w = 0$$

where n is a positive integer and h a constant; they correspond respectively to $n = 2$, $n = 1$ and have been considered by HERMITE, PICARD and HALPHEN.

We have to consider two cases, namely $d_1 = 0$ and $d_1 \neq 0$.

a. $d_1 = 0$. Then equation (27.12) becomes

$$\ddot{d}_0 - 2\wp(x;\ 0;\ g_3)\, d_0 = 0.$$

This Lamé's equation has two distinct solutions

$$d_{01}(x) = e^{-x\zeta(h)} \frac{\sigma(x+h)}{\sigma(x)}, \quad d_{02}(x) = e^{x\zeta(h)} \frac{\sigma(x-h)}{\sigma(x)} \tag{27.13}$$

where h satisfies $\wp(h) = 0$.

The stable equations of this class are

$$\ddot{y} + 6\dot{y}^2 = 6\wp(x;\ 0;\ g_3)(\dot{y} + y^2) + K_1 d_{01}(x) + K_2 d_{02}(x). \tag{27.14}$$

A transformation $x \to \alpha x$, $y \to \beta y$, $c_0 \to \gamma c_0$, $\alpha\beta = 1$, $\alpha^2\gamma = 1$ $\alpha^4 g_3 = 1$ brings equation (27.14) to

$$\ddot{y} + 6\dot{y}^2 = 6\wp(x;\ 0,\ 1)(\dot{y} + y^2) + K_1 d_{01}(x) + K_2 d_{02}(x) \tag{27.15}$$

where $\wp(x)$ is the solution of

$$\ddot{c}_0 = 4c_0^3 - 1.$$

b. $d_1 \neq 0$. Equation (27.11) has two distinct solutions, namely

$$d_{11}(x) = \wp(x), \qquad d_{12}(x) = \wp(x) \int_0^x \frac{dt}{\wp^2(t)}.$$

The homogeneous equation associated with (27.12) has two distinct solutions $d_{01}(x)$ and $d_{02}(x)$ [see (27.13)]; one has

$$d_{01}(x)\dot{d}_{02}(x) - d_{02}(x)\dot{d}_{01}(x) = -\sigma^2(h)\dot{\wp}(h).$$

Finally, one particular solution of equation (27.12) may be obtained by quadratures by using the method of variation of parameters.

V. Equations of class III.

28. The stable equations of this class are of the form

$$\dddot{y} = a(y\ddot{y} + \dot{y}^2) + F(x, y); \tag{28.1}$$

their solutions have only one set of simple parametric poles such that $as = -2$ and $\Theta = 1, 2$.

By a suitable transformation T, one may assume

$$a = -2, \quad c_1 = d_3, \quad a_1 = 0$$

so that $s = 1$. One has only to determine λ, μ, φ by setting

$$a(\Lambda - \Phi) + \dot{a} = 0,$$

$$a(4\Lambda + \Phi) + c_1 + \frac{2d_3}{a} = 0,$$

$$a\mu - 3(\Lambda + \Phi) + a_1 = 0.$$

With these simplifications, equation (28.1) takes the form

$$\dddot{y} + 2(y\ddot{y} + \dot{y}^2) = c_1(y\dot{y} + y^3) + c_0\dot{y} + d_2y^2 + d_1y + d_0; \tag{28.2}$$

its associated equation in u is

$$\begin{aligned} z^2\ddot{u} - 2z\dot{u} + 2u + (d_2 - c_0)z + d_1z^2 + d_0z^3 - c_1uz \\ - c_0uz^2 + 3zu^2 - z^2(3u\dot{u} - u^3) = 0. \end{aligned} \tag{28.3}$$

Because $\Theta = 1, 2$, one sets

$$u = P + z^2v \tag{28.4}$$

where

$$P = \alpha + \beta z.$$

One sees immediately that $\alpha = 0$; because $\Theta = 1$, β remains arbitrary and may be assumed to be a constant.

Our problem is now readily solved. In fact, substitution of u given by (28.4) into (28.3) yields

$$z^4\ddot{v} + 2z^3\dot{v} + A_0 + A_1 z + A_2 z^2 + O(z^3) = 0$$

where A_0, A_1, A_2 are determined by

$$A_0 + A_1 z + A_2 z^2 + O(z^3)$$

$$= z^2\ddot{P} - 2z\dot{P} + 2P + (d_2 - c_0)z + d_1 z^2 - c_1 zP - c_0 z^2 P$$

$$+ 3zP^2 - 3z^2 P\dot{P} + z^2\dot{P}^3.$$

One obtains

$$A_0 = 0, \quad A_1 = d_2 - c_0, \quad A_2 = d_1 - c_1\beta;$$

the conditions for stability are thus $A_1 = A_2 = 0$ and therefore because β is arbitrary,

$$d_2 = c_0, \quad d_1 = c_1 = 0.$$

The stable equations of class III are thus [see (28.2)]

$$\dddot{y} + 2(y\ddot{y} + \dot{y}^2) = c_0(\dot{y} + y^2) + d_0 \tag{28.5}$$

where c_0 and d_0 are arbitrary analytic functions of x.

To integrate this equation, observe that it is equivalent to the differential system

$$\begin{cases} \dot{y} + y^2 = u, \\ \ddot{u} = c_0 u + d_0 \end{cases}$$

or to the differential system

$$\begin{cases} y = \dfrac{\dot{t}}{t}, \quad \ddot{t} = ut, \\ \ddot{u} = c_0 u + d_0. \end{cases}$$

Therefore, $y(x)$ is stable.

VI. Equations of class IV.

29. The stable equations of this class are of the form

$$(29.1)\qquad \dddot{y} = ay\ddot{y} + a\dot{y}^2 - \frac{a^2}{3}\, y^2\dot{y} + F(x,\ y).$$

The solutions of this equation have two sets of simple parametric poles determined by

$$(29.2)\qquad \begin{cases} as_1 = -3, & \Theta = 0,\ 2; \\ as_2 = -6, & \Theta = -3,\ 2. \end{cases}$$

The associated equation in u is

$$(29.3)\qquad \begin{cases} z^2\ddot{u} - z\dot{u}(4 + as) + u\left(12 + 5as + \dfrac{a^2s^2}{3}\right) \\ + 2a_1 - c_1 s + d_3 s^2 + z(d_2 s - c_0) + d_1 z^2 + \dfrac{d_0}{s} z^3 \\ + (3a_1 - c_1 s) zu + (7 + 2as) zu^2 - c_0 z^2 u + a_1 z^2 u^2 \\ - a_1 z^2 \dot{u} - z^2(3u\dot{u} - u^3) = 0 \end{cases}$$

where s is equal to s_1 or s_2.

By a suitable transformation T, one may assume

$$(29.4)\qquad a = -3, \quad a_1 - c_1 + 2d_3 = 0, \quad a_1 = 0$$

so that $s_1 = 1$ and $s_2 = 2$. One has only to determine λ, μ, φ by setting

$$\Lambda - \Phi + \frac{\dot{a}}{a} = 0,$$

$$3\Lambda - a\mu + a_1 + \frac{3c_1}{a} + \frac{18d_3}{a^2} = 0,$$

$$3(\Lambda + \Phi) - a\mu = a_1 .$$

With these simplifications, equation (29.3) becomes

$$
(29.5)\quad \left\{ \begin{aligned}
& z^2\ddot{u} - z\dot{u}(4-3s) + u(12-15s+3s^2) \\
& - c_1 s + d_3 s^2 + z(d_2 s - c_0) + d_1 z^2 + \frac{d_0}{s} z^3 \\
& - c_1 s z u + (7-6s) z u^2 - c_0 z^2 u - z^2(3u\dot{u} - u^3) = 0.
\end{aligned} \right.
$$

Now suppose $s = 1$, so that

$$
(29.6)\quad \begin{aligned}
& z^2\ddot{u} - z\dot{u} - c_1 + d_3 + z(d_2 - c_0) + d_1 z^2 + d_0 z^3 \\
& - c_1 z u + z u^2 - c_0 z^2 u - z^2(3u\dot{u} - u^3) = 0.
\end{aligned}
$$

Because $\Theta = 0,\ 2$, a condition for stability is $c_1 = d_3$ and consequently [see (29.4)],

$$
(29.7)\qquad c_1 = d_3 = 0.
$$

Then, substitute

$$
u = P + z^2 v,
$$

$$
P = \alpha + \beta z,
$$

into (29.6) and rewrite the result as

$$
z^4\ddot{v} + 3z^3\dot{v} + A_0 + A_1 z + A_2 z^2 + O(z^3) = 0
$$

where A_0, A_1, A_2 are defined by

$$
\begin{aligned}
& A_0 + A_1 z + A_2 z^2 + O(z^3) \\
& = z^2\ddot{P} - z\dot{P} + z(d_2 - c_0) + d_1 z^2 + zP^2 - c_0 z^2 P - 3P\dot{P}z^2 + z^2 P^3.
\end{aligned}
$$

Accordingly, the conditions for stability are determined by

$$
A_0 = 0, \quad A_1 = 0, \quad A_2 = 0.
$$

Note that A_0 is identically zero, so that α remains arbitrary; we assume it to be a constant. Then, from $A_1 = 0$,

$$\beta = \alpha^2 + d_2 - c_0 \tag{29.8}$$

and from $A_2 = 0$,

$$\dot{\beta} + \alpha^3 - \alpha\beta - c_0\alpha + d_1 = 0. \tag{29.9}$$

Substituting β given by (29.8) into (29.9) yields

$$\dot{d}_2 - \dot{c}_0 - \alpha d_2 + d_1 = 0;$$

because α is arbitrary, one finally obtains

$$d_2 = 0, \quad d_1 = \dot{c}_0. \tag{29-10}$$

It remains to consider the parametric poles corresponding to $s = 2$. Because of (29.7) and (29.10), equation (29.5) takes the form

$$z^2\ddot{u} + 2z\dot{u} - 6u - c_0 z + d_1 z^2 + \frac{d_0}{2} z^3 - 5zu^2 - c_0 z^2 u - z^2(3u\dot{u} - u^3) = 0.$$

In agreement with $\Theta = -3, 2$, one sets

$$u = P + z^2 v,$$

$$P = \alpha + \beta z$$

and determines A_0, A_1, A_2 by

$$A_0 + A_1 z + A_2 z^2 + O(z^3)$$

$$= z^2\ddot{P} + 2z\dot{P} - 6P - c_0 z + d_1 z^2 + \frac{d_0}{2} z^3 - 5zP^2 - c_0 z^2 P - 3z^2 P\dot{P} + z^2 P^3.$$

The condition for stability is given by $A_0 = A_1 = A_2 = 0$.

From $A_0 = 0$, it follows that $\alpha = 0$. Then $A_1 = 0$, $A_2 = 0$ yield respectively

$$4\beta + c_0 = 0, \quad 4\dot{\beta} + d_1 = 0$$

or

$$d_1 = \dot{c}_0$$

which is already satisfied [see (29.10)].

The stable equations of this class are thus

$$\dddot{y} = -3y\ddot{y} - 3\dot{y}^2 - 3y^2\dot{y} + c_0\dot{y} + \dot{c}_0 y + d_0 \tag{29.11}$$

where c_0 and d_0 are arbitrary analytic functions of x.

To integrate these equations, set $d_0 = \dot{q}$ and integrate with respect to x; then

$$\ddot{y} = -3y\dot{y} - y^3 + c_0 y + q + K. \tag{29.12}$$

Equation (29.12) is of the second order, of type I [see Part I, equation (20.4)]; by setting $y = \frac{\dot{v}}{v}$, it reduces to a linear equation

$$\dddot{v} = c_0\dot{v} + (q + K)v$$

of the third order; therefore, $y(x)$ is stable.

VII. Equations of class V.

30. - The stable equations of this class are of the form

$$\dddot{y} = ay\ddot{y} + 2a\dot{y}^2 - \frac{a^2}{2}y^2\dot{y} + F(x, y). \tag{30.1}$$

The solutions of equation (30.1) have two sets of simple parametric poles determined by

$$\begin{cases} as_1 = -2, \quad \Theta = 0, \; 3; \\ as_2 = -6, \quad \Theta = -4, \; 3. \end{cases} \tag{30.2}$$

The associated equation in u is

$$\begin{cases} z^2\ddot{u} - z\dot{u}(4 + as) + u\left(12 + 7as + \frac{a^2s^2}{2}\right) + 2a_1 - c_1 s + d_3 s^2 \\ + z(d_2 s - c_0) + d_1 z^2 + \frac{d_0 z^3}{s} + uz(3a_1 - c_1 s) - c_0 z^2 u - a_1 z^2\dot{u} \\ + a_1 z^2 u^2 + zu^2(7 + 3as) - z^2(3u\dot{u} - u^3) = 0 \end{cases} \tag{30.3}$$

where s is equal to s_1 or s_2.

By a suitable transformation T, one may assume

$$a = -2, \quad a_1 = 0, \quad c_1 = 0 \tag{30.4}$$

so that $s_1 = 1$ and $s_2 = 3$. One has only to determine λ, μ φ by setting

$$\Lambda - \Phi + \frac{\dot{a}}{a} = 0,$$

$$3(\Lambda + \Phi) - a\mu - a_1 = 0,$$

$$6\Lambda + \Phi - a\mu + \frac{c_1}{a} = 0.$$

With these simplifications, equation (30.3) becomes

$$\begin{cases} z^2\ddot{u} - z\dot{u}(4 - 2s) + u(12 - 14s + 2s^2) + d_3 s^2 + z(d_2 s - c_0) + d_1 z^2 \\ + \dfrac{d_0}{s} z^3 - c_0 z^2 u + zu^2(7 - 6s) - z^2(3u\dot{u} - u^3) = 0. \end{cases} \tag{30.5}$$

Now suppose $s = 1$ and note that because $\Theta = 0$, a condition for stability is $d_3 = 0$; then equation (30.5) is

$$z\ddot{u} - 2\dot{u} + d_2 - c_0 + d_1 z + d_0 z^2 - c_0 zu + u^2 - z(3u\dot{u} - u^3) = 0. \tag{30.6}$$

In agreement with $\Theta = 0, 3$, set

$$u = P + z^3 v,$$

$$P = \alpha + \beta z + \gamma z^2$$

where α remains arbitrary and will be assumed to be a constant parameter. Further, determine A_0, A_1, A_2 by setting

$$z\ddot{P} - 2\dot{P} + d_2 - c_0 + d_1 z + d_0 z^2 - c_0 zP + P^2 - 3zP\dot{P} + zP^3$$

$$= A_0 + A_1 z + A_2 z^2 + O(z^3)$$

and note that β, γ are given by

$$A_0 = A_1 = 0 \tag{30.7}$$

and that the condition for stability is

$$A_2 = 0. \tag{30.8}$$

One obtains by using (30.7)

$$2\beta = \alpha^2 + d_2 - c_0, \tag{30.9}$$

$$2\gamma = - d_2\alpha + d_1; \tag{30.10}$$

then (30.8) yields

$$\ddot{\beta} + 2\dot{\gamma} - \alpha\dot{\beta} - 2\alpha\gamma + d_0 - d_2\beta = 0. \tag{30.11}$$

The left hand member of (30.11) is a polynomial in α of degree 2; because α is arbitrary, one has

$$\begin{cases} d_2 = 0, \\ 2d_1 = \dot{c}_0, \\ 2d_0 = \ddot{c}_0 - 2\dot{d}_1 = 0. \end{cases} \tag{30.12}$$

By virtue of these conditions and with c_0 being replaced by $2c_0$, equation (30.1) becomes

$$\dddot{y} = - 2y\ddot{y} - 4\dot{y}^2 - 2y^2\dot{y} + 2c_0\dot{y} + \dot{c}_0 y \tag{30.13}$$

where c_0 is an arbitrary analytic function of x.

To integrate this equation, multiply both members by 2y and note that

$$2y\dddot{y} = \frac{d}{dx}(2y\ddot{y} - \dot{y}^2);$$

equation (30.13) is thus equivalent to

$$\frac{d}{dx}(2y\ddot{y} - \dot{y}^2) = - \frac{d}{dx}(4y^2\dot{y} + y^4) + 2\frac{d}{dx}(c_0 y^2)$$

and also to

$$\ddot{y} = \frac{\dot{y}^2}{2y} - 2y\dot{y} - \frac{y^3}{2} + c_0 y + \frac{K}{2y}. \tag{30.14}$$

This equation of the second order is of Type III, class $E.16$ and may be reduced by the transformation $x \to \alpha x$, $y \to \beta y$, $\alpha\beta = 1$, $K\alpha^4 = -1$ to the canonical form

$$\ddot{y} = \frac{\dot{y}^2}{2y} - 2y\dot{y} - \frac{y^3}{2} + \alpha^2 c_0 y - \frac{1}{2y};$$

it is reducible to a linear equation of the fourth order by setting $y = \frac{\dot{w}}{w}$. Therefore $y(x)$ given by (30.14) or (30.13) is stable.

31. - It may be verified that no new condition for stability arises from the second set of parametric poles.

Taking into account the conditions for stability obtained in the preceding article, one may write equation (30.5) as follows (note that $s = 3$)

$$z^2\ddot{u} + 2z\dot{u} - 12u - 2c_0 z + \dot{c}_0 z^2 - 2c_0 z^2 u - 11zu^2 - z^2(3u\dot{u} - u^3) = 0.$$

Since $\Theta = -4, 3$, one sets

$$u = P + z^3 v,$$

$$P = \alpha + \beta z + \gamma z^2$$

and determines A_0, A_1, A_2, A_3 by setting

$$z^2\ddot{P} + 2z\dot{P} - 12P - 2c_0 z + \dot{c}_0 z^2 - 2c_0 z^2 P - 11zP^2 - 3z^2 P\dot{P} + z^2 P^3$$

$$= A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4).$$

Obviously, $\alpha = 0$ and $A_0 = 0$; then $A_1 = 0$, $A_2 = 0$ give respectively

$$5\beta + c_0 = 0, \quad 30\gamma = \dot{c}_0.$$

The condition for stability is then $A_3 = 0$ or

$$\ddot{\beta} + 6\dot{\gamma} - 2c_0\beta - 10\beta^2 = 0$$

and is an identity.

VIII. Equations of class VI.

32. - The stable equations of this class are of the form

$$\dddot{y} = a y \ddot{y} + 5a\dot{y}^2 - a^2y^2\dot{y} + F(x, y). \tag{32.1}$$

The solutions of these equations have two sets of simple parametric poles determined by

$$\left\{ \begin{array}{ll} as_1 = -1, & \Theta = 0, 4 \quad ; \\ as_2 = -6, & \Theta = -6, 5. \end{array} \right. \tag{32.2}$$

The associated equation in u is

$$\left\{ \begin{array}{l} z^2\ddot{u} - (4 + as)z\dot{u} + (12 + 13as + a^2s^2)u + 2a_1 - c_1s + d_3s^2 \\ + z(d_2s - c_0) + z^2d_1 + \dfrac{d_0}{s} z^3 + zu(3a_1 - c_1s) - c_0z^2u \\ - z^2a_1(\dot{u} - u^2) + zu\,(7 + 6as) - z^2(3u\dot{u} - u^3) = 0 \end{array} \right. \tag{32.3}$$

where s is equal to s_1 or s_2.

By a suitable transformation T, one may assume

$$a = 1, \qquad a_1 = c_1 = 0 \tag{32.4}$$

and consequently, $s_1 = -1$, $s_2 = -6$. One has only to determine λ, μ, φ by setting

$$\Lambda - \Phi + \frac{\dot{a}}{a} = 0,$$

$$3\Lambda + 3\Phi - a\mu - a_1 = 0,$$

$$12\Lambda + \Phi - 2a\mu + \frac{c_1}{a} = 0.$$

For convenience, we shall write $3c$ instead of c_0. Then because of (32.4), equation (32.3) may be rewritten as

$$\left\{ \begin{array}{l} z^2\ddot{u} - (4 + as)z\dot{u} + (12 + 13as + a^2s^2)\,u + d_3s^2 + z(d_2s - 3c) \\ + z^2d_1 + \dfrac{d_0}{s} z^3 - 3cz^2u + (7 + 6as)zu^2 - z^2(3u\dot{u} - u^3) = 0. \end{array} \right. \tag{32.5}$$

Now suppose $s = -1$ so that

$$(32.6) \quad z^2\ddot{u} - 3z\dot{u} + d_3 - (3c + d_2)z + d_1 z^2 - d_0 z^3 - 3cz^2 u + zu^2 - z^2(3u\dot{u} - u^3) = 0 .$$

Since $\Theta = 0, 4$, a condition for stability is

$$(32.7) \qquad d_3 = 0 .$$

Further, set

$$u = P + z^4 v ,$$

$$P = \alpha + \beta z + \gamma z^2 + \delta z^3$$

where α is an arbitrary parameter, and determine A_1, A_2, A_3, A_4 by setting

$$z\ddot{P} - 3\dot{P} - (3c + d_2) + d_1 z - d_0 z^2 - 3czP + P^2 - z(3P\dot{P} - P^3)$$

$$= A_1 + A_2 z + A_3 z^2 + A_4 z^3 + O(z^4) .$$

In agreement with the general theory, β, γ, δ are given by

$$(32.8) \qquad A_1 = 0, \quad A_2 = 0, \quad A_3 = 0$$

and the condition for stability by

$$(32.9) \qquad A_4 = 0 .$$

The equations (32.8) yield respectively

$$(32.10) \qquad \begin{cases} 3\beta = \alpha^2 - 3c - d_2 , \\ 4\gamma = \alpha d_2 + d_1 - \dot{\beta} , \\ 3\delta = d_2(\beta - \alpha^2) - d_1\alpha - d_0 + \ddot{\beta} + \dot{\gamma} . \end{cases}$$

Finally, $A_4 =$ gives

$$(32.11) \qquad \ddot{\gamma} + 3\dot{\delta} + \alpha\dot{\gamma} - \alpha\delta - 5\beta\gamma - 3c\gamma + \alpha^2\gamma = 0$$

which is a polynomial in α of degree 3. Since α is arbitrary, the coefficients of this polynomial are identically zero. The coefficient of α^3 is $\frac{1}{18}\, d_2$: therefore

$$d_2 = 0 . \tag{32.12}$$

With this simplification, one has

$$\left\{\begin{aligned} 3\beta &= \alpha^2 - 3c , \\ 4\gamma &= d_1 + \dot c , \\ 3\delta &= -d_1\alpha - d_0 - \ddot c + \frac{1}{4}\,(\dot d_1 + \ddot c) \end{aligned}\right. \tag{32.13}$$

and (32.11) may be rewritten as

$$\frac{1}{6}\,(d_1 - \dot c)\,\alpha^2 + \frac{1}{6}\,(2d_0 + 3\ddot c - 5\dot d_1)\, x + \frac{1}{2}\,(\ddot d_1 - \dddot c) - \dot d_0 + \frac{1}{2}\, c(d_1 + \dot c) = 0$$

from which it follows that

$$d_1 = \dot c, \quad d_0 = \ddot c, \quad \dot d_0 = c\dot c = \dddot c . \tag{32.14}$$

The stable equations of this class are thus of the form

$$\dddot y = y\ddot y + 5\dot y^2 - y^2\dot y + 3c\dot y + \dot c y + \ddot c \tag{32.15}$$

where c is given by

$$\dddot c = c\dot c . \tag{32.16}$$

It is readily seen that c satisfies

$$\dot c^2 = \frac{1}{3}\, c^3 + K_2 c + K_3 \tag{32.17}$$

and is an elliptic function of x.

Moreover,

$$\left\{\begin{aligned} &(\ddot y - y\dot y - y^3 + cy + \dot c)^2 = \frac{8}{3}\left(\dot y - y^2\right)^2\left(\dot y + \frac{y^2}{2} + \frac{3c}{2}\right) \\ &+ 4(\dot y - y^2)\,(2cy^2 + \dot c y + \ddot c) + 4c^2y^2 + 4c\dot c y + 2\dot c^2 + K \end{aligned}\right. \tag{32.16}$$

is an integral of (32.15) [see Chazy [2, *a*]].

IX. Equations of class VII.

33. - The stable equations of this class are of the form

$$\dddot{y} = ay\ddot{y} + 2a\dot{y}^2 + 2a^2y^2\dot{y} + F(x, y). \tag{33.1}$$

The solutions of this equation have two sets of simple parametric poles determined by

$$\begin{cases} as_1 = -1, & \Theta = 1, 3; \\ as_2 = 3, & \Theta = 3, 5. \end{cases} \tag{33.2}$$

The associated equation in u is

$$\begin{cases} z^2\ddot{u} - (4 + as)z\dot{u} + (12 + 7as - 2a^2s^2)u + 2a_1 - c_1s + d_3s^2 \\ + (d_2s - c_0)z + d_1z^2 + \dfrac{d_0}{s}z^3 + (3a_1 - c_1s)uz - c_0z^2u \\ + (7 + 3as)z\dot{u}^2 - a_1z^2(\dot{u} - u^2) - z^2(3u\dot{u} - u^3) = 0. \end{cases} \tag{33.3}$$

By a suitable transformation T, one may assume

$$a = 1, \tag{33.4}$$

$$2a_1 + c_1 + d_3 = 0, \tag{33.5}$$

$$2a_1 - 3c_1 + 9d_3 = 0 \tag{33.6}$$

or

$$a = 1, \quad 4a_1 + 3c_1 = 0, \quad 2a_1 + 3d_3 = 0. \tag{33.7}$$

One has only to determine λ, μ, φ by setting

$$\Lambda - \Phi + \frac{\dot{a}}{a} = 0,$$

$$6\Lambda - 9\Phi + 16a\mu + 4a_1 + \frac{3c_1}{a} = 0,$$

$$6\Phi - 2a\mu - 2a_1 - \frac{3d_3}{a^2} = 0.$$

Now assume $a = 1$ and $s = -1$; then (33.3) may be rewritten as

$$z^2\ddot{u} - 3z\dot{u} + 3u - (c_0 + d_2)z + d_1 z^2 - d_0 z^3 + (3a_1 + c_1)zu - c_0 z^2 u - a_1 z^2(\dot{u} - u^2)$$
$$+ 4zu^2 - z^2(3u\dot{u} - u^3) = 0\,.$$

Set

$$u = P + z^3 v\,,$$
$$P = \alpha + \beta z + \gamma z^2\,,$$

and determine A_0, A_1, A_2, A_3 by setting

$$z^2\ddot{P} - 3z\dot{P} + 3P - (c_0 + d_2)z + d_1 z^2 - d_0 z^3$$
$$+ (3a_1 + c_1)zP - c_0 z^2 P - a_1 z^2(\dot{P} - P^2) + 4zP^2 - z^2(3P\dot{P} - P^3)$$
$$= A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4)\,.$$

Because of (33.5), $A_0 = 0$ gives $\alpha = 0$; since $\Theta = 1$, β is an arbitrary parameter and a condition for stability follows from $A_1 = 0$, namely

(33.8) $$c_0 + d_2 = 0\,.$$

Then $A_2 = 0$ yields

(33.9) $$\gamma = (2a_1 + c_1)\beta + d_1$$

and $A_3 = 0$ in turn results in

(33.10) $$\dot{\gamma} + (a_1 + c_1)\gamma - c_0\beta - d_0 = 0\,.$$

From (33.9–10) and since β is arbitrary, one obtains two additional conditions for stability, namely

(33.11) $$2\dot{a}_1 + \dot{c}_1 + (a_1 + c_1)(2a_1 + c_1) = c_0\,,$$

(33.12) $$\dot{d}_1 + (a_1 + c_1)d_1 = d_0\,.$$

34. - We proceed further by considering the second set of simple parametric poles, namely $s_2 = 3$, $\Theta = 3, 5$.

For convenience, we write $4c$ and $12d$ instead of c_1 and d_1 respectively; then according to the preceding section

$$\left\{\begin{aligned} &a_1 = -3c\,, \quad d_3 = 2c\,, \quad c_0 + d_2 = 0\,, \\ &c_0 = -2(\dot{c} + c^2)\,, \quad d_0 = 12(\dot{d} + cd) \end{aligned}\right. \tag{34.1}$$

so that equation (33.3) may be rewritten as follows

$$z^2\ddot{u} - 7z\dot{u} + 15u + 8(\dot{c} + c^2)z + 12dz^2 + 4(\dot{d} + cd)z^3 - 21cuz$$
$$+ 2(\dot{c} + c^2)z^2u + 16zu^2 + 3cz^2(\dot{u} - u^2) - z^2(3u\dot{u} - u^3) = 0\,.$$

Because $\Theta = 3, 5$, one sets

$$u = P + z^2 v\,,$$
$$P = \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4$$

[obviously $\alpha = 0$] and determine A_1, A_2, A_3, A_4, A_5 by

$$z^2\ddot{P} - 7z\dot{P} + 15P + 8(\dot{c} + c^2)z + 12dz^2 + 4(\dot{d} + cd)z^3 - 21czP$$
$$+ 2(\dot{c} + c^2)z^2P + 16zP^2 + 3cz^2(\dot{P} - P^2) - z^2(3P\dot{P} - P^3)$$
$$= A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + A_5 z^5 + O(z^6)\,.$$

The coefficients β, γ, δ, ε of P are given by

$$A_1 = 0\,, \qquad A_2 = 0\,, \qquad A_3 = 0\,, \qquad A_4 = 0\,;$$

the condition for stability is $A_5 = 0$.

Since $\Theta = 3$, δ remains arbitrary. Then, $A_1 = 0$, $A_2 = 0$ yield respectively

$$\beta = -(\dot{c} + c^2)\,, \tag{34.2}$$

$$3\gamma = 5\dot{\beta} + 18c\beta - 12d\,. \tag{34.3}$$

Further, $A_3 = 0$ gives

$$\ddot{\beta} - 3\dot{\gamma} + 8\beta^2 + 4(\dot{d} + cd) - 15c\gamma + 2(\dot{c} + c^2)\beta + 3c\dot{\beta} = 0$$

or by virtue of (34.4),

$$(34.4) \qquad \ddot{\beta} + 6(\dot{c} + 4c^2)\beta - 4(\dot{d} + 4cd) + 10c\dot{\beta} = 0\,.$$

From $A_4 = 0$, it follows

$$(34.5) \qquad \varepsilon = \ddot{\gamma} + 11\beta\gamma - 12c\delta + 3c\dot{\gamma}\,.$$

The condition for stability $A_5 = 0$ yields

$$(34.6) \qquad \dot{\varepsilon} + 2\beta\dot{\gamma} + \dot{\beta}\gamma + 12\beta\delta + 6\gamma^2 - 9c\varepsilon + 3c\beta\gamma = 0\,,$$

Because δ is arbitrary, the coefficient of δ in (34.6) is zero; therefore

$$(34.7) \qquad \dot{c} = 4c^2\,.$$

By virtue of this relation, one obtains

$$(34.8) \qquad \beta = -5c^2\,, \qquad \dot{\beta} = -40c^3\,, \qquad \ddot{\beta} = -40.12c^4\,,$$

$$(34.9) \qquad 3\gamma = -290c^3 - 12d$$

and from (34.4)

$$(34.10) \qquad \dot{d} + 4cd + 280c^4 = 0\,.$$

Then by using (34.7) and (34.10), one has

$$\dot{\gamma} = 16cd - 40c^4\,,$$

$$\ddot{\gamma} = -16 \cdot 320\,c^5\,,$$

$$\varepsilon = -12c\delta + 4 \cdot 67c^2d + \frac{230}{3}\,c^5\,,$$

$$\dot{\varepsilon} = -12c\dot{\delta} + 4^2 \cdot 67c^3d - \frac{40}{3} \cdot 5513\,c^6\,.$$

Substitution in (34.6) gives

$$12d + 420c^3d - \frac{5}{3} \cdot 931c^6 = 0. \tag{34.11}$$

By differentiating (34.11) and because of (34.10), one obtains

$$12\,d^2 + 420\,c^3d + 19355\,c^6 = 0, \tag{34.12}$$

Comparison of (34.11-12) shows that $c = 0$, $d = 0$.

Therefore, the stable equation of this class is

$$\dddot{y} = y\ddot{y} + 2\dot{y}^2 + 2y^2\dot{y}.$$

X. **Equations of class** VIII.

35 - The stable equations of this class are of the form

$$\dddot{y} = cy^2\dot{y} + F(x, y). \tag{35.1}$$

The solutions of this equation have two sets of simple parametric poles determined by $cs^2 = 6$, $\Theta = 2, 3$.

The associated equation in u is

$$\left\{\begin{aligned} & z^2\ddot{u} - 4z\dot{u} + 6u + 2a_1 - c_1s + d_3s^2 \\ & - z(c_0 - d_2s) + d_1z^2 + \frac{d_0z^3}{s} + uz(3a_1 - c_1s) - c_0z^2u \\ & + 7u^2z + a_1z^2u^2 - a_1z^2\dot{u} - z^2(3u\dot{u} - u^3) = 0. \end{aligned}\right. \tag{35.2}$$

By a suitable transformation T, one may assume

$$c = 6 \tag{35.3}$$

and therefore $s = \pm 1$ and $2\ a_1 - c_1s + d_3s^2 = 0$ or

$$c_1 = 0, \qquad 2a_1 + d_3 = 0. \tag{35.4}$$

One has only to determine λ, μ, φ by setting

$$2\Lambda - 2\Phi + \frac{\dot{c}}{c} = 0\,,$$

$$-3\Phi + a_1 + \frac{3d_3}{c} = 0\,,$$

$$3c\mu + c_1 = 0\,.$$

With these simplifications, equation (35.2) becomes

$$z^2\ddot{u} - 4z\dot{u} + 6u - z(c_0 - d_2 s) + d_1 z^2 + d_0 s z^3$$

$$+ 3a_1 u z - c_0 z^2 u + 7u^2 z + a_1 z^2 u^2 - a_1 z^2 \dot{u} - z^2(3u\dot{u} - u^3) = 0\,.$$

Set

$$u = P + z^3 v\,,$$

$$P = \alpha + \beta z + \gamma z^2$$

and determine A_0, A_1, A_2, A_3 by setting

$$z^2\ddot{P} - 4z\dot{P} + 6P - z(c_0 - d_2 s) + d_1 z^2 + d_0 s z^3$$

$$+ 3a_1 z P - c_0 z^2 P + 7zP^2 + a_1 z^2 P - a_1 z^2 \dot{P} - 3z^2 P\dot{P} + z^2 P^3$$

$$= A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4)\,.$$

Obviously $\alpha = 0$ and γ is arbitrary. Then $A_1 = 0$, $A_2 = 0$ yield respectively

(35.5)
$$\begin{cases} 2\beta = c_0 - d_2 s\,, \\ 2\dot{\beta} = d_1 + 2a_1\beta \end{cases}$$

so that, since $s = \pm 1$,

(35.6)
$$\begin{cases} \dot{d}_2 = a_1 d_2\,, \\ d_1 = \dot{c}_0 - a_1 c_0. \end{cases}$$

The condition for stability, $A_3 = 0$, is then

$$\ddot{\beta} + 2\beta^2 + d_0 s + a_1\gamma - c_0\beta - a_1\dot{\beta} = 0. \tag{35.7}$$

Because γ is arbitrary, this yields

$$a_1 = 0$$

and consequently [see (35.6)].

$$\dot{d}_2 = 0 \quad \text{or} \quad d_2 = 2K,$$

$$d_1 = \dot{c}_0.$$

From (35.7), one obtains in the usual way

$$\ddot{c}_0 + d_2^2 = 0,$$

$$2d_0 = c_0 d_2.$$

Therefore,

$$d_1 = \dot{c}_0 = -4K^2x + K_1,$$

$$c_0 = -2K^2x^2 + K_1x + K_2,$$

$$d_0 = K(-2K^2x^2 + K_1x + K_2).$$

Now we have to consider several cases according to the values of K, K_1, K_2.

i. Suppose $K = 0$; then

$$c_0 = K_1x + K_2, \qquad d_2 = d_0 = 0.$$

a. When $K_1 = 0$, the stable equation is

$$\dddot{y} = 6y^2\dot{y} + K_2\dot{y}. \tag{35.8}$$

b. When $K_1 \neq 0$, the stable equation is

$$\dddot{y} = 6y^2\dot{y} + (K_1x + K_2)\dot{y} + K_1y. \tag{35.9}$$

ii. Suppose $K \neq 0$; the stable equation of this class is

$$\dddot{y} = 6y^2\dot{y} + (-2K^2x^2 + K_1x + K_2)\dot{y} + 2Ky^2 + (-4K^2x + K_1)y + K(-2K^2x^2 + K_1x + K_2)\,. \tag{35.10}$$

On using a linear transformation, one may suppose $K_1 = 0$.

Then by $x \to \alpha x$, $y \to \beta y$ where $\alpha\beta = -1$, $K\alpha^2 = 1$ and $K_2\alpha^2 = -4h$ where h is a constant (arbitrary), one obtains

$$\dddot{y} = 6y^2\dot{y} - (2x^2 + 4h)\dot{y} - 2y^2 - 4xy + 2x^2 + 4h\,. \tag{35.11}$$

Substitution of $y = u + x$ brings (35.11) to

$$\dddot{u} = 6u^2\dot{u} + 12u\dot{u}x + 4(x^2 + h)\dot{u} + 4u^2 + 4xu\,.$$

This equation is obtained by differentiating with respect to x, the equation

$$2u\ddot{u} = \dot{u}^2 + 3u^4 + 8xu^3 + 4(x^2 + h)u^2 + 2K \tag{35.12}$$

in order to eliminate the second constant K.

Equation (35.12) is a stable equation of the second order [see Part I, table I, eq. 4]; therefore, $y(x)$ gives by (35.11) is also stable.

PART III

Equations of order four.

1. - This paper is the third part of a group of studies concerning differential equations with fixed critical points. We shall again use the method, notations and terminology employed in Parts I and II.

We write the equations concerned in the form

$$y^{iv} = ay\dddot{y} + b\dot{y}\ddot{y} + cy^2\ddot{y} + dy\dot{y}^2 + ey^3\dot{y} + fy^5 + F(x, y)\,, \tag{1.1}$$

where

$$F(x, y) = a_0\dddot{y} + (c_1y + c_0)\ddot{y} + d_0\dot{y}^2 + (e_2y^2 + e_1y + e_0)\dot{y} + f_4y^4 + f_3y^3 + f_2y^2 + f_1y + f_0\,; \tag{1.2}$$

a, b, c, d, e. f, with or without subscripts, are analytic functions of x in a certain domain D.

The reduced equation corresponding to (1.1) is easily determined by setting $x = x_0 + \varepsilon t$, where x_0 is a point in D and $\varepsilon \neq 0$ a parameter; one finds $y^{iv} = 0$.

The only value of y for which Cauchy's general existence theorem does not apply to equation (1.1) is $y = \infty$. To determine necessary conditions for the absence of parametric critical points for equation (1.1), suppose that in a neighborhood of $x = x_0$, $y(x)$ takes the form

$$y(x) = \frac{s(x)}{(x - x_0)^r}, \tag{1.3}$$

where $r > 0$ and $s(x_0) \neq 0$; $s(x)$ is a holomorphic function of x.

Substitute $y(x)$ given by (1.3) into (1.1) and note that

$$\dot{y}(x) = -\frac{rs(x_0)}{(x - x_0)^{r+1}}[1 + O(x - x_0)],$$

$$\ddot{y}(x) = \frac{r(r+1)s(x_0)}{(x - x_0)^{r+2}}[1 + O(x - x_0)],$$

$$\dddot{y}(x) = -\frac{r(r+1)(r+2)s(x_0)}{(x - x_0)^{r+3}}[1 + O(x - x_0)],$$

$$y^{iv}(x) = \frac{r(r+1)(r+2)(r+3)s(x_0)}{(x - x_0)^{r+4}}[1 + O(\bar{x} - x_0)].$$

i. First, suppose that at least one of $a(x_0)$, $b(x_0)$, $c(x_0)$, $d(x_0)$, $e(x_0)$, $f(x_0)$ is not zero; then the dominant terms arise from y^{iv}, $y\dddot{y}$, $\dot{y}\ddot{y}$, $y^2\ddot{y}$, $y\dot{y}^2$, $y^3\dot{y}$, y^5.

It is easily seen that $r = 1$ and that $s(x_0)$ must satisfy the equation

$$fs^4 - es^3 + (2c + d)s^2 - 2(3a + b)s - 24 = 0; \tag{1.4}$$

thus $y(x)$ has at most four sets of parametric poles.

In addition, every solution of the equation

$$y^{iv} = ay\dddot{y} + b\dot{y}\ddot{y} + cy^2\ddot{y} + dy\dot{y}^2 + ey^3\dot{y} + fy^5, \tag{1.5}$$

where a, b, c, d, e, f are constant, must be single-valued.

These conditions restrict the possible values of a, b, c, d, e, f, as will be seen in the following sections.

ii. Second, suppose $f(x_0) = e(x_0) = 0$, $2c(x_0) + d(x_0) = 0$, $3a(x_0) + b(x_0) = 0$. Substitution of (1.3) into (1.1) shows that every solution of the equation

$$y^{iv} = a(y\dddot{y} - 3\dot{y}\ddot{y}) + c(y^2\ddot{y} - 2y\dot{y}^2)$$

must be single-valued. However this is known to be impossible except when $a = c = 0$ [see CHAZY].

iii. Third, suppose $a = b = c = d = e = f = 0$. The dominant terms arise from y^{iv}, $y\ddot{y}$, $\dot{y}^2$, y^3. Then $r = 2$ and the corresponding equation is

$$y^{iv} = c_1 y\ddot{y} + d_0\dot{y}^2 + f_3 y^3 ; \tag{1.6}$$

c_1, d_0, f_3 are constant and every solution of (1.6) must be single-valued. Moreover, $s(x_0)$ must satisfy the equation

$$f_3 s^2 + 2(3c_1 + 2d_0)s - 120 = 0 . \tag{1.7}$$

iv. Fourth, suppose $c_1 = d_0 = f_3 = 0$. The dominant terms arise from y^{iv}, $y\dot{y}$. Then $r = 3$ and the corresponding equation is

$$y^{iv} = e_1 y\dot{y} , \tag{1.8}$$

where e_1 is a constant.

The integral of (1.8) satisfies

$$\dddot{y} = \frac{e_1}{2}y^2 + K$$

and is a multiple-valued function except when $e_1 = 0$. [see Part II, eq. (5.12)].

v. Fifth, suppose $e_1 = 0$. The dominant terms arise from y^{iv} and y^2 so that $r = 4$. The corresponding equation is

$$y^{iv} = f_2 y^2 , \tag{1.9}$$

where f_2 is a constant.

This equation is not stable except when $f_2 = 0$. To prove this, assume $f_2 = 4 \cdot 5 \cdot 6 \cdot 7 = 840$ (by replacing y by αy). The equation

$$y^{iv} = 840\, y^2$$

is satisfied by $y = x^{-4}$. Then set $y = x^{-4} + \varepsilon z$; $z(x)$ is determined by

$$z^{iv} = 840 \left(\frac{2}{x^4}\, z + \varepsilon z^2 \right) .$$

According to the general theorem of stability, $z(x\,;\,\varepsilon)$ given by

$$z(x\,;\,\varepsilon) = z_0(x) + \varepsilon z_1(x) + \ldots$$

must be single-valued together with $z_0(x)$, $z_1(x)$, ...

In particular, $z_0(x)$ is determined by

$$z_0^{iv} - 1680\, \frac{z_0}{x^4} = 0\,;$$

the related indicial equation is

$$\Theta(\Theta - 1)\,(\Theta - 2)\,(\Theta - 3) - 1680 = 0$$

or

$$(\Theta + 5)\,(\Theta - 8)\,(\Theta^2 - 3\Theta + 42) = 0$$

and has two complex roots. Therefore, $z_0(x)$ and also $z(x)$ are multiple-valued functions of x.

2. - In the following sections, we consider the cases $r = 1$ [eq. (1.5)] and $r = 2$ [eq. (1.6)].

When $r = 2$, set

$$y = s z^{-2}, \qquad \dot{z} = 1 + u z\,, \tag{2.1}$$

where s is a constant and note that

$$z^2 \dot{y} = -\, 2s \left(\frac{1}{z} + u \right),$$

$$z^3 \ddot{y} = 2s \left(\frac{3}{z} - z\dot{u} + 5u + 2zu^2 \right),$$

$$z^4 \dddot{y} = -2s\left(\frac{12}{z} + z^2\ddot{u} - 7z\dot{u} + 27u - 6z^2u\dot{u} + 19zu^2 + 4z^2u^3\right),$$

$$z^5 y^{iv} = 2s\left[\frac{60}{z} + 168u + z(-48\dot{u} + 165u^2)\right.$$

$$+ z^2(9\ddot{u} - 71u\dot{u} + 65u^3)$$

$$\left. + z^3(-\dddot{u} + 6\dot{u}^2 + 8u\ddot{u} - 24u^2\dot{u} + 8u^4)\right].$$

Substitute $y(x)$ given by (2.1) into (1.6) i.e.

$$y^{iv} = cy\ddot{y} + d\dot{y}^2 + fy^3 \tag{2.2}$$

[we omit the subscripts]; then, $s(x)$ is determined by

$$fs^2 + 2(3c + 2d)s - 120 = 0. \tag{2.3}$$

Moreover the roots of the indicial equation corresponding to

$$z^3\dddot{u} - 9z^2\ddot{u} + (48 - cs)z\dot{u} - (168 - 5cs - 4ds)u = 0,$$

i.e.,

$$\Theta(\Theta - 1)(\Theta - 2) - 9\Theta(\Theta - 1) + (48 - cs)\Theta - 168 + 5cs + 4ds = 0 \tag{2.4}$$

must be integers.

Set $\Theta = \chi - 1$; then (2.4) becomes

$$\chi^3 - 15\chi 2 + (86 - cs)\chi - 240 + 2s(3c + 2d) = 0. \tag{2.5}$$

Now, we have to consider two cases according as f is or is not zero.

i. $f = 0$. Equation (2.3) is

$$2(3c + 2d)s = 120 \tag{2.6}$$

and the related stable differential equations have only one set of double parametric poles.

On taking (2.6) into account, equation (2.5) may be rewritten as

$$\chi^3 - 15\chi^2 + (86 - cs)\chi - 120 = 0. \tag{2.7}$$

Because the roots χ_1, χ_2, χ_3 of this equation must be integers, our problem is now: to determine all the integral solutions of

$$\chi_1 \chi_2 \chi_3 = 120, \qquad \chi_1 + \chi_2 + \chi_3 = 15.$$

With these values of χ_1, χ_2, χ_3, one finds

$$cs = 86 - (\chi_1 \chi_2 + \chi_2 \chi_3 + \chi_3 \chi_1).$$

The only solutions of our problem are:

a) $$\chi_1 = 4, \ \chi_2 = 5, \ \chi_3 = 6;$$

$$cs = 12, \qquad ds = 12, \qquad c = d;$$

the desired equation is

$$y^{iv} = c(y\ddot{y} + \dot{y}^2). \tag{2.8}$$

By a transformation $y \to \alpha y$, one may assume $c = 12$ so that (2.8) becomes

$$y^{iv} = 12\,(y\ddot{y} + \dot{y}^2).$$

To prove that this equation is stable, integrate with respect to x and find

$$\dddot{y} = 12 y\dot{y} + K,$$

$$\ddot{y} = 6y^2 + Kx + K_1$$

i.e. a stable equation. [see Part I, Table I, eq. 1].

b) $$\chi_1 = -2, \quad \chi_2 = -3, \quad \chi_3 = 20;$$

$$cs = 180, \qquad ds = -480, \qquad 8c + 3d = 0;$$

the desired equation is

$$y^{iv} = \frac{c}{3}\,(3y\ddot{y} - 8\dot{y}^2). \tag{2.9}$$

We shall not consider this equation in this paper.

ii. $f \neq 0$. Let s_1 and s_2 be two solutions of equation (2.3); then

$$s_1 + s_2 = -\frac{2}{f}(3c + 2d), \tag{2.10}$$

$$s_1 s_2 = -\frac{120}{f}, \tag{2.11}$$

$$240 - 2(3c + 2d)s_1 = 120\left(1 - \frac{s_1}{s_2}\right) \tag{2.12}$$

so that equation (2.5) becomes

$$\chi^3 - 15\chi^2 + (86 - cs_1)\chi - 120\left(1 - \frac{s_1}{s_2}\right) = 0. \tag{2.13}$$

The roots of this equation must be integers and similarly for

$$\lambda^3 - 15\lambda^2 + (86 - cs_2)\lambda - 120\left(1 - \frac{s_2}{s_1}\right) = 0. \tag{2.14}$$

Now set

$$120\left(1 - \frac{s_1}{s_2}\right) = p, \qquad 120\left(1 - \frac{s_2}{s_1}\right) = q$$

so that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{120}. \tag{2.15}$$

Our next problem is thus: to find the integral solutions of (2.15). This problem has a number of solutions; we leave it for another occasion.

3. - Suppose $r = 1$ and set

$$y = sz^{-1}, \qquad \dot{z} = 1 + zu. \tag{3.1}$$

Substitute $y(x)$ given by (3.1) into the reduced equation

$$y^{iv} = ay\dddot{y} + b\dot{y}\ddot{y} + cy^2\ddot{y} + dy\dot{y}^2 + ey^3\dot{y} + fy^5 \tag{3.2}$$

and determine $s(x)$ by

$$fs^4 - es^3 + (2c + d)s^2 - 2(3a + b)s - 24 = 0. \tag{3.3}$$

Moreover the roots of the indicial equation corresponding to

$$z^3\dddot{u}-(5+as)z^2\ddot{u}+[20+(4a+b)s-cs^2]\,z\dot{u}-[60+(12a+5b)s-(3c+2d)s^2+es^3]u=0\,,$$

i.e.,

$$\text{(3.4)}\quad \begin{aligned}&\Theta(\Theta-1)(\Theta-2)-(5+as)\Theta(\Theta-1)+[20+(4a+b)s-cs^2]\Theta\\ &\qquad -[60+(12a+5b)s-(3c+2d)s^2+es^3]=0\,,\end{aligned}$$

must be integers.

Set $\Theta=\chi-1$; then (3.4) becomes

$$\text{(3.5)}\quad \begin{aligned}&\chi^3-(11+as)\chi^2+[46+(7a+b)s-cs^2]\chi\\ &-[96+6(3a+b)s-2(2c+d)s^2+es^3]=0\,.\end{aligned}$$

We have to consider several cases according to the values of a, b, c, d, e, f.

i. Suppose that $f=e=c=d=0$ and that a and b are not both zero. Then s is given by

$$(3a+b)s+12=0\,;$$

moreover, the solutions χ_1, χ_2, χ_3 of

$$\chi^3-(11+as)\chi^2+(34+4as)\chi-24=0$$

must be integers.

One obtains two solutions, namely,

a. $$\chi_1=1\,,\qquad \chi_2=4\,,\qquad \chi_3=6\,;$$
$$a=0\,,\qquad bs=-12\,.$$

The corresponding equation

$$\text{(3.6)}\qquad y^{iv}=b\dot{y}\ddot{y}$$

is stable. To prove this, observe that a trivial transformation enables one to assume $b=-12$ and brings (3.6) to

$$\text{(3.7)}\qquad y^{iv}+12\dot{y}\ddot{y}=0\,.$$

Now integrate (3.7) with respect to x and find

$$\dddot{y} = -6\dot{y}^2 + K;$$

therefore $\dot{y} = z$ is a solution of $\ddot{z} = -6z^2 + K$ and is an elliptic function $\wp(x)$. Thus, $y(x)$ is stable.

Note that $\Theta = 0,\ 3,\ 5$.

b. $\chi_1 = 2, \quad \chi_2 = 3, \quad \chi_3 = 4;$

$as = -2, \quad bs = -6, \quad 3a = b.$

The correlated equation is

$$y^{iv} = a(y\dddot{y} + 3\dot{y}\ddot{y}). \tag{3.8}$$

To provo that this equation is stable, observe that a trivial transformation enables one to assume $a = -2$, $b = -6$ and brings (3.8) to

$$y^{iv} = -2y\dddot{y} - 6\dot{y}\ddot{y}. \tag{3.9}$$

This equation is equivalent to the differential system

$$z = \dot{y} + y^2, \qquad \dddot{z} = 0;$$

therefore, $y(z)$ is stable.

Note that $\Theta = 1,\ 2,\ 3$.

ii. Suppose that $f = e = 0$ and that c and d are not bothzero. Then s is determined by

$$(2c + d)s^2 - 2(3a + b)s - 24 = 0. \tag{3.10}$$

Moreover, the solutions χ_1, χ_2, χ_3 and $\lambda_1, \lambda_2, \lambda_3$ of

$$\chi^3 - (11 + as_1)\chi^2 + [46 + (7a + b)s_1 - cs_1^2]\chi - P(s_1) = 0, \tag{3.11}$$

$$\lambda^3 - (11 + as_2)\lambda^2 + [46 + (7a + b)s_2 - cs_2^2]\lambda - P(s_2) = 0, \tag{3.12}$$

respectively, must be integers; we have set

$$P(s) = 96 + 6(3a + b)s - 2(2c + d)s^2. \tag{3.13}$$

From (3.10), one obtains

$$2c + d = -\frac{24}{s_1 s_2}, \tag{3.14}$$

$$2(3a + b) = -\frac{24(s_1 + s_2)}{s_1 s_2} \tag{3.15}$$

and thus

$$\left\{ \begin{array}{l} p \equiv P(s_1) = 24\left(1 - \frac{s_1}{s_2}\right), \\ q \equiv P(s_2) = 24\left(1 - \frac{s_2}{s_1}\right). \end{array} \right. \tag{3.16}$$

Therefore,

$$24(p + q) = pq$$

or

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{24}. \tag{3.17}$$

Since χ_i and λ_i must be integers, p and q are also integers. Our next problem is thus: to determine all the integral solutions of (3.17).

The solutions of this problem are given in the following table together with the corresponding value of $\frac{s_1}{s_2}$ given by (3.16).

p	q	s_1/s_2	p	q	$-s_1/s_2$
6	– 8	3/4	25	24.25	1/24
8	– 12	2/3	26	24.13	1/12
12	– 24	1/2	27	24.9	1/8
15	– 40	3/8	28	24.7	1/6
16	– 48	1/3	30	24.5	1/4
18	– 3.24	1/4	32	24.4	1/3
20	– 5.24	1/6	33	8.11	3/8
21	– 7.24	1/8	36	72	1/2
22	– 11.24	1/12	40	60	2/3
23	– 23.24	1/24	42	84	3/4
			48	48	1

4. - Suppose that a solution (p, q) of (3.17) is known; then (3.16) gives the value of $\frac{s_1}{s_2}$.

Corresponding to these values of p and q, there are equations (3.11) and (3.12) whose solutions (χ_1, χ_2, χ_3) and $(\lambda_1, \lambda_2, \lambda_3)$ must be integers; therefore,

$$\chi_1 \chi_2 \chi_3 = p, \qquad \lambda_1 \lambda_2 \lambda_3 = q \tag{4.1}$$

and these relations give a finite number of possible values for (χ_1, χ_2, χ_3) and $(\lambda_1, \lambda_2, \lambda_3)$.

With these particular values of χ_i and λ_i, one has

$$\left\{ \begin{aligned} as_1 &= \Sigma \chi_i - 11, \\ as_2 &= \Sigma \lambda_i - 11 \end{aligned} \right. \tag{4.2}$$

and consequently a value of s_1/s_2. The desired solutions are those which correspond to the value of s_1/s_2 previously given by (3.16), [see the preceding table].

Further, bs_1 and bs_2 are determined by (3.15); then, cs_1^2 and cs_2^2 are given by [cf. (3.11-12)]

$$cs_1^2 = 46 + (7a + b)s_1 - \Sigma \chi_i \chi_j, \tag{4.3}$$

$$cs_2^2 = 46 + (7a + b)s_2 - \Sigma \lambda_i \lambda_j. \tag{4.4}$$

Finally, ds_1^2 and ds_2^2 are determined by (3.14).

Except for $(p, q) = (12, -24)$ and $(20, -5.24)$, the values obtained for s_1/s_2 are not consistent with those given by (3.16).

To bring out the chief features of the method, we consider the case $(p, q) = (12, -24)$.

From (3.16), one has $s_1/s_2 = 1/2$. The integral solutions of (4.1) are

$$\left\{ \begin{array}{llll} \chi_1 = 1, & \chi_2 = 3, & \chi_3 = 4, & as_1 = -3, \\ \lambda_1 = -2, & \lambda_2 = 3, & \lambda_3 = 4, & as_2 = -6; \end{array} \right. \tag{4.5}$$

$$\left\{ \begin{array}{llll} \chi_1 = -1, & \chi_2 = -3, & \chi_3 = 4, & as_1 = -11, \\ \lambda_1 = -1, & \lambda_2 = -4, & \lambda_3 = -6, & as_2 = -22; \end{array} \right. \tag{4.6}$$

(4.7) $$\begin{cases} \chi_1 = -1, & \chi_2 = 3, \quad \chi_3 = -4, \quad as_1 = -13, \\ \lambda_1 = -1, & \lambda_2 = -2, \quad \lambda_3 = -12, \quad as_2 = -26. \end{cases}$$

Now we consider (4.5). From (3.15), one has

(4.8) $$bs_1 = -9, \qquad bs_2 = -18.$$

Then (4.3 —4) yield

(4.9) $$cs_1^2 = -3, \qquad cs_2^2 = -12$$

and finally [cf. (3.14)]

(4.10) $$ds_1^2 = -6, \qquad ds_2^2 = -24.$$

From (4.5, 4.8 —9), one concludes that

(4.11) $$b = 3a, \qquad 3c = -a^2, \qquad 3d = -2a^2.$$

The desired equation is thus

$$y^{iv} = ay\dddot{y} + 3a\dot{y}\ddot{y} - \frac{a^2}{3}y^2\ddot{y} - \frac{2}{3}a^2y\dot{y}^2$$

or on writing $3a$ instead of a,

(4.12) $$y^{iv} = 3ay\dddot{y} + 9a\dot{y}\ddot{y} - 3a^2y^2\ddot{y} - 6a^2y\dot{y}^2;$$

for (4.12), the two sets of simple parametric poles are given by

(4.13) $$\begin{cases} as_1 = -1, & \Theta = 0,\ 2,\ 3; \\ as_2 = -2, & \Theta = -3,\ 2,\ 3. \end{cases}$$

For (4,6), one obtains successively

$$bs_1 = 15, \qquad bs_2 = 30;$$

$$cs_1^2 = -3, \qquad cs_2^2 = -112$$

which is not consistent with $2s_1 = s_2$ Thus (4.6) does not furnish a solution for our problem.

The same inconsistency arises for (4.7).

It remains to verify that equation (4.12) is stable. By a suitable transformation, one may assume $a = 1$ which brings (4.12) to

$$y^{iv} = 3y\dddot{y} + 9\dot{y}\ddot{y} - 3y^2\ddot{y} - 6y\dot{y}^2 . \tag{4.14}$$

Integration of (4.14) with respect to x yields

$$\dddot{y} = 3y\ddot{y} + 3\dot{y}^2 - 3y^2\dot{y} + K ,$$

where K is as usual an arbitrary constant. This is a stable equation of the third order and of class iv [see Part II, eq. (29.11)].

5. - For $p = 20$, $q = -5.24$, $6s_1 = s_2$, one obtains two possible solutions, namely

$$\text{(5.1)} \quad \begin{cases} \chi_1 = 1 , & \chi_2 = 4 , \quad \chi_3 = 5 , \quad as_1 = -1 , \\ \lambda_1 = 4 , & \lambda_2 = -5 , \quad \lambda_3 = 6 , \quad as_2 = -6 ; \end{cases}$$

$$\text{(5.2)} \quad \begin{cases} \chi_1 = -1 , & \chi_2 = -2 , \quad \chi_3 = 10 , \quad as_1 = -4 , \\ \lambda_1 = 1 , & \lambda_2 = 6 , \quad \lambda_3 = -20 , \quad as_2 = -24 . \end{cases}$$

For (5.2), one has

$$bs_1 = -2 , \qquad bs_2 = -12 ;$$

$$cs_1^2 = 44 , \qquad cs_2^2 = 0 ,$$

which are inconsistent with $6s_1 = s_2$.

For (5.1), one finds

$$bs_1 = -11 , \qquad bs_2 = -66 ;$$

$$cs_1^2 = -1 , \qquad cs_2^2 = -36 ; \qquad ds_1^2 = -2 .$$

Therefore,

$$b = 11a , \qquad c = -a^2 , \qquad d = -2a^2$$

and the desired equation is

$$y^{iv} = ay\dddot{y} + 11a\dot{y}\ddot{y} - a^2 y^2 \ddot{y} - 2a^2 y \dot{y}^2 . \tag{5.3}$$

However, this equation is not stable. In fact, by a suitable transformation, one may assume $a = 1$ which brings equation (5.3) to

$$y^{iv} = y\dddot{y} + 11\dot{y}\ddot{y} - y^2 \ddot{y} - 2y\dot{y}^2 . \tag{5.4}$$

Integration of (5.4) with respect to x yields

$$\dddot{y} = y\ddot{y} + 5\dot{y}^2 - y^2 \dot{y} + K .$$

This equation of the third order and class vi is not stable [see Part II; eq. (32.15)].

6. - Suppose that $f \neq 0$. Then s is determined by

$$fs^4 - es^3 + (2c + d)s^2 - 2(3a + b)s - 24 = 0 . \tag{6.1}$$

Moreover the solutions of

$$\chi^3 - (11 + as)\chi^2 + [46 + (7a + b)s - cs^2]\chi - P(s) = 0 , \tag{6.2}$$

where s is replaced by s_1, s_2, s_3, s_4 (the roots of (6.1)) must be integers; we have set

$$P(s) = 96 + 6(3a + b)s - 2(2c + d)s^2 + es^3 . \tag{6.3}$$

On using (6.1), one obtains with obvious notations

$$\left\{\begin{aligned} &\Sigma s = \frac{e}{f} , \qquad \Sigma ss = \frac{2c + d}{f} , \\ &\Sigma sss = \frac{2(3a + b)}{f} , \qquad s_1 s_2 s_3 s_4 = -\frac{24}{f} \end{aligned}\right. \tag{6.4}$$

A simple calculation shows that

$$P(s_1) = 24 \left(1 - \frac{s_1}{s_2}\right)\left(1 - \frac{s_1}{s_3}\right)\left(1 - \frac{s_1}{s_4}\right) \tag{6.5}$$

and similarly for $P(s_2)$, $P(s_3)$, $P(s_4)$.

It is left to the reader to verify the relation

$$\frac{1}{P(s_1)} + \frac{1}{P(s_2)} + \frac{1}{P(s_3)} + \frac{1}{P(s_4)} = \frac{1}{24}\,. \tag{6.6}$$

In order to proceed further with our problem, it becomes necessary to find the integer solutions of the DIOPHANTINE equation (6.6). We hope to have an opportunity to return to this problem on another occasion.

7. - We sum up the results obtained above in the following table where the relevant equations are listed together with the related values of s and Θ.

A. One set of double parametric poles.

Class I, [cf. (2.8)].

$$y^{iv} = c(y\ddot{y} + \dot{y}^2) + F(x,\ y)\,, \tag{7.1}$$

$$F(x,\ y) = a_0\dddot{y} + c_0\ddot{y} + e_0\dot{y} + f_2y^2 + f_1y + f_0\,; \tag{7.2}$$

$$cs = 12\,, \qquad \Theta = 3,\ 4,\ 5\,. \tag{7.3}$$

B. One set of simple parametric poles.

Class II, [cf. (3.6)].

$$y^{iv} = b\dot{y}\ddot{y} + F(x,\ y)\,, \tag{7.4}$$

$$\begin{aligned} F(x,\ y) = a_0\dddot{y} + (c_1y + c_0)\ddot{y} + d_0\dot{y}^2 + (e_2y^2 + e_1y + e_0)\dot{y} \\ + f_4y^4 + f_3y^3 + f_2y^2 + f_1y + f_0\,; \end{aligned} \tag{7.5}$$

$$bs = -12\,, \qquad \Theta = 0,\ 3,\ 5\,. \tag{7.6}$$

Class III, [cf. (3.8)]

$$y^{iv} = a(y\dddot{y} + 3\dot{y}\ddot{y}) + F(x,\ y)\,, \tag{7.7}$$

where $F(x,\ y)$ is given by (7.5);

$$as = -2\,, \qquad \Theta = 1,\ 2,\ 3\,. \tag{7.8}$$

C. Two sets of simple parametric poles.

Class IV, [cf. (4·12)].

$$y^{iv} = 3ay\dddot{y} + 9a\dot{y}\ddot{y} - 3a^2y^2\ddot{y} - 6a^2y\dot{y}^2 + F(x, y)\,, \tag{7.9}$$

where $F(x, y)$ is given by (7.5);

$$as_1 = -1\,, \qquad \Theta = 0\,, \quad 2,\ 3\,; \tag{7.10}$$

$$as_2 = -2\,, \qquad \Theta = -3,\ 2,\ 3\,. \tag{7.11}$$

8. - To obtain canonical forms for the stable equations, it is often most convenient to use a transformation $T\,(\lambda,\ \mu,\ \varphi)$, namely,

$$y(x) = \lambda(x)u + \mu(x)\,, \qquad t = \varphi(x) \tag{8.1}$$

which does not alter the main features of the equations concerned [$\lambda(x)$, $\mu(x)$, $\varphi(x)$ are analytic functions of x; see Part I, § 18; Part II, § 21].

We note for future use the following formulas where primes denote differentiations with respect to t, i.e. $u' = \dfrac{du}{dt}$, $u'' = \dfrac{d^2u}{dt^2}$, $u''' = \dfrac{d^3u}{dt^3}$, $u^{(4)} = \dfrac{d^4u}{dt^4}$:

$$\left\{\begin{aligned} \dot{y} &= \lambda\dot{\varphi}u' + \dot{\lambda}u + \dot{\mu}\,,\\ \ddot{y} &= \lambda\dot{\varphi}^2u'' + (2\Lambda + \Phi)\lambda\dot{\varphi}u' + \ddot{\lambda}u + \ddot{\mu}\,,\\ \dddot{y} &= \lambda\dot{\varphi}^3u''' + N\lambda\dot{\varphi}^2u'' + M\lambda\dot{\varphi}u' + \dddot{\lambda}u + \dddot{\mu}\,, \end{aligned}\right. \tag{8.2}$$

where

$$\left\{\begin{aligned} &M = 3\frac{\ddot{\lambda}}{\lambda} + 3\Lambda\Phi + \frac{\dddot{\varphi}}{\dot{\varphi}}\,, \qquad N = 3(\Lambda + \Phi)\,,\\ &\Lambda = \frac{\dot{\lambda}}{\lambda}\,, \qquad \varphi = \frac{\ddot{\varphi}}{\dot{\varphi}}\,; \end{aligned}\right. \tag{8.3}$$

$$\left\{\begin{aligned} y^{iv} &= \lambda\dot{\varphi}^4u^{(4)} + \lambda\dot{\varphi}^3(4\Lambda + 6\Phi)u''' + [M\lambda\dot{\varphi}^2 + (N\lambda\dot{\varphi}^2)^{\cdot}]u''\\ &\quad + \left[\frac{\dddot{\lambda}}{\lambda} + \dot{M} + M\Lambda + M\Phi\right]\lambda\dot{\varphi}u' + \lambda^{iv}u + \mu^{iv}\,. \end{aligned}\right. \tag{8.4}$$

For $F(x, y)$ given by (7.5), one obtains

$$F(x,\ y) = A_0u''' + (C_1u + C_0)u'' + D_0u'^2 + (E_2u^2 + E_1u + E_0)u' + F_4u^4 + F_3u^3 + F_2u^2 + F_1u + F_0\,, \tag{8.5}$$

where

$$A_0 = a_0\lambda\dot{\varphi}^3\,, \qquad C_1 = c_1\lambda^2\dot{\varphi}^2\,,$$

$$C_0 = [a_0N + c_1\mu + c_0]\lambda\dot{\varphi}^2\,, \qquad D_0 = d_0\lambda^2\dot{\varphi}^2\,,$$

$$E_2 = e_2\lambda^3\dot{\varphi}\,,$$

$$E_1 = [c_1(2\Lambda + \Phi) + 2d_0\Lambda + 2e_2\mu + e_1]\lambda^2\dot{\varphi}\,,$$

$$E_0 = [a_0M + c_1(2\Lambda + \Phi)\mu + c_2(2\Lambda + \Phi) + 2d_0\dot{\mu} + e_2\mu^2 + e_1\mu + e_0]\lambda\dot{\varphi}\,,$$

$$F_4 = f_4\lambda^4\,, \qquad F_3 = [e_2\Lambda + 4f_4\mu + f_3]\lambda^3\,,$$

$$F_2 = \left[c_1\frac{\ddot{\lambda}}{\lambda} + d_0\Lambda^2 + 2e_2\Lambda\mu + e_2\dot{\mu} + e_1\Lambda + 6f_4\mu^2 + 3f_3\mu + f_2\right]\lambda^2\,,$$

$$F_1 = \begin{bmatrix} a_0\dfrac{\dddot{\lambda}}{\lambda} + c_1\dfrac{\ddot{\lambda}}{\lambda}\mu + c_1\ddot{\mu} + c_0\dfrac{\ddot{\lambda}}{\lambda} + 2d_0\Lambda\dot{\mu} + e_2\Lambda\mu^2 \\ + 2e_2\mu\dot{\mu} + e_1(\dot{\mu} + \mu\Lambda) + e_0\Lambda + 4f_4\mu^3 + 3f_3\mu^2 + 2f_2\mu + f_1 \end{bmatrix}\lambda\,,$$

$$F_0 = F(x\,;\ \mu)\,.$$

We also note the following result. Suppose

$$y = \frac{s}{z}\,, \qquad \dot{z} = 1 + zu\,,$$

s a constant and $F(x, y)$ given by (7.5); one has

$$\begin{cases} \dfrac{z^4}{s}F(x,\ y) \equiv G(z\,;\ u) \equiv A + (B_1u + B_2)z \\ + [f_2s - e_0 + (3c_0 - e_1s)u + (4a_0 - c_1s)\dot{u} - B_3u^2]z^2 \\ + [f_1 - e_0u + c_0u^2 - c_0\dot{u} + a_0(3u\dot{u} - u^3 - \ddot{u})]z^3 + \dfrac{f_0}{s}z_4\,, \end{cases} \tag{8.6}$$

where

$$(8.7)\qquad \begin{cases} A = -6\ a_0 + 2c_1s + d_0s - e_2s^2 + f_4s^3\,, \\ B_1 = -12a_0 + 3c_1s + 2d_0s - e_2s^2\,, \\ B_2 = 2c_0 - e_1s + f_3s^2\,, \\ B_3 = 7a_0 - c_1s - d_0s\,. \end{cases}$$

Equations of class I.

9. - The stable equations of this class are of the form

$$(9.1)\qquad y^{iv} = a(y\ddot{y} + \dot{y}^2) + F(x,\ y)\,,$$

where

$$(9.2)\qquad F(x,\ y) = a_0\dddot{y} + c_0\ddot{y} + e_1y\dot{y} + e_0\dot{y} + f_2y^2 + f_1y + f_0\,.$$

Set

$$(9.3)\qquad y = \frac{s}{z^2}\,, \qquad \dot{z} = 1 + zu$$

and assume s to be a constant (this is obtained by a transformation T) [the related formulas are given art. 2, (2.1)].

Recall that $as = 12$, $\Theta = 3,\ 4,\ 5$.

On using a transformation T, we may assume

$$(9.4)\qquad a = 12\,, \qquad e_1 + 12a_0 = 0\,, \qquad 6c_0 + f_2 = 0\,.$$

To this effect, one has only to determine λ, μ, φ by

$$\Lambda - 2\Phi + \frac{\dot{a}}{a} = 0\,,$$

$$5\Phi - a_0 - \frac{e_1}{a} = 0\,,$$

$$a\mu + 2\left(\frac{\ddot{\lambda}}{\lambda} + \Lambda^2 + \frac{f_2}{a} + \frac{e_1}{a}\Lambda\right) - [M + \dot{N} + N\Lambda + 2N\Phi] + a_0N + c_0 = 0\,.$$

With these simplifications, equation (9.1) assumes the form

$$y^{iv} = 12(y\ddot{y} + \dot{y}^2) + a_0(\dddot{y} - 12y\dot{y}) + c_0(\ddot{y} - 6y^2) + e_0\dot{y} + f_1 y + f_0 . \tag{9.5}$$

Substitution of $y(x)$ given by (9.3) into (9.5) shows that this equation is equivalent to the differential system

$$\dot{z} = 1 + zu , \tag{9.6}$$

$$\left\{\begin{aligned} & z^3\dddot{u} - 9z^2\ddot{u} + 36z\dot{u} - 60u - 117zu^2 - 65z^2u^3 + 71z^2u\dot{u} \\ & - 6z^3\dot{u}^2 - 8z^3u\ddot{u} + 24z^3u^2\dot{u} - 8z^3u^4 \\ & + a_0[- 15zu + 7z^2\dot{u} - 19z^2u^2 - z^3\ddot{u} + 6z^3u\dot{u} - 4z^3u^3] \\ & + c_0[5z^2u - z^3\dot{u} + 2z^3u^2] - e_0z^2(1 + zu) + \frac{f_1}{2} z^3 + \frac{f_0}{2} z^5 = 0 . \end{aligned}\right. \tag{9.7}$$

For convenience, we denote by $E(u)$ the left hand member of (9.7).

Because $\Theta = 3, 4, 5$, one sets [see Part II, § 3],

$$u = P + z^5 v , \tag{9.8}$$

$$P = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 , \tag{9.9}$$

where δ and ε remain arbitrary and may be assumed to be constant parameters.

Substitute u given by (9.8) into (9.7); one obtains

$$z^5[z^3\dddot{v} + 6z^2\ddot{v} + 6z\dot{v}] + A_0 + A_1z + A_2z^2 + A_3z^3 + A_4z^4 + A_5z^5 + O(z^6) = 0 , \tag{9.10}$$

where the A_i' s are determined by

$$E(P) = A_0 + A_1z + A_2z^2 + A_3z^3 + A_4z^4 + A_5z^5 + O(z^6) . \tag{9.11}$$

We define α, β, γ, δ, ε by setting

$$A_i = 0 , \qquad (i = 0, 1, 2, 3, 4) .$$

Equation (9.10) becomes

$$z^3\dddot{v} + 6z^2\ddot{v} + 6z\dot{v} + A_5 + O(z) = 0 . \tag{9.12}$$

For equation (9.12) to be stable, it is necessary that

$$A_5 = 0 .$$

From $A_0 = A_1 = 0$, one immediately deduces $\alpha = \beta = 0$ which simplifies our problem.

Then $A_2 = 0$, $A_3 = 0$, $A_4 = 0$ yield, respectively,

$$6\gamma + e_0 = 0 , \tag{9.13}$$

$$12\dot{\gamma} - 6a_0\gamma + f_1 = 0 , \tag{9.14}$$

$$\ddot{\gamma} = a_0\dot{\gamma} + c_0\gamma \tag{9.15}$$

while $A_5 = 0$ gives

$$\dddot{\gamma} - 3\gamma^2 + a_0\varepsilon - a_0\ddot{\gamma} + c_0(2\delta - \dot{\gamma}) - e_0\gamma + \frac{f_0}{2} = 0 . \tag{9.16}$$

Because δ and ε are arbitrary, one obtains

$$a_0 = 0 , \qquad c_0 = 0 \tag{9.17}$$

and consequently

$$e_1 = 0 , \qquad f_2 = 0 . \tag{9.18}$$

Therefore, one determines γ, e_0, f_1, f_0 subject to the following conditions

$$\left\{ \begin{aligned} &\ddot{\gamma} = 0 , \\ &6\gamma + e_0 = 0 , \\ &f_1 + 12\dot{\gamma} = 0 , \\ &f_0 = e_0\gamma . \end{aligned} \right. \tag{9.19}$$

Accordingly,

$$\left\{ \begin{aligned} &\gamma = K_1 x + K_2 , \\ &e_0 = -6(K_1 x + K_2) , \qquad f_1 = -12 K_1 , \\ &f_0 = -6(K_1 x + K_2)^2 . \end{aligned} \right. \tag{9.20}$$

The stable equations of class I are thus of the form

$$y^{iv} = 12(y\ddot{y} + \dot{y}^2) - 6(K_1x + K_2)\dot{y} - 12K_1y - 6(K_1x + K_2)^2 . \tag{9.21}$$

We have to consider two cases according as K_1 is or is not zero.

i. $K_1 = 0$; equation (9.21) becomes

$$y^{iv} = 12(y\dot{y})^{\cdot} - 6K_2\dot{y} - 6K_2^2 \tag{9.22}$$

or on integrating with respect to x,

$$\dddot{y} = 12y\dot{y} - 6K_2y - 6K_2^2x + K_3 . \tag{9.23}$$

This equation (9.23), of the third order and of class I [Part II, eq. (22-23)], is stable; therefore equation (9.22) is also stable.

[On setting $y = \dot{z}$, equation (9.23) becomes

$$z^{iv} = 12\dot{z}\ddot{z} - 6K_2\dot{z} - 6K_2^2x + K_3 ; \tag{9.24}$$

see eq. (14.7) below].

ii. $K_1 \neq 0$. A transformation $x \to \alpha x$, $y \to \beta y$, where $\beta\alpha^2 = 1$, $K_1\alpha^4 = 1$ reduces equation (9.21) to the canonical form

$$y^{iv} = 12(y\dot{y})^{\cdot} - 6x\dot{y} - 12y - 6x^2 . \tag{9.25}$$

Now set $y = \dot{z}$; equation (9.25) can then be wvitten

$$z^{v} = 12(\dot{z}\ddot{z})^{\cdot} - 6(x\dot{z})^{\cdot} - 6\dot{z} - 6x^2 .$$

Integration with respect to x yields

$$z^{iv} = 12\dot{z}\ddot{z} - 6x\dot{z} - 6z - 2x^3 + K ,$$

an equation of class II [substitute x, y with αx, βy, $\alpha\beta = -1$, $\alpha^4 = 1$; see eq. (14.10) below].

Equations of class II.

10. - In the remainder of this paper, we consider equations whose solutions have simple parametric poles. We set

$$y = \frac{s}{z}, \qquad \dot{z} = 1 + uz, \tag{10.1}$$

where s is assumed to be a constant (this is obtained by a transformation T).

The stable equation of class II are of the form

$$y^{iv} = b\dot{y}\ddot{y} + F(x, y), \tag{10.2}$$

where $F(x, y)$ is given by (7.5).

The general solution of these equations has one set of simple parametric poles characterized by

$$bs + 12 = 0, \qquad \Theta = 0,\ 3,\ 5. \tag{10.3}$$

Substitute $y(x)$ given by (10.1) into (10.2). In order to simplify notations, we suppose at once $s = 1$ which can be obtained by a transformation T (see below); one has

$$\left\{\begin{aligned} & z^3\dddot{u} - 5z^2\ddot{u} + 8z\dot{u} + A + z[B_2 + B_1 u - 2u^2] \\ & + z^2[f_2 - e_0 + (3c_0 - e_1)u - B_3 u^2 + (4a_0 - c_1)\dot{u} - 3u^3 + 13u\dot{u}] \\ & + z^3[f_1 - e_0 u - c_0\dot{u} + c_0 u^2 + a_0(3u\dot{u} - u^3 - \ddot{u}) - 4u\ddot{u} - 3\dot{u}^2 + 6\dot{u}u^2 - u^4] \\ & + f_0 z^4 = 0, \end{aligned}\right. \tag{10.4}$$

where A, B_1, B_2, B_3 are given by (8.7).

To determine a transformation T in order that

$$b = -12, \qquad c_1 = 0, \qquad B_2 = 0 \tag{10.5}$$

[then $s = 1$], one has only to choose λ, φ such that

$$\Lambda - \Phi + \frac{\dot{b}}{b} = 0,$$

$$\Lambda + \frac{c_1}{b} = 0;$$

then μ is determined by a linear differential equation of the first order in the form

$$b\dot{\mu} + \left(c_1 + 12\,\frac{e_2}{b} + \frac{2\cdot 12\cdot 12}{b^2}\,f_4\right)\mu + M_1 = 0\,,$$

where M_1 depends on λ, φ and on certain coefficients of $F(x,\ y) = 0$.

With these simplifications, equation (10.4) can be written [for convenience, we replace a_0 by a and c_0 by c]

$$(10.6)\quad \left\{\begin{aligned} & z^3\dddot{u} - 5z^2\ddot{u} + 8z\dot{u} + A + z[B_1u - 2u^2] \\ & + z^2[f_2 - e_0 + (3c - e_1)u - B_3u^2 + 4a\dot{u} - 3u^3 + 13u\dot{u}] \\ & + z^3[f_2 - e_0u - c\dot{u} + cu^2 + a(3u\dot{u} - \ddot{u} - \dot{u}^3) - 4u\ddot{u} - 3\dot{u}^2 + 6\dot{u}u^2 - u^4] \\ & + f_0z^4 = 0\,, \end{aligned}\right.$$

where

$$(10.7)\quad \left\{\begin{aligned} A &= -6a + d_0 - e_2 + f_4\,, \\ B_1 &= -12a + 2d_0 - e_2\,, \\ B_2 &= 2c - e_1 + f_3\,, \\ B_3 &= 7a - d_0\,. \end{aligned}\right.$$

We denote by $E(u)$ the left hand member of (10.6).

11. - To proceed further, set

$$(11.1)\quad \left\{\begin{aligned} u &= P + z^5v\,, \\ P &= 2\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4\,, \end{aligned}\right.$$

[2α instead of α simplifies the notation]. Equation (10.6) becomes

$$z\,[z^3\dddot{v} + 10z^2\ddot{v} + 18z\dot{v}] + A_0 + A_1z + A_2z^2 + A_3z^3 + A_4z^4 + A_5z^5 + O(z^6) = 0\,,$$

where the $A_1's$ are determined by

$$E(P) = A_0 + A_1z + A_2z^2 + A_3z^3 + A_4z^4 + A_5z^5 + O(z^6)\,.$$

According to the general theory, α and δ remain arbitrary and are assumed to be constant parameters.

The condition for stability are given by the relations

$$A_i = 0, \qquad (i = 0, 1, 2, 3, 4, 5).$$

A simple but tedious calculation gives the following results.

i. $A_0 = 0$ shows that

$$A = 0. \tag{11.2}$$

ii. $A_1 = 0$ gives

$$\beta = \alpha^2 - \frac{B_1}{4}\alpha. \tag{11.3}$$

iii. From $A_2 = 0$, it follows that

$$6\gamma = \gamma_2\alpha^2 + \gamma_1\alpha + \gamma_0, \tag{11.4}$$

where

$$\begin{cases} \gamma_2 = 4B_3 + 5B_1 - 4a, \\ \gamma_1 = -\frac{1}{2}B_1 + \frac{B_1}{4}(B_1 + 4a) - 2(3c - e_1), \\ \gamma_0 = e_0 - f_2. \end{cases} \tag{11.5}$$

iv. On using (11.3), we write $A_3 = 0$ in the form

$$\begin{cases} 2\ddot{\beta} + 6\dot{\gamma} + 4\alpha\dot{\beta} - 12\alpha\gamma - B_1\gamma + 4B_3\alpha\beta + 2B_1\alpha(\beta + 2\alpha^2) \\ \quad - (2c - e_1)\beta - 4c\alpha^2 + 2e_0\alpha - f_1 \\ \quad - a(2\dot{\beta} + 6\gamma + 12\alpha\beta - 8\alpha^3) = 0. \end{cases} \tag{11.6}$$

The left hand member of (11.6) is a polynomial in α of degree 3; because α is arbitrary, the coefficients of this polynomial must be zero. In particular, on taking into account the coefficient of α^3, one obtains

$$B_1 + B_3 = a. \tag{11.7}$$

We shall consider later on the other conditions for stability given by (11.6).

v. The condition $A_4 = 0$ gives

$$(11.8)\quad \left\{\begin{aligned} & 4\varepsilon = (\ddot\beta + \dot\gamma - 2\alpha\dot\beta - 10\alpha\gamma - 3\beta^2 + 4\alpha^2\dot\beta) - 8\alpha\delta + (B_1 + 6a)\delta \\ & + 6\beta\gamma - 4\alpha^2\gamma - 2B_1\alpha^2\beta - B_3(\beta^2 + 4\alpha\gamma) - e_0\beta - e_1\gamma \\ & + c(\gamma + 2\alpha\beta - \dot\beta) + a(16\alpha\gamma + 5\beta^2 - \ddot\beta + 2\alpha\dot\beta - 4\alpha^2\beta) + f_0, \end{aligned}\right.$$

i.e.,

$$(11.9)\qquad 4\varepsilon = -8\alpha\delta + (B_1 + 6a)\delta + Q(\alpha),$$

where $Q(\alpha)$ is a polynomial in α.

vi. The condition $A_5 = 0$ is

$$(11.10)\quad \begin{aligned} & (\ddot\gamma + 4\alpha\dot\gamma + 4\varepsilon - 4\beta\gamma + 8\alpha^2\gamma)^{\boldsymbol\cdot} + B_1\varepsilon - e_1\delta - B_3(2\beta\gamma + 4\alpha\delta) \\ & - e_0\gamma - c\dot\gamma + a[4\varepsilon + 12\alpha\delta + 10\beta\gamma - 4\alpha^2\gamma - \ddot\gamma - 2\alpha\dot\gamma] = 0. \end{aligned}$$

The coefficient of $\alpha\delta$ in (11.10) is zero; this yields

$$(11.11)\qquad B_1 + 2B_3 = 2a$$

and because of (11.7),

$$(11.12)\qquad B_1 = 0,$$

$$(11.13)\qquad B_3 = a.$$

The coefficient of δ is also zero; therefore

$$(11.14)\qquad e_1 = 6(\dot a + a^2).$$

With these values of B_1, B_3 and on account of (10.7), one obtains

$$(11.15)\qquad d_0 = 6a, \qquad e_2 = 0, \qquad f_4 = 0;$$

further, by virtue of (11.3-5), one has

$$\beta = \alpha^2, \quad 6\gamma = \gamma_1\alpha + \gamma_0, \quad \gamma_1 = -2(3c - e_1). \tag{11.16}$$

12. - With these simplifications, we rewrite $A_3 = 0$ in the form [see (11.6)]

$$6\dot{\gamma} - 12\alpha\gamma - 6a\gamma - 6c\alpha^2 + e_1\alpha^2 + 2e_0\alpha - f_1 = 0. \tag{12.1}$$

This polynomial in α of the second degree is identically zero. Thus

$$e_1 = 2c, \tag{12.2}$$

whence

$$f_3 = 0, \qquad [\text{cf. } B_2 = 0], \tag{12.3}$$

$$\gamma_1 = -2c; \tag{12.4}$$

further,

$$f_2 = \dot{c} - ac, \tag{12.5}$$

$$f_1 = \dot{\gamma}_0 - a\gamma_0. \tag{12.6}$$

13. - Now we return to 4ε [cf. (11.8)] and $A_5 = 0$ [cf. (11.10)]. One obtains

$$\left\{\begin{aligned} 4\varepsilon = &-8\alpha\delta + 6a\delta + \ddot{\gamma} - 10\alpha\dot{\gamma} + 2\alpha^2\gamma - e_0\alpha^2 \\ &+ c(2\alpha^3 - \gamma) + 12a\alpha\gamma + f_0 \end{aligned}\right. \tag{13.1}$$

or

$$4\varepsilon + 8\alpha\delta - 6a\delta = \varepsilon_3\alpha^3 + \varepsilon_2\alpha^2 + \varepsilon_1\alpha + \varepsilon_0, \tag{13.2}$$

where

$$
(13.3)\qquad \begin{cases} \varepsilon_3 = \frac{4}{3} c, \\ \varepsilon_2 = \frac{10}{3} \dot{c} + \frac{1}{3} \gamma_0 - e_0 - 4ac, \\ \varepsilon_1 = 2a\,\gamma_0 - \frac{5}{3} \dot{\gamma}_0 - \frac{1}{3} (\ddot{c} - c^2), \\ \varepsilon_0 = \frac{1}{6} (\ddot{\gamma}_0 - c\gamma_0) + f_0 . \end{cases}
$$

Further we rewrite $A_5 = 0$ as

$$
(13.4)\qquad \begin{cases} (\ddot{\gamma} + 4\alpha\dot{\gamma} + 4\varepsilon + 4\alpha^2\gamma)^{\cdot} - e_1\delta - e_0\gamma - c\dot{\gamma} \\ + a[4\varepsilon + 8\alpha\delta + 4\alpha^2\gamma - \ddot{\gamma} - 2\alpha\dot{\gamma}] = 0 . \end{cases}
$$

This polynomial in α is identically zero. The coefficient of α^3 vanishes because of the value (13.3) of ε_3. The coefficient of α^2 gives

$$
(13.5)\qquad (2\dot{c} - 4ac - f_2)^{\cdot} + a(4\dot{c} - 4ac - f_2) = 0 ;
$$

on using (12.4) and because of

$$
(13.6)\qquad c = 3(\dot{a} + a^2)
$$

[cf. (12.2) and (11.14)], one obtains instead of (13.5),

$$
(13.7)\qquad \ddot{c} = c^2 .
$$

The coefficient of α gives

$$
(13.9)\qquad 3\ddot{\gamma}_0 - 2c\gamma_0 + c\dot{c} - ce_0 - ac^2 = 0 ;
$$

on using the value (11.5) of γ_0 and in view of (12.5), one obtains

$$
(13.9)\qquad \ddot{\gamma}_0 - c\gamma_0 = 0
$$

or by a short calculation,

$$
(13.10)\qquad \ddot{e}_0 - ce_0 = 2a(ac)^{\cdot} .
$$

Finally, the coefficient of α^0 gives

$$6[\dot{f}_0 + af_0] + (2\ddot{\gamma}_0 - c\gamma_0)^{\cdot} - \gamma_0 e_0 - c\dot{\gamma}_0 - ac\gamma_0 = 0 . \tag{13.11}$$

It is not difficult to see that (13.11) reduces to

$$\dot{f}_0 + af_0 = \frac{1}{6}\gamma_0^2 \tag{13.12}$$

[use (13.9), (12.5) and γ_0 given by (11.5)].

14. - We sum up the results obtained in the preceding sections.

For convenience, we replace c by $6c$.

The coefficients of the stable equations of class II are given in terms of a function a determined by

$$\ddot{c} = 6c^2 , \tag{14.1}$$

$$\dot{a} + a^2 = 2c \tag{14.2}$$

or on setting

$$a = \frac{\dot{v}}{v} , \tag{14.3}$$

by

$$\ddot{v} - 2cv = 0 . \tag{14.4}$$

Then

$$\left\{ \begin{aligned} & c_1 = 0 , \qquad d_0 = 6a ; \\ & e_2 = 0 , \qquad e_1 = 12c ; \end{aligned} \right. \tag{14.5}$$

$$\ddot{e}_0 - 6ce_0 = 12a(ac)^{\cdot} ; \tag{14.6}$$

$$\left\{ \begin{aligned} & f_4 = f_3 = 0 , \\ & f_2 = 6(\dot{c} - ac) , \qquad f_1 = \dot{\gamma}_0 - a\gamma_0 , \\ & \dot{f}_0 + af_0 = \frac{1}{6}\gamma_0^2 , \end{aligned} \right. \tag{14.7}$$

where

$$(14.8)\qquad \gamma_0 = e_0 - f_2 ,$$

$$(14.9)\qquad \ddot{\gamma}_0 - 6c\gamma_0 = 0 .$$

The stable equations of this class are thus of the form

$$(14.10)\quad y^{iv} + 12\dot{y}\ddot{y} = a(\dddot{y} + 6\dot{y}^2) + 6c(\ddot{y} + 2y\dot{y}) + e_0\dot{y} + f_2y^2 + f_1y + f_0 .$$

15. - To determine explicitly all the stable equations (14.10), we begin by considering (14.1). One has

$$(15.1)\qquad \dot{c}^2 = 4c^3 + g_3 ,$$

where g_3 is an arbitrary constant. Therefore, three cases are to be considered according to the values of g_3, namely,

i. $g_3 = 0 , \qquad c = 0 ,$

ii. $g_3 = 0 , \qquad c = x^{-2} ,$

iii. $g_3 \neq 0 , \qquad c = \wp(x ;\ 0,\ g_3) .$

Then, we determine a by (14.2) or, which is equivalent, by (14.3-4); one obtains the following results

i. $g_3 = 0 , \qquad c = 0$; then

$$v = K_1x + K_2 ;$$

i. α. when $K_1 = 0$;

$$a = 0 ;$$

i. β. when $K_1 \neq 0$,

$$a = \frac{K_1}{K_1x + K_2} ;$$

ii. $g_3 = 0 ,\ c = x^{-2}$; then

$$\ddot{v} - \frac{2}{x^2}\,v = 0$$

and thus

$$v_1 = x^2 , \qquad v_2 = x^{-1}$$

to which correspond respectively

ii. α. $\qquad a = \frac{2}{x} ,$

ii. β. $\qquad a = -\frac{1}{x} ,$

ii. γ. $\qquad a = \frac{2K_1x^3 - K_2}{x(K_1x^3 + K_2)} .$

iii. $g_3 \neq 0$; then

$$\ddot{v} - 2\wp(x;\, 0,\, g_3)v = 0 \tag{15.2}$$

which is a LAMÉ equation.

16. - We consider the case $g_3 = 0$, $c = 0$. Then

$$f_2 = 0 , \tag{16.1}$$

$$f_1 = \dot{e}_0 - ae_0 , \tag{16.2}$$

$$\dot{f}_0 + af_0 = \frac{1}{6} e_0^2 , \tag{16.3}$$

$$y^{iv} + 12\dot{y}\ddot{y} = a(\dddot{y} + 6\dot{y}^2) + e_0\dot{y} + f_1y + f_0 . \tag{16.4}$$

Further,

$$e_0 = K_3x + K_4 . \tag{16.5}$$

i. Suppose $a = 0$. We consider two cases according as K_3 is or is not zero.

i. α. $K_3 = 0$. Then

$$f_1 = 0 , \tag{16.6}$$

$$f_0 = \frac{1}{6} K_4^2x + K_5 . \tag{16.7}$$

Equation (16.4) is

$$y^{iv} + 12\dot{y}\ddot{y} = K_4\dot{y} + \frac{1}{6}K_4^2 x + K_5\,; \tag{16.8}$$

on replacing x by αx and y by βy, where $\alpha\beta = 1$, $K_4\alpha^3 = 6$, one brings (16.8) to the canonical form

$$y^{iv} + 12\dot{y}\ddot{y} = 6\dot{y} + 6x + K\,. \tag{16.9}$$

[Set $\dot{y} = z$; then (16.9) assumes the form

$$\dddot{z} + 12 z\dot{z} = 6z + 6x + K$$

which is a stable equation of the third order and of class I (eq. 22.23)].

i. β. $K_3 \neq 0$. Then

$$f_1 = K_3\,, \tag{16.10}$$

$$f_0 = \frac{1}{18\,K_3}(K_3 x + K_4)^3 + K_5\,. \tag{16.11}$$

Equation (16.4) becomes

$$y^{iv} + 12\dot{y}\ddot{y} = (K_3 x + K_4)\dot{y} + K_3 y + \frac{1}{18 K_3}(K_3 x + K_4)^3 + K_5\,. \tag{16.12}$$

A trivial transformation enables one to suppose $K_4 = 0$; then a substitution $(x, y\,;\ \alpha x, \beta y)$ where $\alpha\beta = 1$, $K_3\alpha^4 = 6$, brings equation (16.12) to the canonical form

$$y^{iv} + 12\dot{y}\ddot{y} = 6x\dot{y} + 6y + 2x^3 + K\,, \tag{16.13}$$

where K is an arbitrary constant.

ii. Suppose $a = \dfrac{K_1}{K_1 x + K_2}$ or by a trivial substitution

$$a = \frac{1}{x}\,. \tag{16.14}$$

Then e_0 is still given by (16.5) and $f_1 = -\dfrac{K_4}{x}$. Further,

$$\dot{f}_0 + \frac{1}{x} f_0 = \frac{1}{6}(K_3 x + K_4)^2\,;$$

therefore, when $K_3 = 0$,

$$f_0 = \frac{K_4^2}{12}\, x + \frac{K_5}{x} \tag{16.15}$$

and when $K_3 \neq 0$,

$$f_0 = \frac{1}{24}\, K_3^2 x^3 + \frac{1}{9}\, K_3 K_4 x^2 - \frac{K_4}{x}\, y + \frac{1}{12}\, K_4^2 x + \frac{K_5}{x} \tag{16.16}$$

When $K_3 = 0$, the stable equations (16.4) are thus of the form

$$y^{iv} + 12\,\dot{y}\,\ddot{y} = \frac{1}{x}\,(\dddot{y} + 6\,\dot{y}^2) + K_4 \dot{y} + \frac{K_4^2}{12}\, x + \frac{K_5}{x} \tag{16.17}$$

which may be brought to the canonical forms

$$y^{iv} + 12\,\dot{y}\,\ddot{y} = \frac{1}{x}\,(\dddot{y} + 6\,\dot{y}^2) + \frac{1}{x}\,, \tag{16.18}$$

$$y^{iv} + 12\,\dot{y}\,\ddot{y} = \frac{1}{x}\,(\dddot{y} + 6\,\dot{y}^2) + 6\,\dot{y} - \frac{6}{x}\, y + 3x + \frac{K}{x}\,, \tag{16.19}$$

[For (16.19), $\alpha\beta = 1$, $K_4 \alpha^3 = 6$].

On setting

$$z = \dddot{y} + 6\,\dot{y}^2\,,$$

equation (16.18) becomes

$$\dot{z} = \frac{1}{x}\, z + \frac{1}{x}$$

and thus

$$z = K_6 x - 1\,.$$

Therefore equation (16.18) is equivalent to

$$\dddot{y} + 6\,\dot{y}^2 = Kx - 1$$

which is a stable equation of the third order [see Part II, eq. (27.2)].

When $K_3 \neq 0$, the stable equations (16.4) are of the form

$$(16.20)\qquad \begin{aligned} y^{iv} + 12\dot{y}\ddot{y} &= \frac{1}{x}(\dddot{y} + 6\dot{y}^2) + (K_3 x + K_4)\dot{y} - \frac{K_4}{x}y \\ &+ \frac{1}{24}K_3^2 x^3 + \frac{1}{9}K_3 K_4 x^2 + \frac{1}{12}K_4^2 x + \frac{K_5}{x}. \end{aligned}$$

Equation (16.20) may be brought to canonical forms, namely,

i. when $K_4 = 0$,

$$(16.21)\qquad y^{iv} + 12\dot{y}\ddot{y} = \frac{1}{x}(\dddot{y} + 6\dot{y}^2) + x\dot{y} + \frac{x^3}{24} + \frac{K}{x};$$

ii. when $K_4 \neq 0$,

$$(16.22)\qquad \left\{ \begin{aligned} & y^{iv} + 12\dot{y}\ddot{y} = \frac{1}{x}(\dddot{y} + 6\dot{y}^2) + (x + K)\dot{y} - \frac{K}{x}y \\ & + \frac{1}{24}x^3 + \frac{1}{9}Kx^2 + \frac{1}{12}K^2 x + \frac{K_1}{x}. \end{aligned} \right.$$

Equations of class III.

17. - The stable equations of class III are of the form

$$(17.1)\qquad y^{iv} = a(y\dddot{y} + 3\dot{y}\ddot{y}) + F(x, y),$$

where $F(x, y)$ is given by (7.5).

The general solution of these equations has one set of simple parametric poles characterized by

$$(17.2)\qquad as = -2, \qquad \Theta = 1, 2, 3.$$

Substitute $y(x)$ given by (10.1) into (17.1), suppose $s = 1$ [which is obtained by a T transformation; see below] and for convenience, define A, B_1, B_2, B_3 by (8.7); one obtains

$$(17.3)\qquad \left\{ \begin{aligned} & z^3\dddot{u} - 3z^2\ddot{u} + 6z\dot{u} - 6u + A + z[B_2 + B_1 u - 12u^2] \\ & + z^2[f_2 - e_0 + (3c_0 - e_1)u - B_3 u^2 + (4a_0 - c_1)\dot{u} - 7u^3 + 13u\dot{u}] \\ & + z^3[f_1 - e_0 u - c_0\dot{u} + c_0 u^2 + a_0(3u\dot{u} - u^3 - \ddot{u}) - 4u\ddot{u} - 3\dot{u}^2 + 6\dot{u}u^2 - u^4] \\ & + f_0 z^4 = 0. \end{aligned} \right.$$

We denote by $E(u)$ the left hand member of (17.3).

Now determine a T-transformation in order that

$$a = -2, \qquad c_1 = 0, \qquad A = 0.$$

To this effect, one has only to choose λ, φ, μ such that

$$\Lambda - \Phi + \frac{\dot a}{a} = 0,$$

$$6\Lambda + 3\Phi + \frac{c_1}{a} = 0,$$

$$2\Lambda - 3\Phi + a\mu - a_0 + \frac{1}{3a}\left(2c_1 + d_0 + \frac{2e_2}{a} + \frac{4f_4}{a^2}\right) = 0.$$

Set

$$u = P + z^3 v,$$

$$P = \beta z + \gamma z^2$$

[$\alpha = 0$ because $A = 0$]. Equation (17.3) becomes

$$z^3[z^3 \dddot v + 6z^2 \ddot v + 6z\dot v] + A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4) = 0,$$

where A_0, A_1, A_2, A_3 are determined by

$$E(P) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4).$$

According to the general theory, β and γ remain arbitrary and are assumed to be constant parameters.

The conditions for stability are given by

$$(17.4) \qquad A_i = 0, \qquad (i = 0,\ 1,\ 2,\ 3).$$

For convenience, we rewrite the values of A, B_1, B_2, B_3, i. e.,

$$(17.5) \qquad \begin{cases} A = -6a_0 + d_0 - e_2 + f_4, \\ B_1 = -12a_0 + 2d_0 - e_2, \\ B_2 = 2c_0 - e_1 + f_3, \\ B_3 = 7a_0 - d_0. \end{cases}$$

18. - We consider the relations for stability (17.4).

i. Because $A = 0$, one has $A_0 = 0$ and therefore

$$\alpha = 0. \tag{18.1}$$

ii. From $A_1 = 0$, one deduces

$$B_2 = 0. \tag{18.2}$$

iii. Further, $A_2 = 0$ has the form

$$(4a_0 + B_1)\beta + f_2 - e_0 = 0$$

and because β is arbitrary

$$4a_0 + B_1 = 0, \tag{18.3}$$

$$f_2 - e_0 = 0. \tag{18.4}$$

iv. Finally, $A_3 = 0$ has the form

$$(6a_0 + B_1)\gamma + (2c_0 - e_1)\beta + f_1 = 0$$

and therefore

$$6a_0 + B_1 = 0, \tag{18.5}$$

$$2c_0 - e_1 = 0, \tag{18.6}$$

$$f_1 = 0. \tag{18.7}$$

From (18.3) and (18.5), one obtains

$$a_0 = 0, \qquad B_1 = 0. \tag{18.8}$$

From the preceding results, one easily deduces

$$\left\{ \begin{array}{llll} e_1 = 2c_0, & e_2 = 2d_0; & & \\ f_1 = 0, & f_2 = e_0, & f_3 = 0, & f_4 = d_0. \end{array} \right. \tag{18.9}$$

The stable equations of class III are thus of the form

$$\left\{ \begin{array}{l} y^{iv} + 2y\dddot{y} + 6\dot{y}\ddot{y} = c_0(\ddot{y} + 2y\dot{y}) + d_0(\dot{y}^2 + 2y^2\dot{y} + y^4) \\ \qquad\qquad + e_0(\dot{y} + y^2) + f_0, \end{array} \right. \tag{18.10}$$

where c_0, d_0, e_0, f_0 are arbitrary analytic functions of x.

Equation (18.10) is equivalent to the differential system

$$z = \dot y + y^2 \,, \tag{18.11}$$

$$\dddot z = c_0 \dot z + d_0 z^2 + e_0 z + f_0 \,. \tag{18.12}$$

When y is stable, z is also stable. However, in order to be stable, equation (18.12) must be linear and $d_0 = 0$ [see Part II. art. 5, eq. (5.12)].

The stable equations of class III are thus of the form

$$y^{iv} + 2y\dddot y + 6\dot y\,\ddot y = c_0(\ddot y + 2y\dot y) + e_0(\dot y + y^2) + f_0 \,, \tag{18.13}$$

where c_0, e_0, f_0 are arbitrary analytic functions of x.

They are equivalent to the differential system

$$\begin{cases} y = \dfrac{\dot w}{w} \,, \qquad \ddot w = zw \,, \\ \dddot z = c_0 \dot z + e_0 z + f_0 \,; \end{cases} \tag{18.14}$$

therefore z, w and y are stable.

Equations of class IV.

19. - The stable equations of class IV are of the form

$$y^{iv} = 3ay\,\dddot y + 9a\dot y\,\ddot y - 6a^2 y\dot y^2 - 3a^2 y^2\,\ddot y + F(x,\, y) \,, \tag{19.1}$$

where $F(x, y)$ is given by (7.5).

The general solution of these equations has two sets of simple parametric poles characterized by

$$as_1 = -1 \,, \qquad \Theta = 0,\ 2,\ 3 \,, \tag{19.2}$$

$$as_2 = -2 \,, \qquad \Theta = -3,\ 2,\ 3 \,. \tag{19.3}$$

Substitute $y(x)$ given by (10.1) into (19.1), suppose that s is a constant [which is obtained by a transformation T] and for convenience, define A, B_1,

B_2, B_3 by (8.7); one obtains

$$(19.4)\quad \begin{cases} z^3\dddot{u}-(5+3as)z^2\ddot{u}+(20+21as+3a^2s^2)z\dot{u}-(60+81as+21a^2s^2)u+A \\ +z[B_2+B_1u-(50+57as+9a^2s^2)u^2] \\ +z^2[f_2s-e_0+(4a_0-c_1s)\dot{u}+(3c_0-e_1s)u-B_3u^2+(25+18as)u\dot{u}-(15+12as)u^3] \\ +z^3[-a_0\ddot{u}-c_0\dot{u}-e_0u+c_0u^2+f_1+a_0(3u\dot{u}-u^3)-4u\ddot{u}-3\dot{u}^2+6\dot{u}u^2-u^4] \\ +\dfrac{f_0}{s}z^4=0. \end{cases}$$

Now determine a transformation T in order that $a=-1$, $a_0=c_1=0$; to this effect, one has only to choose λ, φ, μ such that

$$\Lambda-\Phi+\frac{\dot{a}}{a}=0,$$

$$4\Lambda+6\Phi-3a\mu-a_0=0,$$

$$18\Lambda+9\Phi-6a\mu+\frac{c_1}{a}=0.$$

20. - We first consider the set of simple parametric poles characterized by

$$s_1=1, \qquad \Theta=0,\ 2,\ 3.$$

Equation (19.4) takes the form

$$(20.1)\quad \begin{cases} z^3\dddot{u}-2z^2\ddot{u}+2z\dot{u}+A+z(B_2+B_1u-2u^2) \\ +z^2[f_2-e_0+(3c_0-e_1)u+d_0u^2+7u\dot{u}-3u^3] \\ +z^3[-c_0\dot{u}-e_0u+c_0u^2+f_1-4u\ddot{u}-3\dot{u}^2+6\dot{u}u^2-u^4] \\ +f_0z^4=0 \end{cases}$$

where

$$(20.2)\qquad A=d_0-e_2+f_4,$$

(20.3) $$B_1 = 2d_0 - e_2 ,$$

(20.4) $$B_2 = 2c_0 - e_1 + f_3 .$$

Denote by $E(u)$ the left hand member of (20.1) and set

$$u = P + z^3 v ,$$

$$P = \alpha + \beta z + \gamma z^2 .$$

Equation (20.1) becomes

$$z^3[z^3 \dddot{v} + 7z^2 \ddot{v} + 8z\dot{v}] + A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4) = 0$$

where the A_i's are determined by

$$E(P) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4) .$$

According to the general theory, α and γ remain arbitrary and are assumed to be constant parameters.

The conditions for stability are given by

(20 5) $$A_i = 0 , \qquad (i = 0, 1, 2, 3) .$$

21. - From (20.5), one obtains the following results.

i. The condition $A_0 = 0$ gives

(21.1) $$A = 0 .$$

ii. From $A_1 = 0$, one deduces

(21.2) $$\beta = \alpha^2 - \frac{B_1}{2} \alpha - \frac{B_2}{2} .$$

iii. Then $A_2 = 0$ yields

(21.3) $$2\dot{\beta} - B_1 \beta + \left(\frac{3}{2} B_1 - d_0\right) \alpha^2 + \left(\frac{3}{2} B_2 + e_1 - 3c_0\right) \alpha + e_0 - f_2 = 0 .$$

This polynomial in α is identically zero; thus

(21.4) $$B_1 = 2d_0 ,$$

(21.5) $$\dot{B}_1 - \frac{B_1^2}{2} - \frac{3}{2} B_2 + 3c_0 - e_1 = 0 ,$$

(21.6) $$\dot{B}_2 - \frac{B_1 B_2}{2} + f_2 - e_0 = 0 .$$

iv. Finally $A_3 = 0$ gives

(21.7) $$\begin{cases} \ddot{\beta} - 2\alpha\dot{\beta} + \gamma B_1 + (2d_0 - B_1)\alpha\beta + \beta(2c_0 - e_1 - B_2) \\ -\frac{B_1}{2}\alpha^3 + \alpha^2\left(c_0 - \frac{1}{2}B_2\right) - e_0\alpha + f_1 = 0 . \end{cases}$$

Because γ is an arbitrary parameter, one has

(21.8) $$B_1 = 0 \cdot$$

From (21.4), one deduces

(21.9) $$d_0 = 0 .$$

Then (20.2-4), (21.1) and (21.5) yield

(21.10) $$e_2 = 0 , \qquad f_4 = 0 , \qquad e_1 = 3f_3 ,$$

(21.11) $$B_2 = 2(c_0 - f_3) .$$

With these simplifications, we rewrite (21.2-3), (21.6-7) as follows

(21.12) $$\beta = \alpha^2 - \frac{B_2}{2} ,$$

(21.13) $$2\dot{\beta} + e_0 - f_2 = 0 ,$$

(21.14) $$\ddot{\beta} - 2\alpha\dot{\beta} - f_3\beta + f_3\alpha^2 - e_0\alpha + f_1 = 0 ,$$

(21.15) $$\dot{B}_2 = e_0 - f_2 .$$

From (21.13-14), one obtains easily

$$\ddot{\beta} - f_2\alpha + \frac{1}{2} f_3 B_2 + f_1 = 0 . \tag{21.16}$$

Because α is arbitrary, (21.16) yields

$$f_2 = 0 \tag{21.17}$$

and accordingly

$$\dot{B}_2 = e_0 , \tag{21.18}$$

$$2f_1 = \ddot{B}_2 - f_3 B_2 . \tag{21.19}$$

22. - Consider now the second set of simple parametric poles characterized by $s = 2$, $\Theta = -3, 2, 3$. Taking into account the values obtained in the preceding section, one rewrites (19.4) in the form

$$(22.1)\quad \left\{ \begin{aligned} & z^3\dddot{u} + z^2\ddot{u} - 10z\dot{u} + 18u + z(B_2 + 28u^2) \\ & + z^2[-e_0 + (3c_0 - 2e_1)u - 11u\dot{u} + 9u^3] \\ & + z^3[-c_0\dot{u} - e_0 u + c_0 u^2 + f_1 - 4u\ddot{u} - 3\dot{u}^2 + 6\dot{u}u^2 - u^4] \\ & + \frac{f_0}{2} z^4 = 0 . \end{aligned} \right.$$

Set again

$$u = P + z^3 v ,$$

$$P = \alpha + \beta z + \gamma z^2$$

and denote by $E(u)$ the left hand member of (22.1). Then determine the A_i's by setting

$$E(P) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + O(z^4) ;$$

the conditions for stability are given by

$$A_i = 0 , \qquad (i = 0, 1, 2, 3) .$$

One obtains the following results.

i. The condition $A_0 = 0$, gives $\alpha = 0$.

ii. From $A_1 = 0$, one deduces

$$8\beta + B_2 = 0. \tag{22.2}$$

iii. Then $A_2 = 0$ yields

$$8\dot{\beta} + e_0 = 0. \tag{22.3}$$

Conditions (22.2–3) are consistent because of (21.18).

iv. Finally, $A_3 = 0$ gives

$$4\ddot{\beta} + 8\beta^2 + 2(c_0 - e_1)\beta + f_1 = 0$$

or

$$\ddot{B}_2 - \frac{1}{4} B_2^2 + \frac{1}{2} B_2(c_0 - e_1) = 2f_1. \tag{22.4}$$

From (21.19) and (22.4), one obtains

$$B_2[B_2 - 2(c_0 - e_1) - 4f_3] = 0;$$

hence,

$$B_2 = 0, \quad \text{i.e.} \quad c_0 = f_3, \tag{22.5}$$

or

$$e_1 = 3f_3, \quad \text{i.e.} \quad (21.10). \tag{22.6}$$

Therefore we have to consider two cases.

i. $c_0 = f_3$. Then $f_1 = 0$ and $e_0 = 0$ [see (22.4) and (21.18)].

The stable equation is of the form

$$y^{iv} + 3y\dddot{y} + 9\dot{y}\ddot{y} + 6y\dot{y}^2 + 3y^2\ddot{y} = \tag{22.7}$$

$$= c_0(\ddot{y} + 3y\dot{y} + y^3) + f_0$$

where c_0 and f_0 are arbitrary analytic functions of x.

That this equation is really stable may be shown as follows.
Set

$$u = \ddot{y} + 3y\dot{y} + y^3 \tag{22.8}$$

and obtaim

$$\ddot{u} = c_0 u + f_0 . \tag{22.9}$$

The linear equation (22.9) is stable. On setting $y = \dfrac{\dot{v}}{v}$, equation (22.8) reduces to

$$\dddot{v} = uv \tag{22.10}$$

and is also stable. [see Part I, eq. (23.1)].

Equation (22.7) is therefore equivalent to the differential system

$$\begin{cases} \dot{v} = vy , \\ \dddot{v} = uv , \\ \ddot{u} = c_0 u + f_0 \end{cases} \tag{22.11}$$

and is stable.

ii. $e_1 = f_3$. For convenience, write f instead of f_3 and set $c = c_0 - f_3$. Then

$$B_2 = 2c , \qquad e_0 = 2\dot{c} , \qquad f_1 = \ddot{c} - fc$$

[see 21.18–19].

The desired equation is

$$\begin{cases} y^{iv} + 3y\dddot{y} + 9\dot{y}\ddot{y} + 6y\dot{y}^2 + 3y^2\ddot{y} \\ = c\ddot{y} + 2\dot{c}\dot{y} + \dot{c}y + f(\ddot{y} + 3y\dot{y} + y^3) - cfy + f_0 . \end{cases} \tag{22.12}$$

Now set

$$u = \ddot{y} + 3y\dot{y} + y^3 - cy , \tag{22.13}$$

[see Part I, eq. (23.1)].

Equation (22.12) may be rewritten as

$$\ddot{u} = fu + f_0 .$$

On setting

$$y = \frac{\dot{v}}{v} ,$$

(22.13) takes the form

$$\dddot{v} = c\dot{v} + uv .$$

Therefore, equation (22.12) is equivalent to the differential system

$$(22.14) \qquad \begin{cases} \dot{v} = vy , \\ \ddot{u} = fu + f_0 ; \\ \dddot{v} = c\dot{v} + uv \end{cases}$$

and is stable; c, f, f_0 are arbitrary analytic functions of x.

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[4] G. Valiron:

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[5] F. J. Bureau:

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