On the Inversion of Some Differentiable Mappings with Singularities between Banach Spaces (*) (**).

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Sunto. – *Il classico metodo di P. Levy e R. Caccioppoli è esteso al caso di applicazioni tra spazi di Banach che presentino singolarità. Viene fatta un'applicazione ad un problema non lineare in cui si ha modo di valutare esattamente il numero delle soluzioni.*

Some classical methods for the inversion of the nonlinear mappings between Banach spaces begin with the local inversion theorem, passing then to prove the invertibility in the large by various methods. In this direction it is well-known that P. Levy [1] and R. Caccioppoli [2] obtained very interesting results with many applications.

The purpose of this research is to prove that the basic idea of these methods can still be usefully employed, studying the singular set (i.e. the set where the differential is not invertible) and its image (which we call critical set). The case that we treat is the most simple in this direction: namely the case in which both the singular and the critical set are differentiable manifolds of codimension 1.

We apply the obtained results to the study of the boundary-value problem

$$\begin{cases} \Delta u + f(u) = g \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where Ω is a bounded open set sufficiently smooh, and f is a function which is linearly increasing as the argument tends to $+\infty$ and $-\infty$, but without symmetry (see § 3).

This method gives very exact results on the number of the solutions. It is interesting to observe that for our problem Leray and Schauder's method—at least in its more obvious use—gives no useful result, since the topological degree is zero.

§ 1. – This section is devoted to some simple purely topological properties concerning the inversion of mappings.

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We recall:

1.1. DEFINITION. – *A mapping $\Phi: X \to Y$ (X, Y topological spaces) is said* proper *if for every compact set $K \subset Y$, the set $\Phi^{-1}(K)$ is compact in X.*

1.2. DEFINITION. – *Let X and Y be topological spaces. A continuous mapping $\Phi: X \to Y$ is said locally invertible in $u_0 \in X$ if there exist a neighborhood U of u_0 and a neighborhood V of $y_0 = \Phi(u_0)$ such that Φ induces a homeomorphism between U and V.*

We set $N(y) = \# \Phi^{-1}(\{y\})$ (cardinal number of $\Phi^{-1}(\{y\})$).

1.3. PROPOSITION. – *Let X and Y be metrizable topological spaces, $\Phi: X \to Y$ a proper, continuous mapping which is locally invertible in every point. Then the function $y \mapsto N(y)$ is finite and locally constant.*

The fact that $N(y)$ is finite follows obviously from the fact that $\Phi^{-1}(\{y\})$ is discrete and compact. Moreover it is easy to check that $N(y)$ is upper and lower semicontinuous: then it is locally constant.

As a corollary of this proposition, we obtain that if Y is connected, then $N(y)$ is constant. Moreover, if X and Y are arcwise connected and Y is simply connected (that is every loop in Y is homotopic to a constant), then we get that $N(y) = 1$ and Φ is a global homeomorphism of X onto Y. This is the wellknown « Global inversion Theorem »; it can be proved by a method similar to the classical one, which is used for the « Monodromy Theorem » in the theory of the analytic functions.

For our purpose it is fundamental to study the set of the points at which the mapping is not locally invertible.

1.4. DEFINITION. – *Let $\Phi: X \to Y$ be a continuous mapping (X and Y topological spaces). We say that $u \in X$ is a singular point if Φ is not locally invertible in u; $y \in Y$ is said to be a critical point if $y = \Phi(u)$, for some singular point $u \in X$.*

We shall speak also, with obvious meaning, of *singular set* and of *critical set.* Clearly, the *singular set is a closed subset of X.*

The following proposition is a trivial consequence of 1.3.

1.5. PROPOSITION. – *Let X and Y be a topological metrizable spaces and $\Phi: X \to Y$ a continuous proper mapping. We denote the singular set by W. Then $N(y)$ is constant on every connected component of $Y \setminus \Phi(W)$.*

In order to obtain this proposition from 1.3, it is enough to consider the mapping $\Phi: X \setminus \Phi^{-1}\Phi(W) \to Y \setminus \Phi(W)$. It is proper and moreover it is clear that it is invertible at every point $u \in X \setminus \Phi^{-1}\Phi(W)$.

§ 2. – Now we consider the differentiable mappings between BANACH spaces.

2.1. – DEFINITION. – *Let X and Y real Banach spaces, A an open set of X. We say that $\phi: A \to Y$ is a $\mathcal{C}^k$ ($k \geq 1$) mapping if it is k-times differentiable and if the*

r-th derivative: $\phi^{(r)}$ $(1 \leqslant r \leqslant k)$ *is a continuous mapping from A in the set of the r-linear mappings of X in Y (with the usual norm). We shall use the symbol $\phi^{(r)}(u_0)[v_1][v_2] \dots [v_r]$ to denote the value that this r-linear mapping assumes when it is evaluated on the arguments $(v_1, \dots, v_r)$, for u_0 fixed.*

2.2. DEFINITION. – *We say that the C^k-mapping ϕ is locally invertible in $u_0 \in X$ if ϕ induces a C^k-diffeomorphism between an open neighborhood U of u_0 and an open neighborhood V of $y_0 = \phi(u_0)$.*

It is known that ϕ is locally invertible in u_0 if and only if the linear mapping $\phi'(u_0)\colon X \to Y$ is invertible.

The notion of local invertibility which we use here is the translation at a differential level of what we have introduced in the previous section at a purely topological level.

Clearly it is not the same as the previous one; on the other side, from now on we shall use only differentiable mappings and therefore the meaning of « local invertibility » well be always the latter. So, for example, we shall way that u_0 is a singular point for ϕ if $\phi'(u_0)$ is not invertible.

2.3. REMARK. – Obviously all the results proved in the previous section still hold if the meaning of local invertibility and of singular set is what we have pointed out in Definition 2.2.

Now we introduce some notions that we shall use in the study of the singular and critical set of a differentiable mapping.

2.4. DEFINITION. – *Let X be a Banach space. A set $M \subset X$ is said a C^k-manifold, of codimension 1, if for every point $u_0 \in M$ there exist a neighborhood U of u_0 and a C^k-functional $\Gamma\colon U \to \mathbf{R}$ such that*

a) $\Gamma'(u_0) \neq 0$;

b) $M \cap U = \{u\colon u \in U,\ \Gamma(u) = 0\}$.

It is easy to prove that a C^k-diffeomorphism transforms a C^k-manifold of codimension 1 in a manifold of the same type. Moreover it is possible, locally, to find a diffeomorphism that transforms such a manifold in a linear manifold of codimension 1.

It is interesting to see how a smooth manifold of codimension 1 disconnects the space.

2.5. PROPOSITION. – *Let M be a closed connected C^k-manifold $(k \geqslant 1)$, of codimension 1 in the Banach space X. Then $\complement M$ has at most 2 components.*

PROOF. – We suppose that there are 3 open not empty, disjoint sets A_1, A_2, A_3 such that $X \setminus M = A_1 \cup A_2 \cup A_3$. We remark that since $X \setminus M$ is open then each A_i is open not only relatively to $X \setminus M$, but also to X.

We denote by $\mathcal{F}_i$ $(i = 1, 2, 3)$ the boundary of A_i; they are not empty for if

A_i has empty boundary, it is an open-closed set of X. Obviously we have also $\mathcal{F}_i \subset M$.

In virtue of the properties of M, for every $u_0 \in M$ we can find an open neighborhood U of u_0 such that $U \cap \complement M$ has exactly 2 connected components: then only two of the sets A_i can have a non-empty intersection with U. It follows that $U \cap M$ can be contained at most in two sets $\mathcal{F}_i$.

Moreover, if it is $u_0 \in \mathcal{F}_i$ then one of the two connected components of $U \cap \complement M$ is contained in A_i; then every point on $U \cap M$ is a boundary point for A_i, that is, it belongs to $\mathcal{F}_i$. Thus the $\mathcal{F}_i$'s are closed-open sets of M. Since, by hypothesis, M is connected, we have $M = \mathcal{F}_i$. But this is not consistent with the fact, we proved above, that every point of M belongs atxmost to 2 sets $\mathcal{F}_i$. Q.E.D.

2.6. REMARK. – We do not know if, under the hypothesis of proposition 2.5., the connected components of $\complement M$ are always two.

We study now a situation, for us specially interesting, in which the singular and the critical set are differentiable manifolds of codimension 1.

2.7. THEOREM. – *Let X and Y be Banach spaces, A an open subset of X, $\phi: A \to Y$ a mapping of class C^k with $k \geq 2$.*

We suppose hat $u_0 \in A$ is such that:

I) *$\phi'(u_0)$ has kernel of dimension 1 and image of codimension 1.*

II) *If $v_0 \in X$ is a non-zero vector such that $\phi'(u_0)v_0 = 0$ and γ_0 is a functional on Y such that $\mathrm{Im}(\phi'(u_0)) = \{z: \langle z, \gamma_0\rangle = 0\}$ then the linear functional*

$$z \mapsto \langle \phi''(u_0)[z][v_0], \gamma_0\rangle$$

is not identically zero.

Then the singular set W of ϕ is, in a neighborhood of u_0, a C^{k-1}-manifold of codimension 1.

If the condition II) is replaced by:

II*) $\langle \phi''(u_0)[v_0][v_0], \gamma_0\rangle \neq 0$

then we can find an open neighborhood U of u_0 such that $\phi(W \cap U)$ is a C^{k-1}-manifold of codimension 1.

The proof of the theorem is based on the following Perturbation Lemma which is well known (it is contained in general results concerning the FREDHOLM operators). For the reader's convenience, we prefer to prove this lemma completely.

2.8. LEMMA. – *Let $T_0: X \to Y$ (X, Y Banach spaces) be a linear continuous mapping. We suppose that* Ker T_0 *and* Coker T_0 *have dimension* 1. *Then every linear mapping T near enough to T_0 (in the usual norm) either is an isomorphism of X in Y, or it has Kernel and Cokernel of dimension* 1.

PROOF OF THE LEMMA. – Let $\{\lambda v_0\}_{\lambda\in R}$ be the Kernel of T_0 and γ_0 a nonzero functional defined on Y such that $\mathrm{Im}(T_0)=\{z\colon \langle z,\gamma_0\rangle=0\}$. Let δ be a functional on X such that $\langle v_0,\delta\rangle=1$, and s an element of Y such that $\langle s,\gamma_0\rangle=1$.

The mapping $p\colon u\mapsto u-v_0\langle u,\delta\rangle$ is a projector in X; let $\hat{X}$ be the linear invariant subspace (obviously of codimension 1) related to p. In the same way, the mapping $\pi\colon y\mapsto y-s\langle y,\gamma_0\rangle$ is a projector in Y; the invariant subspace related to them, $\hat{Y}$, is equal to $\mathrm{Im}(T_0)$. Let π' be the coniugate projector of π.

Let $T\colon X\to Y$ be any linear mapping; in order to study whether T is invertible, we put $T=T_0+S$ and $v=\lambda v_0+w$, with $w\in\hat{X}$. We have to study the equation

$$T_0 w+S(\lambda v_0+w)=g \tag{2}$$

with g given in Y. Applying to (2) the projectors π and π', we have the equivalent system

$$\begin{cases} T_0 w+\pi S(\lambda v_0+w)=\pi g\,, \\ \pi' S(\lambda v_0+w)=\pi' g\,. \end{cases} \tag{3}$$

Since the operator T_0 is an isomorphism of $\hat{X}$ onto $\hat{Y}$, for S small enough, so is also $T_0+\pi S$. We put $A=(T_0+\pi S)^{-1}$.

Then the first equation of (3) is equivalent to

$$w=A(\pi g-\pi S\lambda v_0)\,. \tag{4}$$

Replacing in the second one, we have:

$$\pi' S(\lambda v_0-A\pi S\lambda v_0)=\pi' g-\pi' SA\pi g$$

that is

$$\lambda\langle Sv_0-SA\pi Sv_0,\gamma_0\rangle=\langle g-SA\pi g,\gamma_0\rangle\,.$$

Now, we distinguish two cases:

a) $\langle Sv_0-SA\pi Sv_0,\gamma_0\rangle\neq 0$ (that is the vector $Sv_0-SA\pi Sv_0$ does not belong to $\mathrm{Im}\,T_0$).

In this case, $\forall g\in Y$, the system (3) has a unique solution and hence $T\colon X\to Y$ is an isomorphism.

b) $\langle Sv_0-SA\pi Sv_0,\gamma_0\rangle=0$.

In this case, in order to obtain a solution, we must have $\langle g-SA\pi g,\gamma_0\rangle=0$. We note that, for S small enough, the functional $g\mapsto\langle g-SA\pi g,\gamma_0\rangle$ is different from zero. Then λ can assume arbitrary values, and the kernel of T has dimension 1.

2.9. REMARK. – In the case b) for the proper solutions of the homogeneous system we have $\lambda\neq 0$. Then to represent the kernel of T we can set $\lambda=1$. From (4), for $g=0$, we obtain

$$\|w\|\leqslant\|A\|\,\|\pi\|\,\|S\|\,\|v_0\|\,.$$

This relation implies that Ker T, still in case b), can be associated with the vector $v_0 + w$, with $v_0 \neq 0$, constant, and w which tends to zero as T tends to T_0.

PROOF OF THEOREM 2.7. – We put $T_0 = \phi'(u_0)$, $T = \phi'(u)$; the meaning of the other symbols is the same as in the previous Lemma.

We set: $S(u) = \phi'(u) - \phi'(u_0)$ and we denote by $A(u)$ the inverse mapping of $\phi'(u_0) + \pi S(u)$ (as mapping of $\hat{X}$ in $\hat{Y}$). It is important to remark that $u \mapsto A(u)$ is of class C^{k-1}. A point u belongs to W if and only if the case b) holds. We put

$$B(u) = \big(\phi'(u) - \phi'(u_0)\big) - \big(\phi'(u) - \phi'(u_0)\big) A(u)\pi\big(\phi'(u) - \phi'(u_0)\big)$$

then condition b) yields

$$\langle B(u)v_0, \gamma_0\rangle = 0 .$$

In order to prove that this relation is the equation of a C^{k-1}-manifold of codimension 1, we observe that $B(u)$ is of class C^{k-1} and that the differential of the functional

$$u \mapsto \langle B(u)v_0, \gamma_0\rangle$$

evaluated at the point u_0 is given by

$$z \mapsto \langle \phi''(u_0)[z][v_0], \gamma_0\rangle .$$

This linear functional, by hypothesis II), is different from zero. Thus we have proved the first statement of Theorem 2.7.

To prove the second statement, we show that under, hypothesis II*), we can build, in a neighborhood of u_0, a diffeomorphism which maps W in $\phi(W)$ (obviously, the diffeomorphism cannot be ϕ itself, since u_0 is a singular point for ϕ!).

Namely we consider the mapping $\psi\colon U \to Y$ (where U is a suitable neighborhood of u_0) so defined

$$u \mapsto \psi(u) = \phi(u) + s\langle B(u)v_0, \gamma_0\rangle .$$

Since the functional $u \mapsto \langle B(u)v_0, \gamma_0\rangle$ is zero on W, then ψ is equal to ϕ on W. The differential of ψ at the point u_0 is:

$$z \mapsto \phi'(u_0)z + s\langle \phi''(u_0)[z][v_0], \gamma_0\rangle .$$

An easy computation show that $\psi'(u_0)$ is invertible if and only if

$$\langle \phi''(u_0)[v_0][v_0], \gamma_0\rangle \neq 0 ,$$

which is exactly condition II*). Hence, if this hypothesis holds, the critical set is—locally—the image of the singular manifold under a diffeomorphism of class C^{k-1}. thus it is a C^{k-1}-manifold of codimension 1.

This achieves the proof of the Theorem.

2.10. DEFINITION. – *Given an application of class* C^k *with* $k \geq 2$, *we shall call* ordinary singular point *a point for which condition* I) *and* II*) *hold.*

If u_0 is an ordinary singular point, then we can compute locally the number of the solutions of the equation $\phi(u) = y$.

2.11. THEOREM. – *Let* $\phi: \Lambda \to Y$ (Λ *open set in a Banach space* X, Y *Banach space*) *be a mapping of class* C^k *with* $k \geq 2$ *and* $u_0 \in \Lambda$ *an ordinary singular point.*

Then, denoted by s *a vector which is transversal to* $\phi(W)$ *in* $y_0 = \phi(u_0)$, *there exist a neighborhood* U *of* u_0 *and an* $\varepsilon \in \boldsymbol{R}$ *such that*

a) $\forall y \in]y_0, y_0 + \varepsilon s]$ *the equation* $\phi(u) = y$ *has 2 solutions in* U;

b) $\forall y \in]y_0, y_0 - \varepsilon s]$ *the equation* $\phi(u) = y$ *has no solution in* U.

PROOF. – Since u_0 is an ordinary singular point, then if U is a suitable neighborhood of u_0, $\phi(W \cap U)$ is, in a neighborhood of $y_0 = \phi(u_0)$, a C^{k-1}-manifold od codimension 1.

Using the same notations of the previous Theorems, we denote by s a vector which is transversal to $\phi(W)$ in the point y_0 and set $y = y_0 + \eta s$, $\eta \in \boldsymbol{R}$.

We can assume, without loss of generality $\langle s, \gamma_0 \rangle = 1$, $u_0 = 0$ and $y_0 = \phi(u_0) = 0$. We put

$$r(u) = \phi(u) - \phi'(0)u ,$$

$$u = \lambda v_0 + w \qquad (\text{with } w \in \hat{X}) .$$

The equation $\phi(u) = y$ yields

$$\phi'(0)w + r(\lambda v_0 + w) = \eta s .$$

We transform this equation into a system using the projectors π and π'.

We have

$$\begin{cases} \phi'(0)w + \pi r(\lambda v_0 + w) = 0 , \\ \pi' r(\lambda v_0 + w) = \eta s . \end{cases} \tag{5}$$

Since the operator $\phi'(0)$ is invertible between $\hat{X}$ and $\hat{Y}$ and $r'(0) = 0$, we obtain from the first equation of (5), in virtue of «DINI's Theorem»:

$$w = \sigma(\lambda)$$

where σ is a function of class C^k, defined in a neighborhood of $0 \in \boldsymbol{R}$ and such that $\sigma'(0) = 0$.

Hence the system is reduced to the following equation

$$\pi' r(\lambda v_0 + \sigma(\lambda)) = \eta s$$

that is

$$\langle r(\lambda v_0 + \sigma(\lambda)), \gamma_0 \rangle = \eta$$

this latter equation can be studied by the following lemma, which is elementary:

2.12. LEMMA. – *Let φ be a real function of class C^2 defined in a neighborhood of $0 \in \boldsymbol{R}$, such that $\varphi(0) = 0$, $\varphi'(0) = 0$, $\varphi''(0) > 0$.*

Then there exist two positive numbers ε, τ such that:

a) $\forall \eta \in]0, \varepsilon]$ *the equation $\varphi(\lambda) = \eta$ has two solutions, of apposite sign, in $[-\tau, +\tau]$;*

b) $\forall \eta \in [-\varepsilon, 0[$ *the equation $\varphi(\lambda) = \eta$ has no solution in $[-\tau, +\tau]$.*

PROOF OF THE THEOREM 2.11. COMPLETED. – We set $\varphi(\lambda) = \langle r(\lambda v_0 + \sigma(\lambda)), \lambda_0 \rangle$. We remark that it results $\varphi(0) = \varphi'(0) = 0$, and $\varphi''(0) \neq 0$. Then by Lemma 2.12, we obtain the statement we wanted to prove. Q.E.D.

§ 3. – Now we study a non linear problem, making use of the general arguments which we have developed in the previous sections.

Let Ω be an open bounded connected subset of $\boldsymbol{R}^N$, $\partial\Omega$ its boundary and $\bar{\Omega} = \Omega \cup \partial\Omega$ its closure.

$C^k(\bar{\Omega})$ will denote the space of the functions which are k-times continuously differentiable on Ω and such that the derivatives can be extended by continuity on $\partial\Omega$. With the usual norm:

$$\|u\|_k = \sup_{0 \leqslant r \leqslant k} \sup_{x \in \Omega} |D^r u(x)| ,$$

$C^k(\bar{\Omega})$ is a BANACH space.

$C^{k,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ will denote the space of the functions $u \in C^k(\bar{\Omega})$ such that the k-th derivatives are HÖLDER-continuous with exponent α in $\bar{\Omega}$. $C^{k,\alpha}(\bar{\Omega})$ is a BANACH space under the norm

$$\|u\|_{k,\alpha} = \|u\|_k + \sup_{x \neq y} \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\alpha} .$$

$C_0^{k,\alpha}(\bar{\Omega})$ will denote the subspace of $C^{k,\alpha}(\bar{\Omega})$ consisting of the functions which are zero on $\partial\Omega$. $L^p(\Omega)$ will denote the space of the mesurable functions u, such that

$|u|^p$ are integrable, with the usual norm

$$\|u\|_{L^p(\Omega)} = \left\{ \int_\Omega |u|^p dx \right\}^{1/p}$$

$L^\infty(\Omega)$ will denote the space of the misurable essentially bounded functions, with the ess. sup. norm.

We shall say that Ω is of class $\mathcal{C}^{k,\alpha}$ if its boundary has, in the neighborhood of every point, a regular parametrization of class $\mathcal{C}^{k,\alpha}$.

Finally we recall that the classical problem:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

has countably infinite many eigenvalues $\{\lambda_n\}$, arranged according to increasing magnitude and considering their respective multiplicity. The last eigenvalue is simple: thus we have $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots$.

Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ be a function of class $\mathcal{C}^2$ satisfying the following conditions:

i) $f(0) = 0$;

ii) $f''(t) > 0 \qquad \forall t$;

iii) $\lim\limits_{t\to-\infty} f'(t) = l' \qquad$ with $0 < l' < \lambda_1$;

iv) $\lim\limits_{t\to+\infty} f'(t) = l'' \qquad$ with $\lambda_1 < l'' < \lambda_2$.

In what follows α shall be a *fixed* number in the interval $]0,1[$.

3.1. THEOREM. – *Let $\Omega \subset \boldsymbol{R}^N$ be a boundet connected open set of class $\mathcal{C}^{2,\alpha}$.*
We assume that the real function f has the properties i), ii), iii) *and* iv).
We consider the boundary-value problem

$$\text{(1)} \qquad \begin{cases} \Delta u + f(u) = g & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 . \end{cases}$$

Where g is given in $\mathcal{C}^{0,\alpha}(\bar\Omega)$ and the solution u is looked for in $\mathcal{C}_0^{2,\alpha}(\bar\Omega)$.

Then there exists in $\mathcal{C}^{0,\alpha}(\bar\Omega)$ a closed connected $\mathcal{C}^1$-manifold M of codim. 1, *such that $\mathcal{C}^{0,\alpha}(\bar\Omega)\setminus M$ consists exactly of* 2 *connected components A_1, A_2 with the following properties:*

a) if $g \in A_1$ then the problem (1) *ha no solution;*

b) if $g \in A_2$ then the problem (1) *has exactly* 2 *solution.*

Moreover if $g \in M$ then the problem (1) *has a unique solution.*

PROOF. – We consider the mapping $\phi\colon \mathrm{C}_0^{2,\alpha}(\bar{\Omega}) \to \mathrm{C}^{0,\alpha}(\bar{\Omega})$ defined by

$$\phi(u) = \Delta u + f(u) \; .$$

From the hypothesis on f (*) it follows that ϕ is of class C^2. The differential of ϕ, evaluated on $u \in \mathrm{C}_0^{2,\alpha}(\bar{\Omega})$ is given by

$$\phi'(u)\colon v \mapsto \Delta v + f'(u) v \; .$$

To complete the proof of Theorem 3.1, we state here some lemmas, which we shall prove in the following sections.

LEMMA A. – *The mapping ϕ is proper.*

LEMMA B. – *The singular set W of ϕ is not-empty, closed and connected. Every point of W is an* ordinary singular point.

LEMMA C. – *If $g \in \phi(W)$, the problem* (1) *has a unique solution.*

PROOF OF THE THEOREM COMPLETED. – First we study the properties of the critical set $\phi(W)$. By Lemma A ϕ is proper and hence, since W is closed and connected, $\phi(W)$ is also closed and connected.

We observe that, by Lemma A and C, ϕ induces a homeomorphism between W and $\phi(W)$.

Since all the points of W are ordinary, then by Theorem 2.7. $\phi(W)$ is a manifold of codimension 1. Thus, by Proposition 2.5 we can say that $\mathrm{C}^{0,\alpha}(\bar{\Omega}) \setminus \phi(W)$ has at most 2 connected components; moreover since ϕ is proper then by Proposition 1.5 we get that the number of the solutions of $\phi(u) = g$ is constant, provided g belongs to the same connected component.

To compute such number, we first observe that, for every neighborhood U of $u_0 \in W$, there exists a neighborhood V of $g_0 = \phi(u_0)$ such that $\phi^{-1}(V) \subset U$. Otherwise, there should exist an open neighborhood U^* of u_0 and a sequence u_n such that $u_n \neq U^*$ and $\lim_n \phi(u_n) = g_0$. Since ϕ is proper, we might exctract a subsequence converging to a point u^* such that $u^* \notin U^*$ and $\phi(u^*) = g_0$, $u^* \neq u_0$: this would be against Lemma C.

On the other hand, since u_0 is an ordinary singular point, by Theorem 2.11 we can compute locally the number of the solutions of the equation $\phi(u) = g$ when g lies on a segment which is transversal to $\phi(W)$ in g_0. These solutions are 2 or 0 according to the side of $\phi(W)$ on which g lies.

Hence *a*) and *b*) of the Theorem are proved.

At last if $g \in \phi(W)$, by Lemma C, the solution of $\phi(u) = g$ is unique; the proof of the Theorem is so completed. Q.E.D.

(*) For what concerns the mapping $u \mapsto f(u)$, we can factorize it in this way: $\mathrm{C}_0^{2,\alpha}(\bar{\Omega}) \to \mathrm{C}^2(\bar{\Omega}) \xrightarrow{f} \mathrm{C}^2(\bar{\Omega}) \to \mathrm{C}^{0,\alpha}(\bar{\Omega})$, where the first and last mapping are inclusions.

§ 4. – This section is devoted to the proof of Lemma A. We will recall some propositions upon the Potential Theory and the Eigenvalue Theory.

4.1. PROPOSITION. – *Let $\Omega \subset \boldsymbol{R}^N$ be a bounded set of class $C^{2,\alpha}$. Let v be the solution of the boundary-value problem*

$$\begin{cases} \Delta v = h & \text{on } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$$

where h is a bounded function. Then, for every fixed α $(0 < \alpha < 1)$ the following estimate holds

$$\|v\|_{1,\alpha} \leqslant k_\alpha \|h\|_{L^\infty}$$

where k_α is a suitable constant.

Now we consider the following eigenvalue problem, which is more general than the one we recalled in the previous section:

$$\begin{cases} \Delta v + \mu\varrho v = 0 & \text{on } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$$

chere ϱ is a measurable function bounded by two positive constants; we consider generalized solutions; Ω is a bounded connected open subset of $\boldsymbol{R}^N$. Then:

4.2. PROPOSITION. – *The eigenvalues are all positive and form a non decreasing sequence tending to $+\infty$:*

$$0 < \mu_1 \leqslant \mu_2 \leqslant \mu_3 \ldots \leqslant \mu_2 \leqslant \ldots .$$

(*We suppose every eigenvalue is repeated as many times as its multiplicity*).
For the proof: [3], Chap. VI, § 1.

4.3. PROPOSITION. – *The first eigenvalue is simple (hence it is $\mu_1 < \mu_2$); the corresponding eigenfunction does not vanish on Ω and its values are of the same sign.*

This proposition is contained in a general theorem concerning the nodes of an eigenfunction [3] (pag. 452).

4.4. PROPOSITION. – *The r-th eigenvalue μ_r is a monotone non increasing function of the coefficient ϱ. Moreover if $\varrho_1(x) < \varrho_2(x)$ a.e., then, denoted by μ_r^1 and μ_r^2 the r-th eigenvalue of $\varrho = \varrho_1$, $\varrho = \varrho_2$ respectively, we have $\mu_r^1 > \mu_r^2$.*

4.5. PROPOSITION. – *There exists a real number $p > 1$ such that the r-th eigenvalue μ_r depends continuously on ϱ in the topology of $L^p(\Omega)$.*

These propositions are an obvious consequence of the variational rappresentation of μ_r ([3], p. 406).

In order to prove Lemma A, we find an a priori-estimate for the solutions of problem (1).

4.6. LEMMA. – *Let u_n be a sequence in $C_0^{2,\alpha}(\bar{\Omega})$ and $\phi(u_n) = \Delta u_n + f(u_n) = g_n$. If the sequence g_n is bounded in $C^{0,\alpha}(\bar{\Omega})$, then the sequence u_n is bounded in $C_0^{0,\alpha}(\bar{\Omega})$.*

PROOF OF LEMMA 4.6. – Suppose that the opposite holds: namely $\lim_{n\to+\infty} \|u_n\|_{0,\alpha} = +\infty$. We set $z_n = u_n \|u_n\|_{0,\alpha}^{-1}$; we have $z_n \in C_0^{2,\alpha}(\bar{\Omega})$ and $\|z_n\|_{0,\alpha} = 1$. We introduce the real function h defined as follows:

$$h(t) = \begin{cases} \dfrac{f(t)}{t} & \text{for } t \neq 0\,, \\ f'(0) & \text{for } t = 0\,. \end{cases}$$

In virtue of the hypothesis on f, h is of class C^1 and is bounded.

From the relation $\Delta u_n + f(u_n) = g_n$, dividing by $\|u_n\|_{0,\alpha}$, we get

$$\Delta z_n + h(u_n) z_n = \frac{g_n}{\|u_n\|_{0,\alpha}}\,. \tag{6}$$

The sequence $g_n \|u_n\|_{0,\alpha}^{-1} - h(u_n) z_n$ is bounded in $L^\infty(\Omega)$. By Proposition 4.1 we have that $\|z_n\|_{0,\alpha}$ is bounded. Therefore we can extract a subsequence converging in $C_0^1(\bar{\Omega})$ (thus also in $C_0^{0,\alpha}(\bar{\Omega})$) to a function z^*; we remark that, since $\|z_n\|_{0,\alpha} = 1$ it must be $\|z^*\|_{0,\alpha} = 1$, by the continuity of the norm. In particular, it is $z^* \neq 0$.

We write (6) in generalized form:

$$-\int_\Omega \sum_i \frac{\partial z_n}{\partial x_i} \frac{\partial w}{\partial x_i}\, dx + \int_\Omega h(u_n) z_n w\, dx = \int_\Omega \frac{g_n}{\|u_n\|_{0,\alpha}} w\, dx \tag{7}$$

for every $w \in \mathcal{D}(\Omega)$ ($\mathcal{D}(\Omega)$ denotes the space of C^∞ functions having compact support contained in Ω).

We observe that in the points $x \in \Omega$ where we have $z^*(x) < 0$ it is $\lim_{n\to+\infty} u_n(x) = -\infty$ and hence $\lim_{n\to+\infty} h(u_n(x)) = l'$; so at the points where we have $z^*(x) > 0$, it results $\lim_{n\to\infty} h(u_n(x)) = l''$. Thus if we set

$$a(x) = \begin{cases} l' & \text{if } z^*(x) < 0 \\ l'' & \text{if } z^*(x) > 0 \\ f'(0) & \text{if } z^*(x) = 0 \end{cases}$$

we have in every point of Ω, $\lim_{n\to\infty} h(u_n(x)) z_n(x) = a(x) z^*(x)$.

Taking the limit, from (7) we obtain (by LEBESGUE's theorem)

$$-\int_{\Omega}\sum_i \frac{\partial z^*}{\partial x_i}\frac{\partial w}{\partial x_i}\,dx+\int_{\Omega} a z^* w\,dx=0 \qquad \forall w\in\mathfrak{D}(\Omega)$$

this relation shows that $\mu=1$ is an eigenvalue of the problem (in generalized sense)

$$\text{(8)}\qquad \begin{cases}\Delta v+\mu a v=0 & \text{on } \Omega\,,\\ v|_{\partial\Omega}=0\,. \end{cases}$$

We prove that $\mu=1$ is the least eigenvalue. If the opposite holds, we have $\mu_r=1$, with $r\geqslant 2$. We compare the problem (8) with the problem

$$\text{(9)}\qquad \begin{cases}\Delta v+\mu\lambda_2 v=0 & \text{on } \Omega\,,\\ v|_{\partial\Omega}=0\,. \end{cases}$$

Since it is $a\leqslant l''<\lambda_2$, then by the Proposition 4.4 we get that the eigenvalues of (9) are strictly less than the corresponding eigenvalues of (8).

But this is impossible, for the problem (9) should have two (at most) eigenvalues les than one, while $\mu=1$ is obviously the second eigenvalue for (9).

Then $v=z^*$ is the first eigenfunction of (8) with eigenvalue $\mu=1$. But then, by Proposition 4.3., z^* is of the same sign on the whole Ω. Now, if we suppose $z^*(x)>0$ on Ω, then the following equation is fullfilled

$$\Delta z^*+l''z^*=0\,,\qquad z^*|_{\partial\Omega}=0\,,$$

what cannot be, since l' is not an eigenvalue for $-\Delta$; on the other hand, we suppose $z^*(x)<0$ on Ω, then we have

$$\Delta z^*+l'z^*=0\,,\qquad z^*|_{\partial\Omega}=0\,.$$

Also this relation cannot be true, since l' is not an eigenvalue for $-\Delta$; this complete the proof of the Lemma 4.6.

PROOF OF THE LEMMA A. – Let u_n be a sequence on $C_0^{2,\alpha}(\bar\Omega)$ such that $\phi(u_n)=$ $=\Delta u_n+f(u_n)=g_n$ is convergent in $C^{0,\alpha}(\bar\Omega)$. By Lemma 4.6, we known that u_n is bounded in $C_0^{0,\alpha}(\bar\Omega)$ and then $\Delta u_n=g_n-f(u_n)$ is a bounded sequence in $C^{0,\alpha}(\bar\Omega)$. But since, under our hypothesis for Ω, the operator Δ is an isomorphism of $C_0^{2,\alpha}$ onto $C^{0,\alpha}$ ([4], p. 335), then we can say that u_n is a bounded sequence in $C_0^{2,\alpha}$. Hence we can extract from u_n a subsequence converging in $C^{0,\alpha}(\bar\Omega)$; then the equation itself shows that this subsequence converges in $C_0^{2,\alpha}(\bar\Omega)$. So Lemma A is completely proved.

Q.E.D.

§ 5. – This section is devoted to prove Lemma B.

First we prove that all the points of the singular manifold W are *ordinary*, namely hypothesis I) and II*) of Theorem 2.7 are fullfilled.

The differential of ϕ in a point u_0 is given—in our case—by the mapping $C_0^{2,\alpha} \to \to C^{0,\alpha}$ so defined

$$v \mapsto \Delta v + f'(u_0)v .$$

It is known that the point $u_0 \in W$ (singular set) if and only if the problem

$$\begin{cases} \Delta v + f'(u_0)v = 0 \\ v|_{\partial\Omega} = 0 \end{cases} \tag{10}$$

has proper solutions. This relation is equivalent to state that $\mu = 1$ is an eigenvalue of $\Delta v + \mu f'(u_0)v = 0 \;\; v|_{\partial\Omega} = 0$.

By Proposition 4.4, it is the least eigenvalue since is $0 < l' < f'(u_0(x)) < l''$, with $0 < l' < \lambda_1 < l'' < \lambda_2$.

Then it is a simple eigenvalue; the kernel of $\phi'(u_0)$ is associated with a non-zero vector $v_0 \in C_0^{2,\alpha}(\bar{\Omega})$. It is known that $\operatorname{Im}\phi'(u_0)$ consists of the elements $g \in C^{0,\alpha}(\bar{\Omega})$ for which it is $\int_\Omega g(x)v_0(x)dx = 0$. Then hypothesis I) of Theorem 2.7 is satisfied.

The functional γ_0 which is associated with $\operatorname{Im}\phi'(u_0)$ is

$$z \mapsto \int_\Omega z(x)v_0(x)\,dx .$$

Now we compute ϕ''; since the second differential of the linear term vanishes, we have

$$\big(\phi''(u_0)[v][w]\big)(x) = f''\big(u_0(x)\big)v(x)w(x) .$$

Then condition II*) of Theorem 2.7. becomes

$$\int_\Omega f''(u_0)v_0^3 dx \neq 0 .$$

This condition is satisfied, since $f''(t) > 0 \;\forall t$ and v_0 is of the same sign on the whole Ω, being the first eigenfunction of (10).

To complete the proof of Lemma B, we must show that W is not empty and connected. We show that W has a cartesian representation on a linear subspace of $C_0^{2,\alpha}(\bar{\Omega})$ of codimension 1.

Namely, let $s \in C_0^{2,\alpha}(\bar{\Omega})$ with $s(x) > 0 \;\forall x \in \Omega$ and let Z be any linear subspace of $C_0^{2,\alpha}(\bar{\Omega})$ of codimension 1, such that $s \notin Z$. Every element $u \in C_0^{2,\alpha}(\bar{\Omega})$ can be represented, of course, in a unique way in the form $u = z + \nu s$, $\nu \in \boldsymbol{R}$, $z \in \boldsymbol{Z}$. We consider the eigenvalue problem:

$$\begin{cases} \Delta v + \mu f'(z + \nu s)\, v = 0 & \text{on } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$$

where z is a fixed element of Z, and $\nu \in \boldsymbol{R}$. We set $\mu(\nu)$ the first eigenvalue. By Proposition 4.5, μ is a continuous function of ν. Since $s(x) > 0$ on Ω, we have $\forall x \in \Omega$:

$$\lim_{\nu \to -\infty} f'\big(z(x) + \nu s(x)\big) = l',$$

$$\lim_{\nu \to +\infty} f'\big(z(x) + \nu s(x)\big) = l''.$$

Moreover since $l' < f'(t) < l''$, we can easy prove that the previous limits are still limits also in L^p norm ($\forall p$). It follows that, by Proposition 4.5:

$$\lim_{\nu \to -\infty} \mu(\nu) = \frac{\lambda_1}{l'} > 1\,,$$

$$\lim_{\nu \to +\infty} \mu(\nu) = \frac{\lambda_1}{l''} < 1\,.$$

Thus there exists a value $\bar{\nu}$ such that $\mu(\bar{\nu}) = 1$. This value is unique since μ is a monotone strictly decreasing function (Proposition 4.4).

Then we have proved that every straight-line $\nu \mapsto z + \nu s$ meets the manifold W in a unique point; it is easy to show that this point depends continuously on Z. (Otherwise, we can recall that W is a differentiable manifold and show that the straightlines $\nu \mapsto z + \nu s$ are transverse to W). Q.E.D.

§ 6. – In this last section we prove Lemma C.

PROOF OF LEMMA C. – Let $u_0 \in W$, $\phi(u_0) = g_0$; we suppose that the equation $\phi(u) = g$ has another solution $\tilde{u}$.

We set

$$\omega(x) = \begin{cases} \dfrac{f\big(\tilde{u}(x)\big) - f\big(u_0(x)\big)}{\tilde{u}(x) - u_0(x)} & \text{where it is } \tilde{u}(x) \neq u_0(x)\,, \\ f'\big(u_0(x)\big) & \text{where it is } \tilde{u}(x) = u_0(x)\,. \end{cases}$$

Then $\tilde{u} - u_0$ is a proper solution of the problem

$$\begin{cases} \Delta v + \mu \omega v = 0 & \text{on } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$$

with $\mu = 1$. Since it is always $l' < \omega(x) < l'' < \lambda_2$, we obtain that $\tilde{u} - u_0$ is the first eigenfunction of this problem. By Proposition 4.3 $\tilde{u}(x) - u_0(x)$ has the same sign on the whole Ω and hence in virtue of the hypothesis $f''(t) > 0$, $\forall t$, it follows that $\omega(x) > f'(u_0(x))$ on Ω.

On the other hand, by hypothesis we have $u_0 \in W$; thus also the problem

$$\begin{cases} \Delta v + \mu f'(u_0) v = 0 & \text{on } \Omega \\ v|_{\partial\Omega} = 0 \,. \end{cases}$$

has $\mu = 1$ as first eigenvalue. This is against Proposition 4.4 and so Lemma C is proved.

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