On the relation between Riemann and Abel summability of trigonometrical series.

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Summary: - Szász established a relation between summability $(R, 1)$ and CESÀRO summability of positive order. These results were generalised by YANO who also established the corresponding Result for summability $(R', 1)$. Generalising the result of YANO, here we shall determine the relation between summability (R', p) and ABEL summability of trigonometrical series for all finite integral values of $p \geq 1$.

1. The series $\sum_{v=1}^{\infty} a_v$ is said to be summable (R', p) to the sum zero if

$$(1.1) \quad F(t) = t \sum_{v=1}^n s_v \left(\frac{\sin vt}{vt} \right)^p$$

converges to the sum zero as $t \rightarrow 0$ for all integral values of $p \geq 1$ and s_n is the n -th partial sum of the series considered.

Hardy and Littlewood [1] proved the following theorem:

THEOREM H-L - If the series $\sum_{v=1}^{\infty} a_v$ is summable $(c, -\alpha)$ for some $\alpha > 0$

then it implies summability $(R, 1)$ of the series $\sum_{v=1}^{\infty} a_v$ to the same sum.

They have also mentioned that the result does not hold for $\alpha = 0$.

Szász [2] extended the above theorem as follows:

THEOREM S. - If the series $\sum_{v=1}^{\infty} a_v$ is $(C, 1 - \alpha)$ summable for some positive $\alpha < 1$ and if

$$(1.2) \quad \sigma_n = \sum_{v=1}^n s_v^{-\alpha} = O(n^{1-\alpha}), \text{ as } n \rightarrow \infty$$

then the series $\sum_{v=1}^{\infty} a_v$ is summable $(R, 1)$ to the same sum.

The Theorem does not hold for $\alpha = 1$.

The above theorems establish a relation between summability $(R, 1)$ and CESÀRO summability of a negative and positive order respectively.

YANO [3] has further generalised these results on summability $(R, 1)$. He also proved the corresponding results for summability $(R', 1)$. Our object here is to find a relation between ABEL summability and summability (R', p) by proving the following theorem:

THEOREM 1. - *If the series $\sum_{v=1}^{\infty} a_v$ is ABEL summable and if*

$$(1.3) \quad \sum_n^{2n} (|a_v| - a_v) = O(1)$$

then the series $\sum_{v=1}^{\infty} a_v$ is summable (R', p) to the same value for all integral values of $p \geq 1$.

2. We shall require the following lemmas:

LEMMA 1. - We write

$$(2.1) \quad \Phi(t) = \left(\frac{\sin t}{t} \right)^p$$

for all positive integral values of p , then

$$(2.2) \quad \left(\frac{\partial}{\partial t} \right)^k \Phi(t) \equiv \Phi^{(k)}(t) = O(1), \text{ as } t \rightarrow 0$$

and

$$(2.3) \quad \Phi^{(k)}(t) = O(t^{-p}), \text{ as } t \rightarrow \infty$$

for all $k = 0, 1, 2, \dots$

This is due to Kanno [4].

LEMMA 2. - If

$$(2.4) \quad \sum_n^{2n} (|a_v| - a_v) = O(1)$$

then

$$(2.5) \quad \sum_1^n a_v \equiv s_n = O(1),$$

$$(2.6) \quad \sum_1^n |a_v| = O(1)$$

and

$$(2.7) \quad \sum_n^{\infty} n^{-1} |a_n| = O(n^{-1}).$$

PROOF OF LEMMA 2. - The first part of Lemma 2 i.e. (2.5) is a corollary of an earlier result due to SZÁSZ [5].

To prove the second part (2.6) we have

$$\begin{aligned} \sum_n^{2n} |a_n| &= \sum_n^{2n} (|a_n| - a_n) + \sum_n^{2n} a_n \\ &= \sum_n^{2n} (|a_n| - a_n) + s_{2n} - s_{n-1} \\ &= O(1), \text{ by virtue of (2.4) and (2.5).} \end{aligned}$$

The third part is due to YANO (see YANO [3]).

This completes the proof of Lemma 2.

LEMMA 3 - If $\sum_{n=1}^{\infty} a_n$ is ABEL summable and if

$$(2.9) \quad \sum_n^{2n} (|a_n| - a_n) = O(1)$$

then the series is summable $(C, 1)$.

This is due to SZÁSZ [6].

LEMMA 4. - if the series

$$(2.10) \quad t \sum_{n=1}^{\infty} s_n \Phi(nt)$$

converges in $0 < t < t_0$, then

$$(2.11) \quad t \sum_{n=1}^{\infty} s_n \Phi(nt) = \sum_1^{\infty} a_n \rho_n(t)$$

Where

$$(2.12) \quad \rho_m(t) = t \sum_{n=m}^{\infty} \Phi(nt)$$

PROOF OF LEMMA 4. - By ABEL's partial summation we have

$$(2.13) \quad t \sum_n^m s_n \Phi(nt) = s_n \rho_n(t) - s_m \rho_m(t) + \sum_{n+1}^m a_n \rho_n(t)$$

and if

$$(2.14) \quad \tau_m(t) = \sum_1^m \nu t \Phi(\nu t)$$

then

$$(2.15) \quad \rho_m(t) = -\frac{\tau_m(t)}{m} + \sum_m^\infty \frac{\tau_\nu(t)}{\nu(\nu+1)}.$$

Now

$$\begin{aligned} \tau_m(t) &= \sum_1^m \nu t \Phi(\nu t) \\ &= \sum_1^m O(\nu t) \quad \text{bi Lemma 1, for } 0 < t < \frac{1}{m} < \frac{1}{m_1}, \\ &= O(t m^2) \\ &= O(t^2 m^2 \frac{1}{t}) \\ (2.16) \quad &= O(\frac{1}{t}), \text{ since } mt < 1. \end{aligned}$$

Hence

$$\begin{aligned} \rho_m(t) &= -\frac{\tau_m(t)}{m} + \sum_m^\infty \frac{\tau_\nu(t)}{\nu(\nu+1)} \\ &= O(\frac{1}{mt}) \quad \text{by (2.16).} \end{aligned}$$

Therefore by Lemma 2, since $s_m = O(1)$, as $m \rightarrow \infty$ we have $\{s_m/m\} \rightarrow 0$ as $m \rightarrow \infty$ for some fixed $t > 0$.

Thus we see that

$$\sum_{m+1}^\infty a_\nu \rho_\nu(t)$$

converges and hence by (2.13) we have

$$t \sum_n^\infty s_\nu \Phi(\nu t) = s_n \rho_n(t) + \sum_{n+1}^\infty a_\nu \rho_\nu(t)$$

which gives (2.11) for $n = 1$.

This completes the proof of Lemma 4.

LEMMA 5. - if

$$(2.17) \quad \Phi (nt) = \left(\frac{\text{Sin } nt}{nt} \right)^p$$

$$(2.18) \quad \Delta_n \Phi (nt) = \Phi (nt) - \Phi (\overline{n-1} t)$$

then

$$(2.19) \quad \Delta_n \Phi (nt) = O \left(\frac{1}{n} \right) \text{ for } t > \frac{1}{n-1}$$

PROOF OF LEMMA 5. - We have

$$\begin{aligned} \Delta_n \Phi (nt) &= \left(\frac{\text{Sin } nt}{nt} \right)^p - \left(\frac{\text{Sin } (n-1) t}{(n-1) t} \right)^p \\ &= \frac{(\text{Sin } nt) \{ (n-1)^p - n^p \}}{n^p (n-1)^p t^p} \\ &\quad + \frac{(\text{Sin } nt)^p - (\text{Sin } (n-1) t)^p}{(n-1)^p t^p} \\ (2.20) \quad &= O \left(\frac{1}{(n-1) t} \right)^p \frac{1}{n} + \frac{\Delta_n (\text{Sin } nt)^p}{(n-1)^p t^p} \end{aligned}$$

Writing $p = 2m + 1$ or $2m$ for the odd or even values of p , respectively and by applying the calculus of finite differences (See JORDON [7]), we have

$$\begin{aligned} \frac{\Delta_n (\text{Sin } nt)^p}{(n-1)^p t^p} &= \left(\frac{1}{(n-1) t} \right)^p \left\{ \frac{(-1)^m}{2^{2m-1}} \sum_{v=0}^{m+1} (-1)^v \binom{2m+1}{v} \text{Sin} \left[\frac{1}{2} t \times \right. \right. \\ &\quad \times (2m+1-2v) \Big] \cdot \text{Sin} \left[(2m+1-2v) \left(nt + \frac{1}{2} t \right) + \right. \\ &\quad \left. \left. + \frac{1}{2} \pi \right] + \frac{(-1)^m}{2^{2m-2}} \sum_{v=0}^m (-1)^v \binom{2m}{v} \text{Sin } (m-v) t \cos \left[(2m - \right. \right. \\ &\quad \left. \left. - 2v) \left(nt + \frac{1}{2} t \right) + \frac{1}{2} \pi \right] \right\} \end{aligned}$$

$$\begin{aligned}
&= O\left(\frac{1}{(n-1)t}\right)^p t \left\{ \sum_{v=0}^{m+1} \left| \binom{2m+1}{v} \frac{1}{2} (2m+1-2v) \right| \right. \\
&\quad \left. \cdot \left| \frac{\sin \frac{1}{2} t (2m+1-2v)}{\frac{1}{2} t (2m+1-2v)} \right| \right. \\
&\quad \left. + \sum_{v=0}^m \left| \binom{2m}{v} (m-v) \right| \cdot \left| \frac{\sin (m-v)t}{(m-v)t} \right| \right\} \\
&= O\left(\frac{1}{(n-1)t}\right)^p t \left\{ \sum_{v=0}^{m+1} \left| \binom{2m+1}{v} \frac{1}{2} (2m+1-2v) \right| \right. \\
&\quad \left. + \sum_{v=0}^m \left| \binom{2m}{v} (m-v) \right| \right\} \\
&\quad \text{Since } t > \frac{1}{n-1}. \\
&= O\left(\frac{1}{(n-1)t}\right)^p O(t) \\
&= O\left(\frac{1}{(n-1)t}\right)^{p-1} O\left(\frac{1}{(n-1)}\right) \\
(2.21) \quad &= O\left(\frac{1}{n}\right) \quad \text{since } t > \frac{1}{n-1}, p \geq 1.
\end{aligned}$$

Hence from (2.20), (2.21) we have

$$\begin{aligned}
\Delta_n \Phi(nt) &= O\left(\frac{1}{n}\right) + \frac{\Delta_n (\sin nt)^p}{(n-1)^p t^p} \\
(2.22) \quad &= O\left(\frac{1}{n}\right).
\end{aligned}$$

This completes the proof of Lemma 5.

3. PROOF OF THEOREM 1. - Since by (1.3) and Lemma 3 the series $\sum_{v=1}^{\infty} a_v$ is summable $(C, 1)$ and if we replace a_i by $a_i - s$, then we have

$$(3.1) \quad \frac{1}{n} \sum_{v=1}^n s_v \equiv \frac{1}{n} S_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

and hence, for a given ϵ so that $0 < \epsilon < \frac{1}{2}$, we choose $n_0(\epsilon)$ such that

$$(3.2) \quad |S_n| < \epsilon^2 n$$

for all $n > n_0(\epsilon)$.

Therefore

$$(3.3) \quad \sum_1^n v^{-1} |S_v| = O(\epsilon^2 n)$$

for all $n > n_0(\epsilon)$.

We now write

$$(3.4) \quad \begin{aligned} t \sum_1^{\infty} s_v \Phi(vt) &= t \left(\sum_1^n + \sum_{n+1}^{\infty} \right) s_v \Phi(vt) \\ &= T_1 + T_2, \text{ say.} \end{aligned}$$

By applying ABEL'S partial summation we have

$$(3.5) \quad T_1 = t S_n \Phi(nt) + \sum_1^{n-1} S_v \Delta_v \Phi(vt).$$

Now for all values of t in the interval $0 < t < \frac{1}{n_0(\epsilon)}$ we choose $n = 1 + (\epsilon t)^{-1}$ so that $n = 1 + (\epsilon t)^{-1} > (\epsilon t)$.

Therefore

$$\begin{aligned} T_1 &= t S_n \Phi(nt) + t \sum_1^{n-1} S_v \Delta_v \Phi(vt) \\ &= O(\epsilon^2) (nt)^{1-p} + O(t) \sum_{v=1}^{n-1} |S_v| v^{-1} \end{aligned}$$

by Lemma 5, Lemma 1 and the fact that $nt > \epsilon^{-1}$,

$$\begin{aligned} &= O(\epsilon^2) O\left(\frac{1}{nt}\right)^{p-1} + (\epsilon^2 nt), \text{ by (3.3)} \\ &= O(\epsilon^2 \epsilon^{p-1}) + O(\epsilon^2 \cdot (t + \epsilon^{-1})), \text{ since } n = 1 + (\epsilon t)^{-1} \\ (3.6) \quad &= O(\epsilon^{p+1}) + O(\epsilon) + O(\epsilon^2 t). \end{aligned}$$

Proceeding on the same lines as in Lemma 4, we have

$$\begin{aligned} t \sum_n^m s_n \Phi(vt) &= s_n \rho_n(t) - s_m \rho_m(t) + \sum_{n+1}^m a_v \rho_v(t) \\ &= O\left(\frac{1}{nt}\right) + O\left(\frac{1}{mt}\right) + \sum_{n+1}^m |a_v| \cdot O\left(\frac{1}{vt}\right) \end{aligned}$$

by Lemma 2.

$$= O\left(\frac{1}{nt}\right) + O\left(\frac{1}{mt}\right).$$

Therefore

$$\begin{aligned} t \sum_n^\infty s_n \Phi(vt) &= O(\epsilon) + o(1), \text{ since } nt > \epsilon^{-1} \\ &= O(\epsilon) \end{aligned}$$

which shows that the series $t \sum_n^\infty s_n \Phi(vt)$ converges as $t \rightarrow 0$ for $\epsilon(t)$ being arbitrarily small as $t \rightarrow 0$.

Therefore, by Lemma 4, we have

$$\begin{aligned} T_2 &= t \sum_{n+1}^\infty s_v \Phi(vt) \\ &= \sum_{n+1}^\infty a_v \rho_v(t) \\ &= \sum_{n+1}^\infty |a_v| \cdot O\left(\frac{1}{vt}\right) \\ &= O\left(\frac{1}{t}\right) \cdot \sum_{n+1}^\infty |a_v| \cdot v^{-1} \\ &= O\left(\frac{1}{tn}\right), \text{ by Lemma 2.} \end{aligned}$$

$$(3.7) \quad = O(\epsilon), \text{ since } nt > \epsilon^{-1}.$$

Thus from (3.4), (3.6) and (3.7) we have

$$\begin{aligned} T_1 + T_2 &= O(\epsilon) + O(\epsilon^{p+1}) + O(\epsilon^2 t) \\ &= o(1), \text{ as } t \rightarrow 0 \end{aligned}$$

since $\epsilon(t)$ is arbitrarily small when t is small.

This completes the proof of Theorem 1.

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