Local Imbedding of Einstein Spaces.

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Summary. — *A special class of hypersurfaces of a Riemannian space is examined, this class being defined by the stipulation that the coefficients of the third fundamental form be expressible as linear combinations of the coefficients of the first and second fundamental forms. It is found that these so-called C-hypersurfaces are umbilical provided that certain conditions (which may depend on dimension) are satisfied. An $(n-1)$ -dimensional Einstein space imbedded in an n -dimensional space of constant curvature is such a C-hypersurface; accordingly the theory may be applied to the problem of the local imbedding of such spaces.*

1. — Introduction.

The feasibility of imbedding an EINSTEIN space of dimension m in an $(m+1)$ -dimensional Euclidean space presents a problem of immediate mathematical as well as physical interest. This question has received sporadic interest in the past (e.g., [2], [8]), and it is well known that such an imbedding is possible only for certain special types of EINSTEIN spaces. However, it has recently been pointed out by SZEKERES [7] that the analysis on which some of these conclusions are based is valid solely for positive definite metrics: this is a direct result of the systematic application of special coordinate systems whose introduction is not permissible when the metric happens to be indefinite. Accordingly the problem of imbedding a 4-dimensional EINSTEIN space in a 5-dimensional Euclidean space with normal hyperbolic metric is analyzed *de novo* with the aid of special algebraic techniques by SZEKERES, whose treatment of this case is exhaustive.

In the present note a similar complex of ideas is treated from a somewhat different point in view. It is readily established that, whenever a hypersurface V_{n-1} of a Riemannian space of constant curvature V_n (with $n > 3$) is an EINSTEIN space, the coefficients of its third fundamental form can be represented as a linear combination of the coefficients of its first and second fundamental forms. Accordingly hypersurfaces possessing this property—henceforth called *C*-hypersurfaces—are examined in some detail (where it should be noted that all 2-dimensional hypersurfaces of any Riemannian manifold are automatically spaces of this kind). It is found that *C*-hypersurfaces are everywhere umbilical provided that certain special criteria are satisfied. The analysis on which this result is based depends significantly on a previously established theory ([4], [5]) of the complete set of n fundamental forms

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of hypersurfaces of n -dimensional Riemannian manifolds. (In fact, certain basic theorems concerning these forms which had originally been derived for positive definite metrics are re-derived here independently of such assumptions).

The main conclusion, then, which emerges from this analysis is the fact that an EINSTEIN space V_{n-1} , imbedded in a space V_n of constant curvature, is everywhere umbilical whenever certain conditions are satisfied. It is found that the latter depend partly on dimension, which necessitates the examination of various special cases.

Our considerations are of a purely local character. The assumptions on which the theory is based may be briefly summarized as follows.

The imbedding of the hypersurface V_{n-1} in our Riemannian space V_n , the latter being referred to local coordinates x^i , is represented parametrically in the form $x^i = x^i(u^\alpha)$, where the u^α denote the parameters of V_{n-1} . [Here, and in the sequel, Latin indices assume the values $1, \dots, n$, while Greek indices range over the values $1, \dots, n-1$; the summation convention is operative in respect of both sets of indices]. We shall assume that the functions $x^i(u^\alpha)$ are of class C^3 , and that the matrix of the projection factors $B_\alpha^i = \partial x^i / \partial u^\alpha$ has maximal rank, namely $n-1$. The induced metric tensor $g_{\alpha\beta}$ of V_{n-1} is defined by

$$(1.1) \quad g_{\alpha\beta} = g_{hj} B_\alpha^h B_\beta^j,$$

where g_{hj} denotes the metric tensor of V_n . According to the equation of GAUSS ([2], p. 149) the curvature tensor of V_{n-1} is given by

$$(1.2) \quad R_{\alpha\gamma\beta\epsilon} = R_{ijhk} B_\alpha^i B_\gamma^j B_\beta^k B_\epsilon^h + e(\Omega_{\alpha\beta} \Omega_{\gamma\epsilon} - \Omega_{\alpha\epsilon} \Omega_{\gamma\beta}),$$

in which R_{ijhk} represents the curvature tensor of V_n , while $\Omega_{\alpha\beta}$ denotes the coefficients of the second fundamental form of V_{n-1} , and $e = \pm 1$ according as the square of the length of the unit normal to V_{n-1} has the value ± 1 . It is assumed throughout that V_{n-1} is such that its normal is nowhere a null vector.

If it is furthermore supposed that the imbedding space V_n is a space of constant curvature i.e., that its curvature tensor satisfies the relation

$$(1.3) \quad R_{jihk} = k(g_{jh}g_{ik} - g_{jk}g_{ih}),$$

in which k is a constant, then the equation (1.2) reduces to

$$(1.4) \quad R_{\alpha\gamma\beta\epsilon} = k(g_{\alpha\beta}g_{\gamma\epsilon} - g_{\alpha\epsilon}g_{\gamma\beta}) + e(\Omega_{\alpha\beta}\Omega_{\gamma\epsilon} - \Omega_{\alpha\epsilon}\Omega_{\gamma\beta}).$$

Thus the RICCI tensor $R_{\alpha\beta} = g^{\gamma\epsilon} R_{\alpha\gamma\beta\epsilon}$ of V_{n-1} can be represented in the form

$$(1.5) \quad R_{\alpha\beta} = k(n-2)g_{\alpha\beta} + e(H\Omega_{\alpha\beta} - C_{(\beta)\alpha\beta}),$$

where $H = g^{\gamma\delta} \Omega_{\gamma\delta}$ is the mean curvature of V_{n-1} and where $C_{(3)\alpha\beta}$ denotes the coefficients of the third fundamental form of V_{n-1} (cf. the general definition (2.1) below).

If it is also assumed that V_{n-1} is an EINSTEIN space, one has

$$(1.6) \quad (n-1)R_{\alpha\beta} = Rg_{\alpha\beta},$$

in which R represents the scalar curvature of V_{n-1} , which is constant whenever $n > 3$. Under these circumstances equation (1.5) becomes

$$(1.7) \quad C_{(3)\alpha\beta} = H\Omega_{\alpha\beta} - \mu g_{\alpha\beta},$$

where μ is given by

$$(1.8) \quad (n-1)\mu = e[R - k(n-1)(n-2)].$$

The relation (1.7) provides the motivation for the following definition. A hypersurface V_{n-1} of a Riemannian space V_n is said to be a *C-hypersurface* if the coefficients of its third fundamental form can be represented as a linear combination of the coefficients of the first and second fundamental forms as in (1.7). Indeed, any 2-dimensional hypersurface V_2 of a Euclidean or Riemannian space is a *C-hypersurface*, for, according to a well known classical theorem ([1], p. 93), the three fundamental forms of V_2 are linearly dependent, the coefficient μ in (1.7) being the Gaussian curvature of V_2 .

Evidently any hypersurface V_{n-1} of a space of constant curvature V_n is a *C-hypersurface* whenever V_{n-1} is an EINSTEIN space; again it will be seen that the coefficient μ in the defining relation (1.7) is then by no means arbitrary. In the next section we shall derive some of the basic properties of *C-hypersurfaces*, which may subsequently be applied directly to the theory of local imbeddings of Einstein spaces in spaces of constant curvature.

2. – Properties of *C-hypersurfaces*.

Since the subsequent analysis depends almost entirely on the theory of the complete set of n fundamental forms of a hypersurface V_{n-1} of a Riemannian space V_n , we shall begin by recalling some relevant facts concerning these forms ([4], [5]). The coefficients $C_{(r)\alpha\beta}$ of the r th fundamental form of V_{n-1} are defined by means of the recurrence relation

$$(2.1) \quad C_{(r)\alpha\beta} = g^{\gamma\delta} \Omega_{\alpha\delta} C_{(r-1)\gamma\beta}, \quad (r = 2, \dots, n),$$

in which $C_{(1)\alpha\beta}$, $C_{(2)\alpha\beta}$ are interpreted as $g_{\alpha\beta}$ and $\Omega_{\alpha\beta}$ respectively. Each of these coefficients gives rise to the so-called associated mean curvatures M_r , which are defined by

$$(2.2) \quad M_r = g^{\alpha\beta} C_{(r+1)\alpha\beta}, \quad (r = 0, \dots, n-1),$$

so that $M_0 = n - 1$ and $M_1 = H$. The r th curvatures H_r of V_{n-1} are, as usual, the r th elementary symmetric functions of the roots $\lambda_1, \dots, \lambda_{n-1}$ of the characteristic equation

$$(2.3) \quad \det(\Omega_{\alpha\beta} - \lambda g_{\alpha\beta}) = 0.$$

It is immediately evident from (2.1) that the eigenvalues of the matrix $(C_{(r)\beta}^{\alpha}) = (g^{\alpha\epsilon} C_{(r)\epsilon\beta})$ are $(\lambda_1)^{r-1}, \dots, (\lambda_{n-1})^{r-1}$, and thus (2.2) yields the expressions

$$M_r = (\lambda_1)^{r-1} + \dots + (\lambda_{n-1})^{r-1}, \quad (r = 1, \dots, n-1).$$

We may now relate the M_r to the H_r by means of NEWTON's formula for elementary symmetric functions ([3], p. 151): this gives

$$(2.4) \quad (-1)^{r+1} r H_r = \sum_{s=0}^{r-1} (-1)^s H_s M_{r-s},$$

in which we have put $H_0 = 1$ (which is a convention to which we shall adhere throughout), while $H_1 = H$.

Our subsequent analysis depends crucially on the fact that the n coefficients (2.1) are linearly dependent: indeed they always satisfy the relation

$$(2.5) \quad C_{(n)\alpha\beta} + \sum_{s=1}^{n-1} (-1)^s H_s C_{(n-s)\alpha\beta} = 0,$$

which is a generalization of the classical theorem referred to above.

[REMARK. – In the proof from first principles of (2.5) as given in [5] a special coordinate system is used: since this may not be permissible in the case of indefinite metrics, an alternative derivation is briefly outlined here. Let us put, in conjunction with (2.3),

$$F(\lambda) = (-1)^{n-1} \det(\Omega_{\beta}^{\alpha} - \lambda \delta_{\beta}^{\alpha}),$$

where $\Omega_{\beta}^{\alpha} = g^{\alpha\epsilon} \Omega_{\epsilon\beta}$. This is a polynomial of degree $n - 1$ in λ . It may be shown that this polynomial can be written in the form

$$F(\lambda) = \lambda^{n-1} + \sum_{r=1}^{n-1} (-1)^r H_r \lambda^{n-r-1}.$$

But according to the CAYLEY-HAMILTON theorem the matrix equation $F(\Omega_{\beta}^{\alpha}) = 0$ holds identically. An application of the recursive definition (2.1) then leads almost directly to the required relation (2.5)].

This state of affairs in turn suggests that we should define the coefficients of the

complete set of associated fundamental forms (cf. [4], p. 105) as follows:

$$(2.6) \quad (-1)^{r+1} P_{(r)\alpha\beta} = \sum_{s=0}^{r-1} (-1)^s H_s C_{(r-s)\alpha\beta}, \quad (r = 1, \dots, n).$$

From (2.2) and (2.4) one may then deduce that

$$(2.7) \quad g^{\alpha\beta} P_{(r)\alpha\beta} = (n-r) H_{r-1}, \quad (r = 1, \dots, n),$$

and

$$(2.8) \quad \Omega^{\alpha\beta} P_{(r)\alpha\beta} = r H_r, \quad (r = 1, \dots, n).$$

Furthermore, if (2.5) is applied to (2.6), it follows immediately that

$$(2.9) \quad P_{(n)\alpha\beta} = 0$$

identically, so that we may write, as usual,

$$(2.10) \quad H_n = 0.$$

These results are valid for an arbitrary hypersurface of a Riemannian space. However, let us now assume that our V_{n-1} is a C -hypersurface, i.e., that the relation (1.7) is satisfied. From (2.1) it is then evident that *each coefficient $C_{(r)\alpha\beta}$ is a linear combination of the $\Omega_{\alpha\beta}$ and $g_{\alpha\beta}$* : we shall therefore write

$$(2.11) \quad C_{(r)\alpha\beta} = A_r \Omega_{\alpha\beta} + B_r g_{\alpha\beta} \quad (r = 1, \dots, n),$$

in which the A_r , B_r are polynomials in H and μ , and where, in particular, $A_1 = 0$, $A_2 = 1$, $A_3 = H$, and $B_1 = 1$, $B_2 = 0$. By substituting (1.7) successively in (2.1) we obtain the following recurrence relations:

$$(2.12) \quad A_{r+1} = H A_r + B_r,$$

together with

$$(2.13) \quad B_{r+1} = -\mu A_r,$$

by means of which A_r , B_r can be calculated for all values of r . In fact, it is easily verified that the polynomials A_r can be represented explicitly in the form

$$(2.14) \quad A_{r+3} = H^{r+1} + \sum_{s=1}^t (-1)^s \binom{r-s+1}{s} H^{r-2s+1} \mu^s,$$

for $r = 1, \dots, n-3$, in which

$$\begin{aligned} t &= \frac{1}{2}r && \text{when } r \text{ is even,} \\ t &= \frac{1}{2}r + \frac{1}{2} && \text{when } r \text{ is odd.} \end{aligned}$$

Also, from (2.2), (2.11), (2.12), and (2.13) it follows that

$$(2.15) \quad M_r = (H^2 - (n-1)\mu) A_{r+1} - (n-1)\mu A_r,$$

and a repeated application of (2.12) and (2.13) gives

$$(2.16) \quad M_{r+1} = H M_r - \mu M_{r-1},$$

from which the associated mean curvatures are most conveniently calculated.

The relation (2.11) is now substitute in the identity (2.5). Because of (2.13) this yields

$$A_n \Omega_{\alpha\beta} + \sum_{s=1}^{n-1} (-1)^s H_s A_{n-s} \Omega_{\alpha\beta} - \mu A_{n-1} g_{\alpha\beta} - \mu \sum_{s=1}^{n-1} (-1)^s H_s A_{n-s-1} g_{\alpha\beta} = 0,$$

or, more succinctly,

$$(2.17) \quad C \Omega_{\alpha\beta} - \mu D g_{\alpha\beta} = 0,$$

in which the coefficients C and D are respectively defined by

$$(2.18) \quad C = \sum_{s=0}^{n-1} (-1)^s H_s A_{n-s}$$

and

$$(2.19) \quad D = \sum_{s=0}^{n-1} (-1)^s H_s A_{n-s-1}.$$

These coefficients may be evaluated explicitly by means of (2.4) and (2.14).

It is evident from (2.17) that our C -hypersurfaces are such that there exists a linear relation between the coefficients $\Omega_{\alpha\beta}$ and $g_{\alpha\beta}$ provided that $C \neq 0$, $\mu \neq 0$. This fact is of fundamental importance to our subsequent analysis.

Let us multiply (2.17) by $\Omega^{\alpha\beta}$, noting that according to (2.1) and (2.15)

$$\Omega_{\alpha\beta} \Omega^{\alpha\beta} = M_2 = H^2 - (n-1)\mu.$$

This gives

$$(2.20) \quad [H^2 - (n-1)\mu] C - \mu H D = 0.$$

Similarly, multiplication of (2.17) by $g^{\alpha\beta}$ yields

$$(2.21) \quad HC - (n-1)\mu D = 0.$$

Clearly the relations (2.20) and (2.21) may be regarded as a system of two linear homogeneous equations in C and D . Thus, either $C = D = 0$, in which case (2.17) is devoid of meaning, or, for $C \neq 0$, $D \neq 0$, it is necessary that the determinant of the system be zero, which requires that

$$(2.22) \quad \mu\{(n-2)H^2 - (n-1)^2\mu\} = 0.$$

Let us assume for the moment that $\mu \neq 0$. * [The case $\mu = 0$ will be treated in the next section]. Then, in order that the system (2.20) and (2.21) have non-trivial solutions, it is necessary that

$$(2.23) \quad (n-2)H^2 - (n-1)^2\mu = 0.$$

However, when $C \neq 0$, the relation (2.17) shows that our V_{n-1} is umbilical. This state of affairs is summarized by the following

THEOREM. — *A C -hypersurface V_{n-1} of a Riemannian space V_n for which the conditions $\mu \neq 0$, $C \neq 0$ are satisfied, is everywhere umbilical, the coefficient μ and the mean curvature H being related according to (2.23).*

In view of the observations of the first section this has the obvious

COROLLARY. — *An $(n-1)$ -dimensional Einstein space satisfying the conditions $\mu \neq 0$, $C \neq 0$, which is imbedded in an n -dimensional space of constant curvature is everywhere umbilical.*

This result appears to be a refinement of a theorem stated for the case of a positive definite metric by SCHOUTEN and STRUIK [6]. Before proceeding, a few general observations concerning the main theorem should be made.

REMARK 1. — It is easily seen that for any given umbilical hypersurface the conditions (1.7) and (2.23) must necessarily be satisfied. For, suppose that we are given a V_{n-1} for which

$$(2.24) \quad \Omega_{\alpha\beta} = \sigma g_{\alpha\beta},$$

with $\sigma \neq 0$, $\sigma = H/(n-1)$. From (2.1) it then follows that

$$(2.25) \quad C_{(\mathfrak{g})\alpha\beta} = H^2(n-1)^{-2}g_{\alpha\beta}.$$

This, however, can be written in the form

$$C_{(3)\alpha\beta} = H\Omega_{\alpha\beta} + \frac{2-n}{(n-1)^2} H^2 g_{\alpha\beta},$$

which indicates that *any umbilical hypersurface is a C-hypersurface for which the value of the coefficient μ is that given by (2.23).*

REMARK 2. — In connection with the stipulation $\mu \neq 0$, $C \neq 0$, as required by the theorem, it should be noted that *the condition $\mu = 0$ implies that $C = 0$* . For, from (2.16) it follows that $M_r = H^r$ whenever $\mu = 0$, and under these circumstances the relation (2.4), with $H_1 = H$, immediately yields $H_2 = H_3 = \dots = H_{n-1} = 0$, while (2.14) reduces to $A_r = H^{r-2}$. When these values are substituted in (2.18) it is seen that

$$C = H^{n-2} - H \cdot H^{n-3} = 0$$

as required. However, it will be seen in the next section that the converse of this statement is not generally valid: for certain values of n it is possible that $C = 0$ even if $\mu \neq 0$.

REMARK 3. — When $H_3, H_4, H_5, \dots$ are evaluated directly from (2.4) for $n = 3, 4, 5, \dots$ respectively, it is found that $H_3 = 0, H_4 = 0, H_5 = 0, \dots$, as required by (2.10), whenever (2.23) is satisfied. Thus one would be inclined to conjecture that the condition (2.23) can be derived directly from (2.10), which would simplify matters considerably. However, this does not appear to be the case in general. For, if we substitute from (2.15) in (2.4) with $r = n$, we find that

$$\begin{aligned} (-1)^{n+1} n H_n &= [H^2 - (n-1)\mu] \sum_{s=0}^{n-1} (-1)^s H_s A_{n-s} - H\mu \sum_{s=0}^{n-1} (-1)^s H_s A_{n-s-1} = \\ &= [H^2 - (n-1)\mu] C - H\mu D, \end{aligned}$$

where, in the second step, we have used (2.18) and (2.19). Thus the condition (2.10) merely implies (2.20).

The implications of the requirement that $C \neq 0$, which appears in the enunciation of the theorem, depend on the dimension of V_{n-1} : thus, in the next section, this condition is examined for specific values of n . This necessitates the calculation of the curvatures H_r as polynomials in H and μ , and to this end we shall now derive a recurrence formula for H_r which is valid for any C -hypersurface. Writing (2.4) in the equivalent form

$$(2.26) \quad r H_r = \sum_{s=1}^r (-1)^{s-1} H_{r-s} M_s,$$

we substitute from (2.16), noting that $M_0 = n - 1$, which gives

$$(2.27) \quad rH_r = -(n-2)HH_{r-1} + H \sum_{s=1}^r (-1)^{s-1} H_{r-s} M_{s-1} - \mu \sum_{s=2}^r (-1)^{s-1} H_{r-s} M_{s-2}.$$

However, a slight rearrangement of (2.26), with $r = n - 1$, yields

$$(2.28) \quad \sum_{s=1}^r (-1)^{s-1} H_{r-s} M_{s-1} = (n-r)H_{r-1},$$

while similarly it is found that

$$(2.29) \quad \sum_{s=2}^r (-1)^{s-1} H_{r-s} M_{s-2} = -(n-r+1)H_{r-2}.$$

Substitution of (2.28) and (2.29) in (2.27) yields the following

LEMMA. — *The r th curvature H_r of any C -hypersurface is given by the recurrence relation*

$$(2.30) \quad rH_r = -(r-2)HH_{r-1} + (n-r+1)\mu H_{r-2}.$$

By means of this formula one may obtain the following table for the r th curvatures of any C -hypersurface as polynomials in H and μ :

$$1!H_1 = H;$$

$$2!H_2 = (n-1)\mu;$$

$$3!H_3 = (n-3)H\mu;$$

$$4!H_4 = 3(n-1)(n-3)\mu^2 - 2(n-3)H^2\mu;$$

$$5!H_5 = -(n-3)(5n+7)H\mu^2 + 6(n-3)H^3\mu;$$

$$6!H_6 = 15(n-1)(n-3)(n-5)\mu^3 + 2(n-3)(5n+39)H^2\mu^2 - 24(n-3)H^4\mu;$$

$$7!H_7 = -3(n-3)(35n^2-196n+41)H\mu^3 - 2(n-3)(7n+303)H^3\mu^2 + 120H^5\mu.$$

3. — Special cases.

The criteria according to which a C -hypersurface is umbilical involve certain stipulations, namely that $\mu \neq 0$ and that $C \neq 0$. We shall now briefly examine some the consequences which result from a violation of these conditions. To this end we shall tabulate explicit expressions for the coefficients C as defined by (2.18)

for different values of n , these having been evaluated directly from the r th curvatures H_r as displayed above

$$\begin{aligned}
 (3.1a) \quad & \text{When } n = 3, \quad C = 0; \\
 (3.1b) \quad & \text{» } n = 4, \quad C = \frac{1}{2}\mu; \\
 (3.1c) \quad & \text{» } n = 5, \quad C = \frac{2}{3}H\mu; \\
 (3.1d) \quad & \text{» } n = 6, \quad C = \frac{3}{8}\mu(2H^2 + \mu); \\
 (3.1e) \quad & \text{» } n = 7, \quad C = \frac{4}{15}H\mu(3H^2 + 4\mu); \\
 (3.1f) \quad & \text{» } n = 8, \quad C = \frac{5}{144}\mu(4H^2 + 9\mu)(6H^2 + \mu).
 \end{aligned}$$

It is now evident that the circumstances which entail that $C = 0$ depend on dimension. [The fact that $C = 0$ identically for $n = 3$ merely reflects the phenomenon according to which any 2-dimensional hypersurface is a C -hypersurface].

Also, for future reference, we recall once more that, if λ is an eigenvalue of the matrix (Ω_β^α) corresponding to a principal direction v^α , then λ^2 is an eigenvalue of $(C_{(3)\beta}^\alpha)$ corresponding to the same principal direction. Thus we have, on the one hand,

$$(3.2) \quad C_{(3)\beta}^\alpha = \lambda^2 v^\alpha,$$

while, on the other, the relation (1.7) yields

$$(3.3) \quad C_{(3)\beta}^\alpha v^\beta = H\Omega_\beta^\alpha v^\beta - \mu v^\beta = (H\lambda - \mu)v^\beta,$$

so that

$$(3.4) \quad \lambda = \frac{1}{2}[H \pm (H^2 - 4\mu)^{\frac{1}{2}}].$$

[In passing we note that, if V_{n-1} is umbilical, in which case H and μ are related according to (2.23), this gives $\lambda = H(n-1)^{-1}$ as required by the general theory].

1) *The case when $\mu = 0$:*

As noted above in Remark 2, we have that $C = 0$ when $\mu = 0$; accordingly this case will be considered first. It was shown that if $\mu = 0$, then $H_2 = H_3 = \dots = H_{n-1} = 0$; thus, from the definition of H_r as the r th elementary symmetric function of the roots $\lambda_1, \dots, \lambda_{n-1}$ of the characteristic equation (2.3), it may be inferred that at most one of these roots is non-vanishing. Accordingly the rank of the matrix (Ω_β^α) is at most unity. [This may also be verified by direct calculation, it being recalled

that H_r is the sum of all principal $(r \times r)$ minors of $\det(\Omega_\beta^\alpha)$. Thus for all (2×2) sub-determinants of $\det(\Omega_\beta^\alpha)$ we must have

$$(3.5) \quad \Omega_{\alpha\beta}\Omega_{\gamma\epsilon} - \Omega_{\alpha\epsilon}\Omega_{\gamma\beta} = 0.$$

When this result is substituted in (1.4), we immediately arrive at the following conclusion: *Any C-hypersurface with vanishing coefficient μ , which is imbedded in space of constant curvature, is itself a space of constant curvature whose curvature k coincides with that of the imbedding space.* In particular, if the imbedding space is flat, then the C-hypersurface is flat whenever $\mu = 0$.

2) *The case $H = 0$:*

It is evident from (3.1c) that $C = 0$ for $H = 0$ when, for instance, $n = 5$. However, by virtue of (1.7) our C-hypersurface then satisfies the condition

$$(3.6) \quad C_{(3)\alpha\beta} = -\mu g_{\alpha\beta}.$$

Accordingly we shall such hypersurfaces *C-umbilical*. Furthermore, from (2.4) it then follows that the roots of the characteristic equation (2.3) are given by $\lambda = \pm(-\mu)^{\frac{1}{2}}$. Let us suppose that these roots have been labelled such that $\lambda_1, \dots, \lambda_p = +(-\mu)^{\frac{1}{2}}$, and $\lambda_{p+1}, \dots, \lambda_{n-1} = -(-\mu)^{\frac{1}{2}}$. Then, since $0 = H = \lambda_1 + \dots + \lambda_{n-1}$, we must have

$$(n - 2p - 1)(-\mu)^{\frac{1}{2}} = 0.$$

Then either $\mu = 0$ (which together with $H = 0$ obviously implies that our V_{n-1} is flat), or $2p = n - 1$, which is possible if and only if n is odd. This leads to the following conclusion: *An even-dimensional C-umbilical hypersurface V_{n-1} for which $H = 0$ is either flat, or $\frac{1}{2}(n - 1)$ principal curvatures have the values $+(-\mu)^{\frac{1}{2}}$, while the remaining principal curvatures have values $-(-\mu)^{\frac{1}{2}}$. All odd-dimensional C-umbilical hypersurfaces for which $H = 0$ are flat.*

3) *Some other cases which imply that $C = 0$:*

For the case $n = 6$ it is evident from (3.1d) that $C = 0$ not only when $\mu = 0$, but also when $\mu = -2H^2$. In the latter case it follows that $\lambda = 2H$, or $\lambda = -H$. By means of an argument similar to that used for case 2) it is found that under these conditions the C-hypersurface is either flat, or such that two of its principal curvatures assume the values $2H$, while the remaining three have the values $-H$.

Similarly, for the case $n = 7$, the condition $C = 0$ is satisfied whenever $\mu = 0$, or $H = 0$, or when the C-hypersurface is such that two of its principal curvatures assume the values $\frac{3}{2}H$, while the remaining four have the values $-\frac{1}{2}H$.

It is not immediately evident whether hypersurfaces whose principal curvatures have prescribed values of this kind necessarily admit meaningful realizations; presumably the latter would depend on dimension as well as on the signature of the underlying metric.

In conclusion we shall apply the results of the above theory to the special case which is of interest from the point of view of the general theory of relativity.

Let V_4 be an EINSTEIN space imbedded in a 5-dimensional Euclidean manifold. Clearly our V_4 is a C -hypersurface with the following local properties:

- (i) If $H \neq 0$, $\mu \neq 0$, the space V_4 is umbilical;
- (ii) If $\mu = 0$ (in which case the curvature scalar R of V_4 vanishes by virtue of (1.8) with $k = 0$), the space V_4 is flat;
- (iii) If $H = 0$, $\mu \neq 0$, the space V_4 is a C -umbilical non-flat hypersurface.

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