Almost periodic properties of ordinary differential equations.

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Sunto - *Si tratta di alcune proprietà dell'equazioni differenziali ordinarie con coefficienti quasi-periodici.*

1. - The present paper deals with various properties of ordinary differential equations with almost periodic coefficients. In n. 2 a theorem of FAVARD, which appeared recently in this journal, is substantially improved by means of an argument due to CAMERON.

In n. 3 we prove that the equation

$$x' = Ax + f(t)$$

has almost periodic solutions, even if the corresponding homogeneous equation has almost periodic solutions, provided the frequencies of these solutions are not arbitrarily close to the frequencies of the almost periodic function $f(t)$. This corresponds in a very natural way to the physical concept of non-resonance. The proof is based on an extension of an inequality due to H. BOHR. The result is then applied to nonlinear equations and to the reducibility of linear equations.

In n. 4 a recent result of R. K. MILLER, which deduces the existence of almost periodic solutions of nonlinear equations from stability properties, is reproved by a quite different method and given a somewhat sharper form.

Since translation numbers play no part in this paper we may adopt BOCHNER'S criterion as our definition of an almost periodic function. A continuous (scalar or vector) function $f(t)$ is said to be *almost periodic* if every sequence $\{h_n\}$ of real numbers contains a subsequence $\{k_n\}$ such that $f(t + k_n)$ converges uniformly on the whole axis $J = (-\infty, \infty)$. Whenever we speak of uniform convergence without further qualification we will always mean uniform convergence on J. A sequence of functions will be said to be *locally uniformly convergent* if it converges uniformly on every compact subinterval of J.

We define the norm of an almost periodic function $f(t)$ by

$$\| f \| = \sup_{-\infty < t < \infty} | f(t) | .$$

The set of all almost periodic functions is a BANACH space with respect to this norm. The *closed hull* of an almost periodic function $f(t)$ is the set of all almost periodic functions $g(t)$ such that $f(t+k_n) \to g(t)$ uniformly for some real sequence $\{k_n\}$.

To any almost periodic (scalar or vector) function $f(t)$ there corresponds a unique formal FOURIER expansion $\Sigma c_n e^{i\lambda_n t}$, where the coefficients c_n are non-zero (complex numbers or vectors). The real numbers λ_n will be called the *frequencies* of $f(t)$. The *frequency module* of $f(t)$ is the set of all (real) numbers which are finite linear combinations of the frequencies with integral coefficients. In other words, the frequency module is the additive group generated by the frequencies. The set of all pure imaginary numbers $i\lambda_n$ will be called the *spectrum* of $f(t)$. By the *extended spectrum* of $f(t)$ we will mean the set of all (pure imaginary) numbers which are finite linear combinations $\Sigma i k_n \lambda_n$ with non-negative integral coefficients k_n, at least one of which is positive. Thus the extended spectrum is the additive semigroup generated by the spectrum.

2. - FAVARD [6], pp. 303, 310, has proved a theorem which may be formulated in the following way:

Let $A(t)$ be a real almost periodic matrix and suppose that for each matrix $B(t)$ in the closed hull of $A(t)$ the linear differential equation

$$y' = B(t)y \tag{1}$$

has exactly s linearly independent bounded solutions. If the equation

$$x' = A(t)x \tag{2}$$

has no nontrivial solution $x(t)$ with $\inf\limits_{-\infty<t<\infty} |x(t)| = 0$, then it has a fundamental matrix $X(t)$ such that

$$X(t)\begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} X^*(t)$$

is almost periodic for some $q \leq s$, where I_q is the $q \times q$ unit matrix.

In the present section we will show that we can always take $q = s$. The method of proof is quite different from FAVARD'S and is essentially due to CAMERON [4], who considered the special case in which the solutions of the equations (1) are *all* bounded.

Let $f(t)$ be an almost periodic function and let $g(t)$ be a function which is bounded and uniformly continuous on $J = (-\infty, \infty)$. By ASCOLI'S theorem any sequence $\{h_n\}$ of real numbers contains a subsequence $\{k_n\}$ such that

$$f(t+k_n) \to f_1(t) \text{ uniformly, } g(t+k_n) \to g_1(t) \text{ locally uniformly.}$$

We will say that f_1 and g_2 are (translation) *transforms* of f and g. Evidently $f_1(t)$ is almost periodic and $g_1(t)$ is bounded and uniformly continuous. If $f_2(t)$, $g_2(t)$ are transforms of $f_1(t)$, $g_1(t)$ then they are also transfnrms of $f(t)$, $g(t)$. For suppose

$$f_1(t + l_n) \to f_2(t) \text{ uniformly, } g_1(t + l_n) \to g_2(t) \text{ locally uniformly.}$$

For each positive integer n we can find an integer $m_n > n$ such that

$$|f(t + k_{m_n} + l_n) - f_1(t + l_n)| < 1/n \text{ for all } t,$$

$$|g(t + k_{m_n} + l_n) - g_1(t + l_n)| < 1/n \text{ for } |t| \leq n.$$

Hence $f_2(t)$, $g_2(t)$ are transforms of $f(t)$, $g(t)$ by the sequence $\{k_{m_n} + l_n\}$.

We will say that a sequence $\{h_n\}$ of real numbers is *stationary* for $f(t)$ if $f(t + h_n) \to f(t)$ uniformly, and for $g(t)$ if $g(t + h_n) \to g(t)$, locally uniformly. Obviously any subsequence of a stationary sequence is stationary. If $f_1(t)$ is a transform of the almost periodic function $f(t)$ by a sequence $\{h_n\}$, then it is also a transform of $f(t)$ by another sequence $\{k_n\}$ if and only if $\{h_n - k_n\}$ is a stationary sequence for $f(t)$. In particular, the stationary sequences for an almost periodic function form a group under term by term addition. It is known that, if $f(t)$ and $f_1(t)$ are almost periodic functions, the frequency module of $f_1(t)$ is contained in the frequency module of $f(t)$ if and only if every sequence stationary for $f(t)$ is also stationary for $f_1(t)$.

LEMMA (1) - *Let $f(t)$ be an almost periodic function and let $g(t)$ be a bounded, uniformly continuous function. Then $g(t)$ is almost periodic and its frequency module is contained in the frequency module of $f(t)$ if and only if every sequence stationary for $f(t)$ is also stationary for every transform of $g(t)$.*

The necessity of the condition will not be required later, but follows readily from the remark preceding the statement of the lemma. To prove its sufficiency, suppose

$$f(t + h_n) \to f_1(t) \text{ uniformly, } g(t + h_n) \to g_1(t) \text{ locally uniformly.}$$

It is sufficient to show that the convergence of the second sequence is necessarily uniform. If this is not the case then for some $\varepsilon > 0$ there exists a sequence $\{t_n\}$ and a subsequence $\{k_n\}$ of $\{h_n\}$ such that

$$|g(t_n + k_n) - g_1(t_n)| \geq \varepsilon > 0.$$

By restricting attention to a suitable subsequence we can suppose further that

$$f_1(t + t_n) \to f_2(t), \ f(t + t_n + k_n) \to f_2(t) \text{ uniformly,}$$

$$\left.\begin{array}{l} g_1(t + t_n) \to g_2(t), \ g(t + t_n + k_n) \to g_3(t) \\ g_2(t - t_n) \to g_4(t), \ g_3(t - t_n - k_n) \to g_5(t) \end{array}\right\} \text{ locally uniformly.}$$

Then $f_2(t - t_n) \to f_1(t)$ uniformly and $g_4(t)$ is the transform of $g_2(t)$ by the sequence $\{-t_n\}$, Since $g_2(t)$ is the transform of $g_1(t)$ by $\{t_n\}$ it follows that $g_4(t)$ is the transform of $g_1(t)$ by a sequence $\{t_{m_n} - t_n\}$ which is stationary for $f_1(t)$ and hence also for $f(t)$. Therefore $g_4 = g_1$. Similarly we can show that $g_5 = g$. Thus

$$g_3(t) \Rightarrow g(t) \Rightarrow g_1(t) \Rightarrow g_2(t)$$

under the succession of sequences $\{-t_n - k_n\}$, $\{k_n\}$, $\{t_n\}$. It follows that $g_2(t)$ is the transform of $g_3(t)$ by a sequence stationary for $f(t)$. Therefore $g_3 = g_2$. But, from the definitions of g_2 and g_3,

$$|g_3(0) - g_2(0)| \geq \varepsilon > 0.$$

Thus we have a contradiction.

Consider now the homogeneous linear differential equation

$$x' = A(t)x, \tag{2}$$

where the $r \times r$ matrix $A(t)$ is almost periodic. We assume that (2) has no nontrivial bounded solution $x(t)$ such that

$$\inf_{-\infty<t<\infty} |x(t)| = 0.$$

Let $X(t)$ be the fundamental matrix for (2) such that $X(0) = I$. The values at $t = 0$ of the bounded solutions of (2) form a subspace V_1, of dimension s say, of the r-dimensional vector space V. Let V_2 be any supplementary subspace and let P be the corresponding projection of V onto V_1. Then a solution $x(t) = X(t)\xi$ of (2) is bounded if and only if $x(0) = \xi \in V_1$, i.e. if and only if $P\xi = \xi$. Thus the matrix $X(t)P$ is bounded:

$$|X(t)P| \leq M \text{ for all } t.$$

A matrix S will be called a *translation matrix* of (2) if there exists a sequence $\{h_n\}$ which is stationary for $A(t)$ such that

$$X(h_n)P \to S \text{ as } n \to \infty.$$

Evidently $S = SP$ and $|S| \leq M$. Furthermore, since $X(t+h_n)P$ is the solution of the equation

$$X' = A(t + h_n)X$$

which takes the value $X(h_n)P$ at $t = 0$, it follows from standard theorems about the continuous dependence of solutions on parameters and initial values that $X(t + h_n)P$ converges locally uniformly to $X(t)S$. Hence $X(t)S$ is bounded and $PS = S$.

Let S and T be translation matrices, corresponding to the stationary sequences $\{h_n\}$ and $\{k_n\}$ respectively. Then their product TS is also a translation matrix. For $X(t + h_n)P$ converges to $X(t)S = X(t)PS$ locally uniformly and $X(t+k_n)P$ converges to $X(t)T$ locally uniformly. Hence $X(t+l_n)P \to X(t)TS$ locally uniformly, where $l_n = h_{m_n} + k_n$. Moreover $\{l_n\}$ is a stationary sequence for $A(t)$.

Let $\{S_m\}$ be a sequence of translation matrices and suppose $S_m \to S$ as $m \to \infty$. Then S is also a translation matrix. For let $\{h_n^{(m)}\}$ be a stationary sequence for $A(t)$ corresponding to S_m. For each positive integer m we can choose n_m so large that

$$|A(t + h_{n_m}^{(m)}) - A(t)| < 1/m \text{ for all } t, \; |X(h_{n_m}^{(m)})P - S_m| < 1/m.$$

Then $\{h_{n_m}^{(m)}\}$ is a stationary sequence for $A(t)$ and S is a corresponding translation matrix.

If S is a translation matrix its positive powers S^n are also translation matrices. Suppose that, for some vector ξ, $S^n\xi \to 0$ as $n \to \infty$. For each positive integer n we can find a real number h_n such that $|X(h_n)P - S^n| < 1/n$. It follows that $X(h_n)P\xi \to 0$ as $n \to \infty$. Thus the bounded solution $X(t)P\xi$ of (2) has infimum zero, which is possible only if it is identically zero, i.e. if $P\xi = 0$.

By replacing $A(t)$ by $T^{-1}A(t)T$, and hence $X(t)$ by $T^{-1}X(t)T$, with a suitable constant invertible matrix T, we can suppose that

$$P = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $S = SP = PS$ implies that

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The fact that $S^n\xi \to 0$ only if $P\xi = 0$ implies that S_1 has no eigenvalues with absolute value less than 1. The fact that $|S^n| \leq M$ for all $n > 0$ implies that

S_1 has no eigenvalues with absolute value greater than 1 and that its eigenvalues with absolute value equal to 1 correspond to linear elementary divisors. Thus S_1 is similar to a diagonal matrix whose diagonal elements have absolute value 1. Now it follows from DIRICHLET'S pigeonhole principle (see [10], p. 170) that the inverse of any such matrix is the limit of a sequence of its positive powers. Therefore, since the set of translation matrices is closed,

$$\begin{pmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

is also a translation matrix. Thus all matrices S_1 which appear as upper left-hand corners of translation matrices form a subgroup of the s-dimensional general linear group. Since this subgroup is bounded, it is conjugate to a to a subgroup of the s-dimensional unitary group by a theorem due to AUERBACH [1]. Thus there exists a constant invertible matrix

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & I \end{pmatrix},$$

which commutes with P, such that all translation matrices have the form

$$S = W \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1},$$

where U_1 is unitary.

Put

$$X_1(t) = X(t) W.$$

Then $X_1(t)$ is a fundamental matrix for (2) and $X_1(t)\xi$ is bounded if and only if $\xi = P\xi$. Let $\{h_n\}$ be a stationary sequence for the almost periodic matrix $A(t)$. We will show that it is also a stationary sequence for the bounded and uniformly continuous function

$$Z(t) = X_1(t) P X_1^*(t + \tau),$$

for each real number τ. Let $\{k_n\}$ be a subsequence of $\{h_n\}$ such that $X_1(k_n)P$ converges as $n \to \infty$. Its limit necessarily has the form

$$W \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where U_1 is unitary. Hence $X_1(t + k_n)P$ converges locally uniformly to

$$X_1(t) \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $Z(t+k_n)$ converges locally uniformly to

$$X_1(t)PX_1^*(t+\tau)=Z(t).$$

Since the limit $Z(t)$ is the same for all subsequences $\{k_n\}$, it follows that the whole sequence $Z(t+h_n)$ converges to $Z(t)$ locally uniformly.

We now make the further assumption that for each equation

$$y'=B(t)y \tag{1}$$

in the closed hull of (2) the bounded solutions form a vector space of the same dimension s. We will show that no equation in the closed hull of (2) has a nontrivial bounded solution with infimum zero. In fact, suppose $A(t+h_n)\to B(t)$ uniformly. By replacing $\{h_n\}$ by a subsequence we can suppose also that $X(h_n)P$ converges, with limit T say. Then $T=TP$ and $X(t+h_n)P$ converges to $Y(t)T$ locally uniformly, where $Y(t)$ is the fundamental matrix for (1) such that $Y(0)=I$. Thus $Y(t)T$ is bounded. If

$$\inf|Y(t)T\xi|=0$$

for some vector ξ then

$$\inf|X(t)P\xi|=0$$

and hence $P\xi=0$. In particular $T\xi=0$ if and only if $P\xi=0$. Thus if $\xi_1,\ldots,\xi_s$ form a basis for $V_1=PV$ then $Y(t)T\xi_1,\ldots,Y(t)T\xi_s$ are linearly independent bounded solutions of (1), no nontrivial linear combination of which has infimum zero. By our assumption there are no other bounded solutions.

To prove that $Z(t)$ is almost periodic it is sufficient, by Lemma 1, to show that any sequence $\{k_n\}$ which is stationary for $A(t)$ is stationary for any transform of $Z(t)$. Let $\{h'_n\}$ be a subsequence of $\{h_n\}$ such that $Y(-h'_n)T$ converges, with limit S_0. Then $Y(t-h'_n)T$ converges locally uniformly to $X(t)S_0$. Thus $Y(t)T$ is the transform of $X(t)P$ by the sequence $\{h_n\}$ and $X(t)S_0$ is the tranform of $Y(t)T$ by $\{-h'_n\}$. It follows that $X(t)S_0$ is the transform of $X(t)P$ by a sequence which is stationary for $A(t)$. Thus S_0 is a translation matrix. Consequently, by the group property, there exists a sequence $\{l_n\}$ stationary for $A(t)$ which transforms $X(t)S_0$ into $X(t)P$. Let $\{k_n\}$ be a sequence stationary for $A(t)$ such that $Y(k_n)T\to Q$, say. Then, since $\{k_n\}$ is also stationary for $B(t)$, $Y(t+k_n)T\to Y(t)Q$ locally uniformly. Thus $Y(t)Q$ is a bounded solution of (1) and we can write $Q=TS$ for some matrix S. Since $T=TP$ we can suppose without loss of generality that $PS=S$. Then

$$X(t)P\Rightarrow Y(t)T\Rightarrow Y(t)TS\Rightarrow X(t)S_0S\Rightarrow X(t)S$$

under the succession of sequences $\{h_n\}$, $\{k_n\}$, $\{-h'_n\}$, $\{l_n\}$. Hence $X(t)S$ is a transform of $X(t)P$ by a sequence stationary for $A(t)$. Thus any transform of $Y(t)T$ by a sequence stationary for $A(t)$ has the form $Y(t)TS$, where S is a translation matrix for (2). Now the transform of $Z(t)$ by $\{h_n\}$ is

$$Y(t)TWW^*T^*Y^*(t+\tau)$$

and this is stationary under any sequence which is stationary for $A(t)$, since

$$SWW^*S^* = WPW^*.$$

Therefore, by Lemma 1, $Z(t)$ is almost periodic and its frequency module is contained in the frequency module of $A(t)$.

Summing up, we have proved

Theorem (1) - *Let $A(t)$ be an almost periodic matrix and suppose that for each matrix $B(t)$ in the closed hull of $A(t)$ the equation*

$$y' = B(t)y \tag{1}$$

has exactly s linearly independent bounded solutions. If the equation

$$x' = A(t)x \tag{2}$$

has no nontrivial bounded solution $x(t)$ with $\inf|x(t)| = 0$ *then each equation* (1) *has a fundamental matrix $Y(t)$ such that*

$$Y(t)\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} Y^*(t+\tau)$$

is almost periodic in t, for each real number τ, and its frequency module is contained in the frequency module of $A(t)$.

3. - It was shown by Bohr [3] that if

$$f(t) = \Sigma\, c_n e^{i\lambda_n t},\quad x(t) = \Sigma\, c_n \lambda_n^{-1} e^{i\lambda_n t}$$

are finite trigonometrical sums with $|\lambda_n| \geq 1$ for every n, then there exists an absolute constant C such that

$$\|x\| \leq C\|f\|.$$

He showed further that the best possible value of C was $\pi/2$. The following lemma extends Bohr's result, although we do not attempt to determine the best possible value of the constant which enters.

LEMMA (2) - *Let* $f(t) = \Sigma\, c_n e^{i\lambda_n t}$,

$$x(t) = \Sigma\, c_n(\alpha - i\lambda_n)^{-1} e^{i\lambda_n t}$$

be finite trigonometrical sums, where α *is real,* $|\lambda_n| \geq \beta$ *for every* n *and* $\alpha^2 + \beta^2 > 0$. *Then there exists an absolute constant* C *such that*

$$\|x\| \leq C(\alpha^2 + \beta^2)^{-1/2} \|f\|.$$

Without loss of generality we can suppose $\alpha^2 + \beta^2 = 1$, since the general case is reduced to this by the change of scale $t \to \gamma t$, where $\gamma = (\alpha^2 + \beta^2)^{-1/2}$. Put

$$\varphi(t) = \begin{cases} (\alpha - it)^{-1} & \text{for } |t| \geq \beta, \\ \alpha + it & \text{for } |t| \leq \beta. \end{cases}$$

Evidently $\varphi(t)$ is everywhere continuous and is continuously differentiable except at $t = \pm\beta$, where its derivative has a jump discontinuity. Moreover $\varphi(t)/t$ is absolutely integrable over $(-\infty, -1]$ and $[1, \infty)$, and the real and imaginary parts of $\varphi(t)$ tend to zero monotonically as $t \to \pm\infty$. Hence ([2], pp. 46-50) the FOURIER transform

$$\psi(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iut} \varphi(t)\, dt$$

exists for every real $u \neq 0$ and FOURIER'S integral theorem holds in the form

$$\varphi(t) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |u| \leq \varepsilon^{-1}} e^{iut} \psi(u)\, du.$$

We will show below that there exists a positive constant C, independent of α and β, such that for every $u \neq 0$

$$\pi\, |\psi(u)| \leq C(1 + u^2)^{-1}. \tag{3}$$

Thus

$$\int_{-\infty}^{\infty} |\psi(u)|\, du \leq C$$

and

$$\varphi(t) = \int_{-\infty}^{\infty} e^{iut} \psi(u)\, du.$$

Since $|\lambda_n| \geq \beta$ we have

$$x(t) = \Sigma\, c_n \varphi(\lambda_n) e^{i\lambda_n t}$$

$$= \Sigma\, c_n e^{i\lambda_n t} \int_{-\infty}^{\infty} e^{iu\lambda_n} \psi(u) du$$

$$= \int_{-\infty}^{\infty} [\Sigma\, c_n e^{i\lambda_n (t+u)}]\, \psi(u) du$$

$$= \int_{-\infty}^{\infty} f(t+u) \psi(u) du.$$

It follows immediately that

$$|x(t)| \leq C \, \| f \| \quad \text{for every } t,$$

which is what we wished to prove.

To prove (3) we write

$$2\pi\psi(u) = \int_{-\beta}^{\beta} e^{-iut} \varphi(t) dt + \int_{|t| \geq \beta} e^{-iut} \varphi(t) dt.$$

Integrating both terms on the right by parts twice and remembering that $\varphi(t)$ and $\varphi'(t)$ tend to zero as $t \to \pm\infty$ we get, for any $u \neq 0$,

$$2\pi\psi(u) = 2\beta u^{-2} \{ (\alpha + i\beta) e^{-i\beta u} + (\alpha - i\beta) e^{i\beta u} \}$$

$$+ 2u^{-2} \int_{|t| \geq \beta} e^{-iut} (\alpha - it)^{-3} dt.$$

Now $\alpha^2 + t^2 \geq (1 + t^2)/2$ for $|t| \geq \beta$, and hence

$$2\pi\, |\psi(u)| \leq 4u^{-2} + 2u^{-2} \cdot 2^{3/2} \int_{-\infty}^{\infty} (1 + t^2)^{-3/2} dt = C_1 u^{-2}.$$

To complete the proof of (3) it is sufficient to show that

$$|\psi(u)| \leq C_2 \quad \text{for} \quad |u| \leq 1.$$

From the definition of $\varphi(t)$ we have

$$2\pi\psi(u) = \int_{-\beta}^{\beta} e^{-iut}(\alpha + it)dt + \int_{|t|\geq\beta} e^{-iut}\alpha(\alpha^2 + t^2)^{-1}dt$$
$$+ \int_{|t|\geq\beta} e^{-iut}\, it(\alpha^2 + t^2)^{-1}dt,$$

and hence

$$2\pi\,|\psi(u)| \leq 2 + 2\int_{-\infty}^{\infty} (1 + t^2)^{-1}dt + 2\,|H|,$$

where

$$H = \int_{\beta}^{\infty} t(\alpha^2 + t^2)^{-1} \sin ut\,dt$$
$$= \int_{\beta}^{\infty} t^{-1} \sin ut\,dt - \alpha^2\int_{\beta}^{\infty} t^{-1}(\alpha^2 + t^2)^{-1} \sin ut\,dt.$$

The first term on the right is equal to

$$\operatorname{sgn} u \int_{\beta|u|}^{\infty} s^{-1} \sin s\,ds,$$

which is bounded for all u. For $|u| \leq 1$ the second term has absolute value at most

$$2\int_{0}^{\infty} (1 + t^2)^{-1}\,dt.$$

This completes the proof.

We first apply this result to a simple scalar differential equation.

LEMMA (3) - *Let $b(t)$ be a scalar almost periodic functon and tet α be a complex number whose distance d from the spectrum of $b(t)$ is positive. Then the equation*

(4) $$x' = \alpha x + b(t)$$

has a unique almost periodic solution $x(t)$ whose spectrum is the same as that of $b(t)$. Moreover

$$\|x\| \leq (C/d)\,\|b\|, \tag{5}$$

where C is the numerical constant of Lemma 2.

Without loss of generality we can suppose that α is real, since if $\alpha = \beta + i\gamma$ the change of variable $x = e^{i\gamma t}y$ reduces the general case to this. Let $b(t)$ have the (not necessarily convergent) FOURIER expansion $\Sigma\, c_n e^{i\lambda_n t}$. We can find a sequence of finite trigonometrical sums

$$b_m(t) = \Sigma\, c_n^{(m)} e^{i\lambda_n t}$$

whose frequencies are among the frequencies of $b(t)$ such that $b_m(t) \to b(t)$ uniformly as $m \to \infty$. Then

$$x_m(t) = -\,\Sigma\, c_n^{(m)} (\alpha - i\lambda_n)^{-1} e^{i\lambda_n t}$$

is a solution of the differential equation

$$x' = \alpha x + b_m(t).$$

Moreover, by Lemma 2,

$$\|x_m\| \leq (C/d)\,\|b_m\|,$$

$$\|x_{m+p} - x_m\| \leq (C/d)\,\|b_{m+p} - b_m\|.$$

Therefore the sequence $\{x_m(t)\}$ converges uniformly and from the corresponding differential equations the sequence of derivatives $\{x'_m(t)\}$ also converges uniformly. Thus $x_m(t) \to x(t)$, where $x(t)$ is almost periodic with spectrum contained in the spectrum of $b(t)$, satisfies (5), and is a solution of the differential equation (4).

If $x(t) \sim \Sigma\, d_n e^{i\lambda_n t}$ then $x'(t) \sim i\,\Sigma\, d_n \lambda_n e^{i\lambda_n t}$, whence it follows from (4) that $d_n = -(\alpha - i\lambda_n)^{-1} c_n$. Thus every frequency of $b(t)$ is also a frequency of $x(t)$. This argument shows that (4) has no other solution whose spectrum is contained in the spectrum of $b(t)$.

We now pass from scalar equations to vector equations. Our result extends a theorem of MALKIN [11], p. 230, who considered only the much simpler case in which $b(t)$ is a finite trigonometrical sum.

THEOREM (2) - *Let $b(t)$ be an almost periodic vector function and let A be a constant $r \times r$ matrix such that the distance d of the set of eigenvalues* *of A from the spectrum of $b(t)$ is positive. Then the differential equation*

$$x' = Ax + b(t) \tag{6}$$

has a unique almost periodic solution $x(t)$ with the same spectrum as $b(t)$. Moreover

$$\|x\| \leq M\chi(C/d)\|b\|, \tag{7}$$

where M is a positive constant depending only on A, C is the numerical constant of Lemma 2, and $\chi(u)$ is the polynomial

$$\chi(u) = u + u^2 + \dots + u^r.$$

We can clearly suppose that A is in JORDAN canonical form and in fact that it consists of a single JORDAN block. Then (6) takes the form

$$\begin{aligned} x_1' &= \alpha x_1 && + b_1(t) \\ x_2' &= \alpha x_2 + x_1 && + b_2(t) \\ &\dots\dots\dots \\ x_r' &= \alpha x_r + x_{r-1} && + b_r(t). \end{aligned}$$

These equations can be solved in succession by Lemma 3 to yield an almost periodic solution $x(t)$ of (6) whose spectrum is contained in the spectrum of $b(t)$ and which satisfies (7). That $x(t)$ is unique and has the same spectrum as $b(t)$ may be shown by the method of undetermined coefficients, as in the proof of Lemma 3.

We are going to make two applications of Theorem 2. The first is a perturbation theorem and the second is a criterion for the reducibility of an almost periodic linear equation. Before doing so, however, there are some preliminary matters to be dealt with.

LEMMA (4) - *Let $f(t, x)$ be almost periodic in t for each x in the ball $S: |x| \leq R$, and further let it be continuous in x uniformly with respect to t, i.e. for each $\varepsilon > 0$ and each $x_0 \in S$ there exists a corresponding $\delta = \delta(\varepsilon, x_0) > 0$ such that*

$$|f(t, x) - f(t, x_0)| < \varepsilon \quad \textit{for all } t \textit{ if } \quad |x - x_0| < \delta.$$

Then $f(t, x)$ is uniformly continuous in (t, x) for $t \in J$, $x \in S$ and is almost periodic in t uniformly for $x \in S$.

For each $x_0 \in S$ $f(t, x_0)$ is almost periodic and therefore uniformly continuous. Hence we can find $\eta = \eta(\varepsilon, x_0) > 0$ such that

$$|f(t_2, x) - f(t_1, x_0)| \leq |f(t_2, x) - f(t_2, x_0)| + |f(t_2, x_0) - f(t_1, x_0)|$$
$$< \varepsilon \text{ if } |t_1 - t_2| < \eta, \ |x - x_0| < \eta.$$

Since S compact it is covered by a finite number of the open balls $|x - x_0| < \frac{1}{2}\eta(\varepsilon, x_0)$. Let $\rho = \rho(\varepsilon)$ be the smallest of the corresponding finitely many radii $\frac{1}{2}\eta(\varepsilon, x_0)$. Then for any two points $x_1, x_2 \in S$ and any two values $t_1, t_2 \in J$

$$|f(t_2, x_2) - f(t_1, x_1)| < 2\varepsilon \quad \text{if} \quad |t_2 - t_1| < \rho, \ |x_2 - x_1| < \rho.$$

Thus f is uniformly continuous.

Let $\{h_n\}$ be any sequence of real numbers and let $\{x_m\}$ be a countable dense subset of S. By the diagonal process we can find a subsequence $\{k_n\}$ of $\{h_n\}$ such that $f(t + k_n, x_m)$ converges uniformly in t for each m. It then follows from the uniform continuity of f that $f(t + k_n, x)$ converges uniformly for $t \in J$, $x \in S$. Thus $f(t, x)$ is almost periodic in t uniformly for $x \in S$.

Under the conditions of the lemma if $\varphi(t)$ is an almost periodic function such that $\|\varphi\| \leq R$ then $f[t, \varphi(t)]$ is almost periodic. For if $\{l_n\}$ is a subsequence of $\{k_n\}$ such that $\varphi(t + l_n)$ converges uniformly then $f[t + l_n, \varphi(t + l_n)]$ converges uniformly.

Since $f(t, x)$ is almost periodic uniformly with respect to x it has a Fourier expansion $\Sigma c_n(x)e^{i\lambda_n t}$ with exponents λ_n independent of x, the coefficients $c_n(x)$ being continuous functions which are not identically zero. The spectrum and extended spectrum of $f(t, x)$ are defined in the same way as for functions independent of x.

Suppose the almost periodic function $\varphi(t)$ has its spectrum in the extended spectrum of $f(t, x)$. We will show that the almost periodic function $f[t, \varphi(t)]$ also has its spectrum in the extended spectrum of f. In fact for any $\varepsilon > 0$ we can find a finite trigonometrical sum

$$g(t, x) = \Sigma\, d_n(x)e^{i\lambda_n t},$$

whose coefficients are continuous functions of x, such that

$$|f(t, x) - g(t, x)| < \varepsilon \quad \text{for} \quad t \in J, \ |x| \leq R.$$

By the Weierstrass approximation theorem we can even suppose that the coefficients $d_n(x)$ are polynomials in the coordinates of x. Also, for any $\delta > 0$

we can find a finite trigonometrical sum $\psi(t) = \Sigma a_n e^{i\mu_n t}$, where $i\mu_n$ belongs to the extended spectrum of f, such that $\|\psi\| \leq R$ and $\|\varphi - \psi\| < \delta$. Then if $\delta < \delta(\varepsilon)$ we will have

$$|f[t, \varphi(t)] - g[t, \psi(t)]| < 2\varepsilon \quad \text{for all } t.$$

Moreover $g[t, \psi(t)]$ is a finite trigonometrical sum with its spectrum in the extended spectrum of f. Since ε is arbitrary it follows that $f[t, \varphi(t)]$ has its spectrum in the extended spectrum of f.

THEOREM (3) - *Let $f(t, x)$ be almost periodic in t for each x and satisfy the Lipschitz condition*

$$(8) \qquad |f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|.$$

Let A be a constant matrix such that the distance d of the set of eigenvalues of A from the extended spectrum of f is positive. Then there exists a positive constant L_0, depending only on A and d, such that if $L \leq L_0$ the differential equation

$$(9) \qquad x' = Ax + f(t, x)$$

has a unique almost periodic solution $x(t)$ whose spectrum is contained in the extended spectrum of f.

The LIPSCHITZ condition (8) implies that the hypotheses of Lemma 4 are satisfied. Thus, for any $R > 0$, $f(t, x)$ is almost periodic in t, uniformly for $|x| \leq R$. Let B denote the BANACH space of all almost periodic functions whose spectrum is contained in the extended spectrum of f. For any $\varphi(t) \in B$ also $f[t, \varphi(t)] \in B$. Therefore, by Theorem 2, the equation

$$x' = Ax + f[t, \varphi(t)]$$

has a solution $x(t) \in B$. This solution is unique, since otherwise the homogeneous equation $x' = Ax$ would have a nontrivial solution in B; but then A would have an eigenvalue in the extended spectrum of f, which is contrary to hypothesis.

Thus we have a mapping $T: \varphi(t) \to x(t)$ of B into itself. If $x_1 = T\varphi_1$ and $x_2 = T\varphi_2$ then $x = x_1 - x_2$ is a solution of the equation

$$x' = Ax + g(t),$$

where

$$g(t) = f[t, \varphi_1(t)] - f[t, \varphi_2(t)]$$

and hence $\|g\| \leq L\|\varphi_1 - \varphi_2\|$. Therefore, by Theorem 2,

$$\|x_1 - x_2\| \leq LM\chi(C/d)\|\varphi_1 - \varphi_2\|.$$

Thus if we set $L_0 = [2M\chi(C/d)]^{-1}$ then for $L \leq L_0$ the mapping T is a contraction. Therefore it has a unique fixed point $x(t) \in B$. Thus the differential equation (9) has a unique almost periodic solution $x(t)$ whose spectrum is contained in the extended spectrum of f.

The proof shows that the same conclusion holds for given $L > 0$ if $d \geq d_0(A, L)$.

THEOREM (4) - *Let A be a constant matrix with eigenvalues $\alpha_1, ..., \alpha_r$ and let $B(t)$ be an almost periodic matrix. If the distance d of the set of differences $\alpha_j - \alpha_k (j, k = 1, ..., r)$ from the extended spectrum of $B(t)$ is positive then the equation*

$$x' = [A + B(t)]x \tag{10}$$

has a fundamental matrix $X(t)$ of the form

$$X(t) = [I + Q(t)]e^{tA}, \tag{11}$$

where $Q(t)$ is an almost periodic matrix whose spectrum is contained in the extended spectrum of $B(t)$.

Since 0 belongs to the set $\{\alpha_j - \alpha_k\}$, every number in the extended spectrum of $B(t)$ has absolute value $\geq d$. Suppose the spectrum of $B(t)$ contained numbers $i\lambda$, $i\mu$ where $\lambda < 0 < \mu$. If λ/μ were rational we could find positive integers m, n such that $\lambda/\mu = -m/n$. Them $n\lambda + m\mu = 0$ would belong to the extended spectrum, which is a contradiction. If λ/μ were irrational we could find positive integers m, n such that $|n\lambda/\mu + m| < d/\mu$. Then $|n\lambda + m\mu| < d$ and we again have a contradiction. Thus the spectrum of $B(t)$ lies entirely on one half of the imaginary axis. Without loss of generality we can suppose that it lies on the positive half.

The change of variables $X = Pe^{tA}$ transforms the matrix equation

$$X' = [A + B(t)]X \tag{12}$$

into

$$P' = AP - PA + B(t)P. \tag{13}$$

Let $C(t)$ be an almost periodic matrix whose spectrum is contained in the extended spectrum of $B(t)$. We can regard the equation

$$U' = AU - AU + C(t) \tag{14}$$

as an equation of the form (6). Since the eigenvalues of the linear transformation $U \to AU - UA$ are $\alpha_j - \alpha_k$ $(j, k = 1, \dots, r)$ the condilions of Theorem 2 are satisfied. Hence the equation (14) has a unique solution $U(t)$ whose spectrum is the same as the spectrum of $C(t)$.

Starting from $U_0(t) = I$ we define a sequence $\{U_n(t)\}$ inductively by taking $U_n(t)$ to be the unique almost periodic solution of the equation

$$U' = AU - UA + B(t)U_{n-1}(t)$$

which has the same spectrum as $B(t)U_{n-1}(t)$. Since the spectrum of a product is contained in the sum of the spectra of the factors, the spectrum of $U_n(t)$ $(n \geq 1)$ is contained in the extended spectrum of $B(t)$ and consists of numbers $i\beta$, where $\beta \geq nd$. Moreover

$$\| U_n \| \leq M\chi(C/d_n) \| B \| \; \| U_{n-1} \|,$$

where d_n is the distance of the set $\{\alpha_j - \alpha_k\}$ from the spectrum of $B(t)U_{n-1}(t)$, i.e. from the spectrum of $U_n(t)$.

Choose a positive integer N so that

$$|\alpha_j - \alpha_k| < Nd \quad \text{for} \quad j, k = 1, \dots, r.$$

Then $d_n \geq (n - N)d$ for all $n > N$ and

$$\chi(C/d_n) \leq \chi\left(\frac{C}{(n-N)d}\right).$$

From the definition of $\chi(u)$ we can choose $N' \geq N$ so large that

$$\chi\left(\frac{C}{(n-N)d}\right) \leq \frac{2C}{dn} \quad \text{for} \quad n > N'.$$

Then for $n > N'$ we have

$$\| U_n \| \leq (\rho/n) \| U_{n-1} \|,$$

where $\rho = 2M \| B \| C/d$. Therefore the series $\Sigma \| U_n \|$ converges at least as fast as an exponential series.

Put

$$P(t) = \sum_{n=0}^{\infty} U_n(t) = I + Q(t).$$

Then $Q(t)$ is almost periodic and its spectrum is contained in the extended spectrum of $B(t)$. It is easily seen that $P(t)$ is a solution of the differential equation (13). Thus $X(t) = P(t)e^{tA}$ is a solution of the equation (12). It only remains to show that it is a fundamental solution.

In the same way we can show that the equation

$$P' = AP - PA - PB(t)$$

has a solution $P_1(t) = I + Q_1(t)$, where $Q_1(t)$ is an almost periodic matrix whose spectrum is contained in the extended spectrum of $B(t)$. It follows that $P_1(t)P(t)$ is a solution of the equation

$$U' = AU - UA.$$

Therefore $P_1(t)P(t) - I = Q_1(t) + Q(t) + Q_1(t)Q(t)$ is a solution of the same equation. Since its spectrum is contained in the extended spectrum of $B(t)$ it must be the zero solution. Thus $P_1(t)P(t) = I$ and the matrix $I + Q(t)$ has an inverse of the same form.

Some results in the direction of Theorem 4 have been given in the literature. PUTNAM and WINTNER [13] considered an r^{th} order scalar equation with the corresponding matrix $A = 0$. SANDOR [14] and GOLOMB [9] allowed $A \neq 0$ but made the additional assumption that $B(t)$ had an absolutely convergent FOURIER expansion.

The representation (11) shows that the equation (10) can be reduced to an autonomous equation by the almost periodic change of variables $x = [I + Q(t)]y$. Il also permits us to draw conclusions about the stability of the equation (10) in the same way that FLOQUET'S theorem enables one to draw conclusions about the stability of periodic equations. For example, (10) will be (uniformly) asymptotically stable for $t \to +\infty$ if and only if all eigenvalues of A have negative real parts.

4. - Recently MILLER [12] has proved an interesting stability criterion for the existence of almost periodic solutions. His proof depends on the theory of dynamical sistems. We give here a quite different proof, based on the properties of asymptotically almost periodic functions, which enables us to weaken the hypotheses of MILLER'S theorem and at the same time strengthen the conclusion.

A continuous function $\varphi(t)$, defined on the half-line $[0, \infty)$, is said to be ***asymptotically almost periodic*** (FRÉCHET [7], [8]) if every sequence $\{h_n\}$ of positive numbers contains a subsequence $\{k_n\}$ such that $\varphi(t+k_n)$ converges uniformly on $[0, \infty)$. Just as for almost periodic functions, there is an equivalent definition in terms of translation numbers. If $p(t)$ is almost periodic and if $\alpha(t)$ is a continuous function which tends to zero as $t \to \infty$ then

$$\varphi(t) = p(t) + \alpha(t) \tag{15}$$

is asymptotically almost periodic. FRÉCHET has shown that, conversely, any asymptotically almost periodic function can be represented in this form and that the representation is unique.

Suppose that $\varphi(t)$ is differentiable and that its derivative $\varphi'(t)$ is also asymptotically almost periodic. If

$$\varphi'(t) = q(t) + \beta(t)$$

is the corresponding representation of the derivative then for any fixed $h > 0$

$$\varphi(t+h) - \varphi(t) = \int_t^{t+h} q(s)ds + \int_t^{t+h} \beta(s)ds.$$

The first therm on the right is almost periodic, since it is bounded and its derivative is almost periodic. The second term is continuous and tends to zero as $t \to \infty$. Therefore, by the uniqueness of the representation,

$$p(t+h) - p(t) = \int_t^{t+h} q(s)ds, \quad \alpha(t+h) - \alpha(t) = \int_t^{t+h} \beta(s)ds.$$

It follows that $p(t)$ and $\alpha(t)$ are differentiable and

$$p'(t) = q(t), \quad \alpha'(t) = \beta(t).$$

THEOREM (5) - *Suppose that $f(t, x)$ is almost periodic in t for each x in the ball $|x| \leq R$ and continuous in x uniformly with respect to t. If the equation*

$$x' = f(t, x) \tag{16}$$

has a solution $\varphi(t)$ such that $|\varphi(t)| \leq M < R$ for $t \geq 0$ which is totally stable on the interval $[0, \infty)$, then it has an almost periodic solution $p(t)$ such that

$$|\varphi(t) - p(t)| + |\varphi'(t) - p'(t)| \to 0 \quad as \quad t \to \infty. \tag{17}$$

In MILLER'S formulation (17) is lacking and the solutions of any equation in the closed hull of (16) are required to be uniquely determined by their initial values.

A solution $\varphi(t)$ of the equation (16) is said to be *totally stable* (= stable under constant disturbances) on the interval $[t_0, \infty)$ if for each $\varepsilon > 0$ there exist corresponding positive numbers $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ with $\delta_1(\varepsilon) < \varepsilon$ such that

(a) $|\psi(t_1) - \varphi(t_1)| < \delta_1$ for some $t_1 \geq t_0$, and

(b) $|\psi'(t) - f[t, \psi(t)]| < \delta_2$ for those $t \geq t_1$ for which

$$|\psi(t) - \varphi(t)| < \varepsilon$$

together imply $|\psi(t) - \varphi(t)| < \varepsilon$ for every $t \geq t_1$.

If $\varphi(t)$ is a totally stable solution of the equation (16) on the interval $[t_0, \infty)$, then clearly $\varphi(t+h)$ is a totally stable solution of the equation

$$x' = f(t+h, x)$$

on the interval $[t_0 - h, \infty)$ with the same functions $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$.

We now proceed with the proof of Theorem 5. By Lemma 4 f is uniformly continuous in (t, x) and almost periodic in t uniformly with respect to x. From any sequence $\{h_n\}$ of positive numbers we can extract a subsequence $\{k_n\}$ such that $\varphi(k_n) \to \xi$ and

$$f(t + k_n, x) \to g(t, x) \quad \text{uniformly for} \quad t \in J, \ |x| \leq R.$$

Now $\varphi(t + k_n)$ is a totally stable solution of the equation $x' = f(t + k_n, x)$ and for any $\varepsilon > 0$ we can find a positive integer $N(\varepsilon)$ such that if $n \geq N(\varepsilon)$, $m \geq N(\varepsilon)$

$$|\varphi(k_n) - \varphi(k_m)| < \delta_1(\varepsilon),$$

$$|f(t + k_n, x) - f(t + k_m, x)| < \delta_2(\varepsilon) \quad \text{for} \quad t \in J, \ |x| \leq R.$$

Here $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ are the functions corresponding to $\varphi(t)$ in the definition of total stability. Then, if $n \geq N(\varepsilon)$, $m \geq N(\varepsilon)$,

$$|\varphi(t + k_m) - \varphi(t + k_n)| < \varepsilon \quad \text{for all} \quad t \geq 0.$$

Thus $\varphi(t + k_n)$ converges uniformly on the interval $[0, \infty)$. This shows that $\varphi(t)$ is asymptotically almost periodic, and so it has a representation (15). For any almost periodic function $p(t)$

$$\|p\| = \overline{\lim_{t \to \infty}} |p(t)|$$

and therefore, in our case, $|p(t)| \leq M$ for all t. Thus $f[t, p(t)]$ is almost periodic and $f[t, p(t) + \alpha(t)] - f[t, p(t)]$ tends to zero as $t \to \infty$. Therefore $\varphi'(t) = f[t, \varphi(t)]$ is asymptotically almost periodic. Moreover, by what has been said above, $p(t)$ is differentiable and $p'(t) = f[t, p(t)]$. This completes the proof.

To apply Theorem 5 we need to know when the solution $\varphi(t)$ is totally stable. As MILLER remarks, a sufficient condition is that $\varphi(t)$ be uniformly asymptotically stable and that the function $f(t, x)$ satisfy a LIPSCHITZ condition:

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2| \quad \text{for} \quad t \in J, \ |x_1| \leq R, \ |x_2| \leq R.$$

Since all proofs which I have seen of this result use the rather difficult theorem that uniform asymptotic stability implies the existence of a LYAPUNOV function the following elementary proof may be of interest.

By the definition of uniform asymptotic stability for each $\varepsilon > 0$ there exist positive numbers $\delta_1(\varepsilon)$ and $T(\varepsilon)$, with $\delta_1(\varepsilon) < \varepsilon$, such that if $x(t)$ is a solution of (16) satisfying

$$|x(t_1) - \varphi(t_1)| < \delta_1 \quad \text{for some} \quad t_1 \geq t_0$$

then

$$|x(t) - \varphi(t)| < \frac{1}{2}\varepsilon \quad \text{for all} \quad t \geq t_1$$

and

$$|x(t) - \varphi(t)| < \frac{1}{2}\delta_1 \quad \text{for all} \quad t \geq t_1 + T.$$

Set

$$\delta_2(\varepsilon) = \frac{1}{2}\delta_1 L(e^{LT} - 1)^{-1}.$$

Let $\psi(t)$ be a differentiable function such that $|\psi(t_1) - \varphi(t_1)| < \delta_1$ and let $x(t)$ be the (unique) solution of (16) such that $x(t_1) = \psi(t_1)$. If

$$|\psi'(t) - f[t, \psi(t)]| < \delta_2 \quad \text{for} \quad t_1 \leq t \leq t_2$$

then the same interval we have ([5], p. 20)

$$|\psi(t) - x(t)| \leq \delta_2 L^{-1}[e^{L(t-t_1)} - 1].$$

Thus

$$|\psi(t) - \varphi(t)| < \frac{1}{2}\varepsilon + \delta_2 L^{-1}[e^{L(t-t_1)} - 1].$$

For $t \leq t_1 + T$ the right side does not exceed $\frac{1}{2}\varepsilon + \frac{1}{2}\delta_1 < \varepsilon$. Hence if

$$|\psi'(t) - f[t, \psi(t)]| < \delta_2$$

for all $t \geq t_1$ for which $|\psi(t) - \varphi(t)| < \varepsilon$ the latter inequality must hold at least throughout the interval $[t_1, t_1 + T]$. Therefore

$$|\psi(t) - \varphi(t)| < \varepsilon \quad \text{for} \quad t_1 \leq t \leq t_1 + T$$

and

$$|\psi(t_1 + T) - \varphi(t_1 + T)| \leq |\psi(t_1 + T) - x(t_1 + T)| + |x(t_1 + T) - \varphi(t_1 + T)|$$
$$< \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1 = \delta_1 .$$

The same argument can now be repeated with $t_1 + T$ in place of t_1. Proceeding in this way we see that

$$|\psi(t) - \varphi(t)| < \varepsilon \quad \text{for all} \quad t \geq t_1 .$$

Thus $\varphi(t)$ is totally stable.

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