On V -Killing Forms in a Locally Conformal Kähler Manifold with the Parallel Lee Form (*).

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Summary. — We shall introduce a 1-form which is called V -Killing form in a locally conformal Kähler manifold with the parallel Lee form and prove that this form has similar properties of C -Killing form in a Sasakian manifold. We obtain that a V -harmonic form is invariant by a V -Killing form.

Introduction.

A Hermitian manifold which is locally conformal to a Kähler manifold is called a locally conformal Kähler manifold (l.c. K -manifold) and characterized as a Hermitian manifold whose structure (φ, g) satisfies with a 1-form α (called the Lee form)

$$\nabla_k \varphi_{ji} = -\dot{\beta}_j g_{ki} + \dot{\beta}_i g_{kj} - \alpha_j \varphi_{ki} + \alpha_i \varphi_{kj} (\dot{\beta}_j = \alpha^r \varphi_{rj}) \quad ([5]).$$

If moreover α is parallel, the structure is similar to a Sasakian structure. Hence we have considered in previous paper [3] certain forms which correspond to $C(C^*)$ -harmonic forms of a Sasakian manifold. In this paper we shall introduce 1-forms which seem to correspond to C -Killing 1-forms in order to make our study parallel with that of a Sasakian manifold. With such 1-forms we shall obtain the analogy of the well known property that in a compact Riemannian manifold, the Lie derivative of a harmonic form with respect to a Killing 1-form is zero.

We can also ascertain that the equalities which have been obtained in [4] almost hold in our manifold. In this paper we shall often use intensionally different calculations from [4].

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1. — Preliminaries.

Throughout this paper, the manifold $M^{2m}(\varphi, g, \alpha)$ is a $2m$ -dimensional compact l.c. K -manifold with the parallel Lee form α , $|\alpha| = 1$.

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In $M^{2m}(\varphi, g, \alpha)$, the following identities are known:

$$\begin{aligned}
 (1.1) \quad & \nabla_j \dot{\beta}_i = -\dot{\beta}_j \alpha_i + \dot{\beta}_i \alpha_j - \varphi_{ji} \quad (= -\nabla_i \dot{\beta}_j), \\
 (1.2) \quad & \nabla_j \dot{\beta}^r \nabla_r \dot{\beta}_i = \alpha_j \alpha_i + \dot{\beta}_j \dot{\beta}_i - g_{ji}, \\
 (1.3) \quad & \nabla_k \nabla_j \dot{\beta}_i = \dot{\beta}_j g_{ki} - \dot{\beta}_i g_{kj} + (\alpha_j \dot{\beta}_i - \dot{\beta}_j \alpha_i) \alpha_k = R_{jik}{}^r \dot{\beta}_r, \\
 (1.4) \quad & \nabla^r \nabla_r \dot{\beta}_i = -2(m-1) \dot{\beta}_i, \quad R_i{}^r \dot{\beta}_r = 2(m-1) \dot{\beta}_i, \\
 (1.5) \quad & R_i{}^r \nabla_r \dot{\beta}_j + R_j{}^r \nabla_r \dot{\beta}_i = 0, \\
 (1.6) \quad & \frac{1}{2} R_{rki}{}^j \nabla^r \dot{\beta}^k = -R_i{}^r \nabla_r \dot{\beta}_j + (2m-3) \nabla_i \dot{\beta}_j.
 \end{aligned}$$

We remark here

$$\dot{\beta}^r \nabla_r \dot{\beta}_j = 0, \quad \alpha^r \nabla_r \dot{\beta}_j = 0, \quad \dot{\beta}^r \nabla_r \nabla_j \dot{\beta}_i = 0$$

which are used often by a tacit understanding.

Let the operators $e(\omega)$, $i(\omega)$, L_ω and Λ_ω with respect to a 1-form ω for a p -form u be as follows:

$$\begin{aligned}
 e(\omega)u &= \omega \wedge u, \quad i(\omega)u = *e(\omega)*u, \\
 L_\omega u &= d\omega \wedge u, \quad \Lambda_\omega u = (-1)^p *L*u,
 \end{aligned}$$

where $*$ is the star operator. For a p -form $u = (1/p!) u_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$ if we denote $e(\omega)u = 1/(p+1)! (e(\omega)u)_{i_1 \dots i_{p+1}} dx^{i_1} \wedge \dots \wedge dx^{i_{p+1}}$ etc., these are written explicitly as

$$\begin{aligned}
 (e(\omega)u)_{i_0 \dots i_p} &= \sum_{k=0}^p (-1)^k \omega_{i_k} u_{i_0 \dots \hat{i}_k \dots i_p}, \\
 (i(\omega)u)_{i_2 \dots i_p} &= \omega^{i_1} u_{i_2 \dots i_p}, \\
 (L_\omega u)_{i_1 \dots i_{p+2}} &= \sum_{k < l} (-1)^{k+l+1} (\nabla_{i_k} \omega_{i_l} - \nabla_{i_l} \omega_{i_k}) u_{i_1 \dots \hat{i}_k \dots \hat{i}_l \dots i_{p+2}}, \\
 (\Lambda_\omega u)_{i_3 \dots i_p} &= \frac{1}{2} (\nabla^r \omega^s - \nabla^s \omega^r) u_{rsi_3 \dots i_p}
 \end{aligned}$$

where $u_{i_1 \dots \hat{i}_k \dots i_p}$ means that i_k is omitted. It is easily known

$$L_\omega e(\omega) = e(\omega) L_\omega, \quad \Lambda_\omega i(\omega) = i(\omega) \Lambda_\omega$$

and, if $\nabla_\omega \omega = 0$, $L_\omega i(\omega) = i(\omega) L_\omega$.

For brevity's sake, in the followings we write often e , i , L and Λ instead of $e(\dot{\beta})$, $i(\dot{\beta})$, $L_{\dot{\beta}}$ and $\Lambda_{\dot{\beta}}$.

As $\dot{\beta}$ is Killing, the Lie derivative with respect to $\dot{\beta}$ for forms

$$\theta(\dot{\beta}) = d\dot{\beta} + \dot{\beta}d$$

can be represented also ([1])

$$\theta(\dot{\beta}) = -(\delta e + e\delta)$$

where $\delta = *d*$, and then it follows naturally

$$\theta(\dot{\beta})i = i\theta(\dot{\beta}), \quad \theta(\dot{\beta})e = e\theta(\dot{\beta}).$$

Now we provide several lemmas. We can ascertain them by straightforward calculation so that the proofs are omitted.

LEMMA 1.1. – In a Riemannian manifold

$$e(\omega)i(\zeta) = -i(\zeta)e(\omega) + (i(\omega)\zeta)I$$

holds good where ω and ζ are any 1-forms and I is the identity.

LEMMA 1.2. – In a Riemannian manifold, for any 1-forms ω, ζ ,

$$\theta(\omega)i(\zeta) - i(\zeta)\theta(\omega) = i([\omega, \zeta])$$

holds good.

LEMMA 1.3. – In a Riemannian manifold, for any 1-forms ω, ζ ,

$$\theta(\omega)i(\zeta) - i(\zeta)\theta(\omega) = -\theta(\zeta)i(\omega) + i(\omega)\theta(\zeta)$$

holds good.

We have studied in [3] a V -harmonic form which is defined as a p -form u such that

$$du = 0, \quad \delta u = e(\dot{\beta}) \wedge u.$$

Regarding to V -harmonic forms, the following properties have been known:

PROPOSITION 1.4. – A form u is V -harmonic if and only if

$$i(\dot{\beta})u = 0 \quad \text{and} \quad \Delta u = L \wedge u.$$

PROPOSITION 1.5. – For any V -harmonic p -form u ,

$$\theta(\dot{\beta})u = 0$$

holds good.

PROPOSITION 1.6. — For a V -harmonic p -form u , if $p < m$,

$$i(\beta)u = 0$$

holds good.

2. — V -Killing forms.

DEFINITION. — A 1-form ω is called V -Killing if it satisfied

$$\theta(\omega)(g_{ij} - \beta_i \beta_j) = 0, \quad \delta\omega = 0$$

i.e.,

$$(*) \quad \nabla_i \omega_j + \nabla_j \omega_i = 2\omega^r (\nabla_r \beta_i \beta_j + \nabla_r \beta_j \beta_i) + \beta_i \nabla_j (\omega_r \beta^r) + \beta_j \nabla_i (\omega_r \beta^r),$$

and

$$\nabla_r \omega^r = 0.$$

Especially a V -Killing form is called special V -Killing if $\omega_r \beta^r = \text{constant}$.

REMARK. — (i) α, β are spacial V -Killing.

(ii) A 1-form ω which is Killing and special V -Killing at the same time is represented as

$$\omega = c_1 \alpha + c_2 \beta \quad (c_1, c_2: \text{constants}).$$

Because transvecting $\beta^j \nabla^i \beta_k$ to the equality $(*)$ for such a 1-form ω , we get at first

$$\omega_k = (i(\alpha)\omega)\alpha_k + (i\omega)\beta_k \quad (i\omega = \text{const}).$$

Besides, as it follows

$$(2.1) \quad \nabla_i (i(\alpha)\omega)\alpha_k + \nabla_k (i(\alpha)\omega)\alpha_i = 0,$$

transvecting (2.1) with α and vectors orthogonal to α by turns, we know finally $\nabla_X (i(\alpha)\omega) = 0$ for any vector X .

LEMMA 2.1. — If ω is a V -Killing form, then

$$(i) \quad \nabla_\beta (\omega_r \beta^r) = 0, \quad \nabla_\alpha (\omega_r \alpha^r) = 0, \quad \nabla_\beta (\omega_r \alpha^r) = 0,$$

$$(ii) \quad \theta(\beta)\omega = 0 \quad \text{i.e.,} \quad \beta^r \nabla_r \omega_j = -\omega_r \nabla_j \beta^r \quad (= \nabla_\omega \beta_j)$$

where ∇_ω means $\beta^r \nabla_r$.

PROOF. – (i) Is easily known if we transvect (*) with g^{ij} , $\alpha^i\alpha^j$, and $\beta^i\alpha^j$ making use of $\beta^r\nabla_r\beta_j = \alpha^r\nabla_r\beta_j = 0$.

(ii) Transvecting β^i to (*), we have

$$\nabla_\beta\omega_j + \beta^r\nabla_j\omega_r = 2\omega^r\nabla_r\beta_j + \nabla_j(\omega_r\beta^r)$$

and then

$$\nabla_\beta\omega_j = \nabla_\omega\beta_j \quad \text{q.e.d.}$$

By virtue of Lemma 1.2, 2.1 and Proposition 1.6, we get

LEMMA 2.2. – For a V -harmonic p -form ($p < m$) u and V -Killing form ω

$$i(\beta)\theta(\omega)u = 0$$

holds good.

Evidently for a 1-form ω with $i\omega = \text{const.}$, (*) is equivalent to

$$(**) \quad \nabla_i\omega_j + \nabla_j\omega_i = 2\omega^r(\nabla_r\beta_i\beta_j + \nabla_r\beta_j\beta_i).$$

In reality, (**) characterizes a special V -Killing form, namely

PROPOSITION 2.3. – If a 1-form ω satisfies (**), then ω is special V -Killing.

PROOF. – It is sufficient to prove that (**) implies $\delta\omega = 0$, $i\omega = \text{const.}$.

$\delta\omega = 0$ is easily known if we transvect g^{ij} to (**). Next, transvect β^i to (**). It follows then

$$\nabla_\beta\omega_j + \nabla_j(i\omega) - \omega_r\nabla_j\beta^r = 2\nabla_\omega\beta_j$$

from which, differentiating it covariantly, we have

$$\beta^r\nabla^j\nabla_r\omega_j + \nabla^j\nabla_j(i\omega) + \omega_r\nabla^j\nabla_j\beta^r = 0.$$

Hence, taking account of Ricci's identity and (1.3), we get

$$\nabla^j\nabla_j(i\omega) = 0$$

which means, as the manifold is compact, $i\omega = \text{const.}$ q.e.d.

As to a decomposition of a V -Killing form, we have

THEOREM 2.4. – (i) $f\beta$ ($f \in C^\infty(M)$) is V -Killing if and only if $\nabla_\beta f = 0$.

(ii) If ω is V -Killing, then $\omega - (i(\beta)\omega)\beta$ is special V -Killing.

Hence, $\omega = (\omega - (i(\dot{\beta})\omega)\dot{\beta}) + (i(\dot{\beta})\omega)\dot{\beta}$ is a decomposition to a special V -Killing form and V -Killing form type of $f\dot{\beta}$.

PROOF. – (i) The necessary and sufficient condition for $f\dot{\beta}$ to be V -Killing is

$$\delta(f\dot{\beta}) = -\dot{\beta}^r \nabla_r f = 0$$

because $\theta(f\dot{\beta})(g_{ij} - \dot{\beta}_i \dot{\beta}_j) = 0$ holds necessarily.

(ii) If ω is V -Killing, as $\nabla_\beta(i\omega) = 0$ (Lemma 2.1), $(i\omega)\dot{\beta}$ is V -Killing, and then so is $\omega - (i\omega)\dot{\beta}$. Furthermore $i(\omega - (i\omega)\dot{\beta}) = 0$ is trivial.

3. – Properties of special V -Killing forms.

In this section we consider especially special V -Killing forms.

The first purpose is to verify that a 1-form $\tilde{\omega}_i = \nabla_i \dot{\beta}^r \omega_r$ for a special V -Killing is closed, because this property makes some formulas practical.

LEMMA 3.1. – For a special V -Killing form ω ,

$$\nabla^r \nabla_r \omega_j + R_j{}^r \omega_r = 2(-\nabla^r \dot{\beta}^i \nabla_r \omega_i + 2m \dot{\beta}^r \omega_r) \dot{\beta}_j - 4\omega_j + \alpha^r \omega_r \alpha_j$$

holds good.

PROOF. – Differentiating (**) covariantly and taking account of Ricci's identity, (1.1), (1.3), $\dot{\beta}^i \nabla_i \nabla_r \dot{\beta}_j = 0$ and $\nabla_\omega \dot{\beta} = \nabla_\beta \omega$, we get

$$\begin{aligned} \nabla^r \nabla_r \omega_j + R_j{}^r \omega_r &= 2\nabla^i \omega^r (\nabla_r \dot{\beta}_i \dot{\beta}_j + \nabla_r \dot{\beta}_j \dot{\beta}_i) + \\ &\quad + 2\omega^r (2(m-1) \dot{\beta}_r \dot{\beta}_j + \nabla_r \dot{\beta}_i \nabla^i \dot{\beta}_j + \nabla^i \nabla_r \dot{\beta}_j \dot{\beta}_i) \\ &= 2(-\nabla_i \dot{\beta}_r \nabla^i \omega^r + 2m(\omega^r \dot{\beta}_r)) \dot{\beta}_j + 4(\omega^r \alpha_r \alpha_j - \omega_j). \quad \text{q.e.d.} \end{aligned}$$

We can now obtain the desired property:

THEOREM 3.2. – If ω is V -Killing form, then the 1-form $\tilde{\omega}_i = \nabla_i \dot{\beta}^r \omega_r$ is closed.

PROOF. – At first we notice if the above statement is verified for a special V -Killing form, then the proof is completed, because $\nu = i\epsilon\omega$ for a V -Killing form ω is special Killing (Th. 2.4) and besides $\tilde{\nu} = \tilde{\omega}$ holds good. Hence we assume that ω be a special V -Killing.

For brevity's sake we put $\nabla_i \dot{\beta}^r = J_i{}^r$. We shall know with direct calculations

$$-\tilde{\omega}^i (\delta d\omega)_i = (\nabla^r (\nabla_r (J_j{}^k \omega_k) - \nabla_j (J_r{}^k \omega_k))) \tilde{\omega}^j = 0,$$

which is the sum of the following seven terms $T_1, \dots, T_7$:

$$T_1 = J^{il}\omega_l(\nabla^r\nabla_r J_j^k)\omega_k = -2\tilde{\omega}^j\tilde{\omega}_j$$

where we use $\nabla^r\nabla_r J_j^k = -2J_j^k$,

$$T_2 = 2\nabla^r J_j^k \nabla_r \omega_k J^{il}\omega_l = 2(-\beta^k \nabla_j \omega_k) \tilde{\omega}^j = 2\tilde{\omega}_j \tilde{\omega}^j$$

where we use $\beta^r \omega_r = \text{const.}$, $\alpha^r J_{rj} = \beta^r J_{rj} = 0$,

$$\begin{aligned} T_3 &= J_j^k (\nabla^r \nabla_r \omega_k) J^{il}\omega_l \\ &= (-R_k^r \omega_r - 4\omega_k)(\omega^k - \alpha^i \omega_i \omega^k - \beta^i \omega_i \omega^k) \\ &= -R^{kr} \omega_k \omega_r + 2(m-1)(\beta^r \omega_r)^2 - 4\tilde{\omega}_j \tilde{\omega}^j \end{aligned}$$

where we use Lemma 3.1,

$$\begin{aligned} T_4 &= -(\nabla^r \nabla_j J_r^k) \omega_k J^{il}\omega_l \\ &= \tilde{\omega}_j \tilde{\omega}^j, \\ T_5 &= -\nabla_j J_r^k \nabla^r \omega_k J^{il}\omega_l \\ &= (-\beta_r \nabla^r \omega_j + \beta^r \nabla_j \omega_r) J^{il}\omega_l \\ &= 0 \end{aligned}$$

where we use Lemma 2.1 (ii),

$$\begin{aligned} T_6 &= -\nabla^r J_r^k \nabla_j \omega_k J^{il}\omega_l \\ &= -2(m-1)\tilde{\omega}_j \tilde{\omega}^j, \\ T_7 &= -J_r^k \nabla^r \nabla_j \omega_k J^{il}\omega_l \\ &= J^{jk} \nabla_r (\nabla_k \omega_j - 2\omega^s (J_{sj} \beta_k + J_{sk} \beta_j)) \tilde{\omega}^j \\ &= (-\tfrac{1}{2} J^{rk} R_{rkj}^s \omega_s + 2J^{rk} (\tilde{\omega}_j J_{rk} + \tilde{\omega}_k J_{rj})) \tilde{\omega}^j \\ &= R_k^l \omega^k \omega_l - 2(m-1)(\beta^i \omega_i)^2 + (2m+1)\tilde{\omega}_j \tilde{\omega}^j \end{aligned}$$

where we use (1.5).

Hence, $(d\tilde{\omega}, d\tilde{\omega}) = (\tilde{\omega}, \delta d\tilde{\omega}) = 0$ is shown. q.e.d.

For V -Killing forms the equality corresponding to the following theorem is considerably complicated. But for special V -Killing forms,

THEOREM 3.3. – For a special V -Killing form ω , we have

$$(3.1) \quad \nabla_k \nabla_i \omega_j - R_{ijk}{}^r \omega_r = -2 \nabla_k \tilde{\omega}_i \dot{\beta}_j + 2 \omega^r (\nabla_r \dot{\beta}_k \nabla_i \dot{\beta}_j + \nabla_r \dot{\beta}_i \nabla_k \dot{\beta}_j).$$

PROOF. – Let ω be a special V -Killing. Then we can write

$$(3.2) \quad \frac{1}{2} \nabla_k (\nabla_i \omega_j + \nabla_j \omega_i) = -\nabla_k (\tilde{\omega}_i \dot{\beta}_j + \tilde{\omega}_j \dot{\beta}_i).$$

Interchanging in (3.2) $k \rightarrow i \rightarrow j \rightarrow k$ which is put as (3.2)', and $k \rightarrow j \rightarrow i \rightarrow k$ (3.2)", we have from (3.2) + (3.2)' – (3.2)",

$$\begin{aligned} & \nabla_k \nabla_i \omega_j - R_{ijk}{}^r \omega_r \\ &= -(\nabla_k \tilde{\omega}_i + \nabla_i \tilde{\omega}_k) \dot{\beta}_j + (\nabla_i \tilde{\omega}_k - \nabla_k \tilde{\omega}_j) \dot{\beta}_i + (\nabla_j \tilde{\omega}_i - \nabla_i \tilde{\omega}_j) \dot{\beta}_k - 2 \tilde{\omega}_k \nabla_i \dot{\beta}_j - 2 \tilde{\omega}_i \nabla_k \dot{\beta}_j. \end{aligned}$$

Hence, by virtue of Theorem 3.2, we can obtain

$$\nabla_k \nabla_i \omega_j - R_{ijk}{}^r \omega_r = -2 \nabla_k \tilde{\omega}_i \dot{\beta}_j - 2 (\tilde{\omega}_k \nabla_i \dot{\beta}_j + \tilde{\omega}_i \nabla_k \dot{\beta}_j). \quad \text{q.e.d.}$$

Now we treat with the main theorem. For it, we provide some more formulas.

LEMMA 3.4. – For a special V -Killing form ω , we have

$$e(\omega) \delta + \delta e(\omega) = -\theta(\omega) - 2e(\dot{\beta}) i(\tilde{\omega}) - 2e(\tilde{\omega}) i(\dot{\beta}).$$

PROOF. – Let u be any p -form. Then

$$\begin{aligned} & (e(\omega) \delta u + \delta e(\omega) u)_{i_1 \dots i_p} \\ &= (e(\omega) \delta u)_{i_1 \dots i_p} - \nabla^r (\omega_r u_{i_1 \dots i_p} + (-1)^k \omega_{i_k} u_{r i_1 \dots \hat{i}_k \dots i_p}) \\ &= -\omega^r \nabla_r u_{i_1 \dots i_p} - (-1)^k \nabla^r \omega_{i_k} u_{r i_1 \dots \hat{i}_k \dots i_p}. \end{aligned}$$

Here if we make use of (**), namely

$$u_{i_1 \dots r \dots i_p} \nabla^r \omega_{i_k} = -u_{i_1 \dots r \dots i_p} \nabla_{i_k} \omega^r - 2(\beta \wedge i(\tilde{\omega}) u)_{i_1 \dots i_p} - 2(\tilde{\omega} \wedge i u)_{i_1 \dots i_p}$$

it follows finally

$$e(\omega) \delta u + \delta e(\omega) u = -\theta(\omega) u - 2\beta \wedge i(\tilde{\omega}) u - 2\tilde{\omega} \wedge i u. \quad \text{q.e.d.}$$

From the definition $A_\omega = i(\omega) \delta + \delta i(\omega)$, we get easily

LEMMA 3.5. – For any 1-form ω ,

$$(3.3) \quad \Delta i(\omega) - i(\omega)\Delta = dA_\omega - A_\omega d + \delta\theta(\omega) - \theta(\omega)\delta.$$

LEMMA 3.6. – For a special V -Killing form ω and a V -harmonic p -form u , $p < m$,

$$(\delta\theta(\omega) - \theta(\omega)\delta) = 2e(\tilde{\omega})\Delta u.$$

PROOF. – At first we remark that $\tilde{\omega}$ is orthogonal to β , $\nabla_{\tilde{\omega}}\beta = \nabla_\beta\tilde{\omega}$, and then

$$i(\tilde{\omega})e = -ei(\tilde{\omega}), \quad i(\tilde{\omega})\theta(\beta) = \theta(\beta)i(\tilde{\omega})$$

because of Lemma 1.1, 1.2.

Now taking account of Lemma 3.4 and $iu = \theta(\beta)u = 0$ for a V -harmonic p -form ($p < m$) (Prop. 1.5, 1.6), we get

$$\begin{aligned} \delta\theta(\omega)u &= -\delta(e(\omega)\delta + 2ei(\tilde{\omega}))u \\ &= -\delta e(\omega)\delta u + 2(\theta(\beta)i(\tilde{\omega}) + e\delta i(\tilde{\omega}))u \\ &= -\delta e(\omega)\delta u + 2e\delta i(\tilde{\omega})u. \end{aligned}$$

Furthermore, because $\tilde{\omega}$ is closed and then

$$A_{\tilde{\omega}}u = 0 \quad \text{i.e.} \quad \delta i(\tilde{\omega})u = -i(\tilde{\omega})\delta u,$$

it follows $\delta\theta(\omega)u = -\delta e(\omega)\delta u$.

On the other hand as

$$\begin{aligned} \theta(\omega)\delta u &= -\delta e(\omega)\delta u - 2(ei(\tilde{\omega})\delta u + e(\tilde{\omega})i\delta u) \\ &= -\delta e(\omega)\delta u - 2e(\tilde{\omega})\Delta u, \end{aligned}$$

the lemma is proved.

q.e.d.

Let ω , u be a V -Killing 1-form, a V -harmonic p -form ($p < m$), respectively. From above lemma, (3.3) is rewritten as

$$\Delta i(\omega)u - i(\omega)\Delta u = dA_\omega u + 2e(\tilde{\omega})\Delta u.$$

Hence we get

$$\begin{aligned} \delta\Delta i(\omega)u &= \Delta i(\omega)u - d\delta i(\omega)u \\ &= (i(\omega)\Delta u + dA_\omega u + 2e(\tilde{\omega})\Delta u) - (dA_\omega u - d i(\omega)\delta u) \\ &= \theta(\omega)e\Delta u + 2e(\tilde{\omega})\Delta u \\ (3.4) \quad &= e\theta(\omega)\Delta u \end{aligned}$$

because we know by direct calculations

$$\theta(\omega)e = e\theta(\omega) - 2e(\bar{\omega})$$

for a special V -Killing 1-form ω .

From (3.4) and $iu = 0$, it follows that

$$\begin{aligned} (di(\omega)u, di(\omega)u) &= (i(\omega)u, \delta di(\omega)u) \\ &= (ii(\omega)u, \theta(\omega)\Delta u) = 0. \end{aligned}$$

Thus we can obtain

$$(3.5) \quad \theta(\omega)u = di(\omega)u = 0$$

for a special form ω and V -harmonic $p(< m)$ -form u .

Besides if we notice

$$\theta(f\beta)u = f\theta(\beta)u + df \wedge iu \quad (f \in C^\infty(M))$$

and Theorem (2.4) (ii), we know (3.5) is valid for a V -Killing 1-form. Namely,

THEOREM 3.8. – In a $2m$ -dimensional compact l.c. K -manifold with the parallel Lee form,

$$\theta(\omega)u = 0$$

holds good for a V -Killing 1-form ω and a V -harmonic p -form u ($p < m$).

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