Pontryagin Type Dualities over Commutative Rings (*) (**).

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Summary. – *See Introduction.*

1. – Introduction.

Before describing the purpose of this work, we need some definitions.

Let R be a commutative ring with $1 \neq 0$ endowed with the discrete topology. We denote by TMR the category of topological and Hausdorff R-modules and by CMR the subcategory consisting of compact modules. Mod-R will be the category of R-modules endowed with the discrete topology.

Let $E \in CMR$ be a faithful module. Following [0], we denote by $\mathfrak{D}(E)$ the full subcategory of Mod-R consisting of the E-discrete modules, that is the modules which are algebraically isomorphic to submodules of direct products of copies of E; and by $\mathcal{C}(E)$ the category of E-compact modules, that is the modules belonging to TMR which are topologically isomorphic to closed submodules of topological products of copies of E.

If $M \in TMR$, a character of M (or also an E-character) is, by definition, a continuous morphism of M into E.

For every $M \in TMR$, the character module $\overline{M}$ of M is the R-module $\mathrm{Chom}_R(M, E)$ endowed with the compact-open topology.

It is easily seen, that, if $M \in \mathfrak{D}(E)$ or $M \in \mathcal{C}(E)$, then $\overline{M} \in \mathcal{C}(E)$ or $\overline{M} \in \mathfrak{D}(E)$, and that the assignation $M \mapsto \overline{M}$ defines functors $\Delta_1\colon \mathfrak{D}(E) \to \mathcal{C}(E)$ and $\Delta_2\colon \mathcal{C}(E) \mapsto \mathfrak{D}(E)$.

We denote by Δ_E the pair (Δ_1, Δ_2) and we say, indifferently, that Δ_E or Δ_1 is a duality if for every $M \in \mathfrak{D}(E)$ and every $M \in \mathcal{C}(E)$, the canonical morphism ω_M of M into its bidual $\overline{\overline{M}}$, defined by $\omega_M(x)(f) = f(x)$ for all $x \in M$ and $f \in \overline{M}$, is a topological isomorphism.

DEFINITION 1. – We say that E has property P_1) if the continuous morphisms of E are multiplications by elements of R.

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DEFINITION 2. – We say that E has property P_2) if E does not have small submodules, that is, there exists a neighbourhood of zero in E such that the only submodule contained in it is the zero module.

DEFINITION 3. – We say that E has property P_3) if E is topologically quasi-injective, that is every character of a closed submodule of E has an extension over E.

The aim of this paper is to characterize the faithful and compact modules $E \in CMR$ for which the functor $\Delta_1: \mathfrak{D}(E) \to \mathcal{C}(E)$ is a duality.

We obtain a generalization of Orsatti's results given in [0] and the main theorem we can prove is the following:

THEOREM. – *Let $E \in CMR$ be a faithful module.*

The following conditions are equivalent:

(*a*) $\Delta_1: \mathfrak{D}(E) \to \mathcal{C}(E)$ *is a duality.*

(*b*) E *has properties* P_1), P_2), P_3).

(*c*) *If* $P = \mathrm{Chom}_{\boldsymbol{Z}}(E, \boldsymbol{K})$ ($\boldsymbol{K}$ denotes the compact group of complex numbers of modulo 1), P *is a projective and finitely generated R-module with endomorphism ring isomorphic to* R.

Moreover, if any of the previous conditions holds, then:

1) $\mathfrak{D}(E) = \text{Mod-}R$,

2) $\mathcal{C}(E) = CMR$,

and, if Γ denotes the Pontryagin duality between Mod-R and CMR then:

3) $\Delta_1(M) = \Gamma(M \otimes_R P)$, *for every* $M \in \mathfrak{D}(E)$,

4) $\Delta_2(M) = \mathrm{Hom}_R(P, \Gamma(M))$, *for every* $M \in \mathcal{C}(E)$,

that is the duality Δ_E is the composition of the equivalence $\otimes_R P: \text{Mod-}R \to \text{Mod-}R$ *with the Pontryagin duality.*

In particular this theorem classifies all character dualities, induced by compact modules, between Mod-R and CMR.

2. – Preliminary results.

Throughout this paragraph we suppose that $E \in CMR$ is a faithful module having properties P_1), P_2), P_3).

LEMMA 2.1. – *Let $n \in N$, B a closed submodule of E^n, ζ a character of B. Then ζ extends to a character $\bar{\zeta}$ of E^n.*

PROOF: (ispired by [L] Lemma 4.1.). – We proceed by induction on n. For $n = 1$, statement of the lemma is property P_3). Let $n > 1$ and let us think of E^n as the topological product $E^1 \times E^{n-1}$. By the inductive hypothesis, the restriction of ζ to $B \cap E^{n-1}$ extends to a character η of E^{n-1}. Now we extend η to a morphism $\mu\colon B + E^{n-1} \to E$, by putting $\mu(b + x) = \zeta(b) + \eta(x) \cdot (b \in B,\ x \in E^{n-1})$.

μ is well defined, because if $b + x = 0$, then $x = -\,b \in B \cap E^{n-1}$ and so $\eta(x) = \zeta(x) = -\,\zeta(b)$.

μ is a continuous morphism, as we can see by examining the following commutative diagram:

$$\begin{tikzcd} B \times E^{n-1} \arrow[r, "\zeta'"] \arrow[d, "\sigma"'] & E \\ B + E^{n-1} \arrow[ur, "\mu"'] & \end{tikzcd}$$

where σ is the sum and ζ' is defined by $\zeta'(b, x) = \zeta(b) + \eta(x)$.

In fact, ζ' and σ are continuous, σ is surjective and moreover, since all modules appearing are compact and Hausdorff, $B + E^{n-1}$ has the quotient topology of σ.

Let now $\pi_1\colon E^1 \times E^{n-1} \to E$ be the canonical projection; it is $B + E^{n-1} = \pi_1(B) + E^{n-1}$, Let μ' and μ'' be the restrictions of μ to $\pi_1(B)$ and E^{n-1} respectively. By the inductive hypothesis μ' extends to a character $\bar{\mu}$ of E Then $\bar{\zeta} = \bar{\mu} \oplus \mu''$ is the required extension of ζ. //

We note that a topological and Hausdorff R-module belongs to $\mathcal{C}(E)$ if and only if it has the weak topology of its characters and it is complete in that topology.

DEFINITION 2.1. – We say that $\mathcal{C}(E)$ has the character extension property (C.E.P.) if for every $M \in \mathcal{C}(E)$, every character of a submodule of M extends over M.

PROPOSITION 2.2. – *$\mathcal{C}(E)$ has the C.E.P.*

PROOF. – Cfr. [0] Prop. 3.6. //

PROPOSITION 2.3. – *If $M \in \mathfrak{D}(E)$ or $M \in \mathcal{C}(E)$, then ω_M is a topological embedding.*

PROOF. – Cfr. [0] Prop. 2.3. //

PROPOSITION 2.4. – *If $M \in \mathfrak{D}(E)$, ω_M is an isomorphism.*

PROOF. – Cfr. [0] Prop. 3.4. //

DEFINITION. – We say that a topological R-module M is topologically quasi-injective in the strong sense (s.q.i.) if M is topologically quasi-injective and if, for every closed submodule N of M and every $x_0 \in M \setminus N$, there exists a character of M which is zero on N and different from zero on x_0.

LEMMA 2.5. – *Let E be s.q.i., then E^n is s.q.i.*

PROOF. – Let B be a closed submodule of E^n, $x_0 \in E^n \setminus B$.

Since E is s.q.i., we proceed by induction on n.

Let $n > 1$ and write E^n as the topological product $E^1 \times E^{n-1}$. We denote by π_1 and π_2 the canonical projections of E^n onto E^1 and E^{n-1} respectively. Then $x_0 = (a, b)$, with $a = \pi_1(x_0)$ and $b = \pi_2(x_0)$.

Put $B_1 = B \cap E^1$, $B_2 = B \cap E^{n-1}$. Without loss in generality we can suppose $a \notin B_1$

We distinguish two cases.

1) $b \in \pi_2(B)$.

Consider $E^1 + B$ and let $c \in E$ such that $(c, b) \in B$. We write $x_0 = (a, b)$ in the from $(a, b) = (a - c, 0) + (c, b)$. Then $a - c \notin B_1$, because otherwise (a, b) belongs to B.

Let $\eta \colon E^1 \to E$ be a character of E such that $\eta(B_1) = 0$ and $\eta(a - c) \neq 0$. We define a morphism $\mu \colon E^1 + B \to E$ by putting $\mu(x + b) = \eta(x)$ $(x \in E^1, b \in B)$. μ is well defined because $\mu|_{B \cap E^1} = \eta|_{B_1} = 0$.

The continuity of μ can be proved, as in the proof of Lemma 2.1., by considering the following commutative diagram:

$$\begin{tikzcd} E^1 \times B \arrow[r, "\zeta"] \arrow[d, "\sigma"'] & E \\ E^1 + B \arrow[ur, "\mu"'] & \end{tikzcd}$$

where σ is the sum and $\zeta(x, b) = \eta(x)$. $(x \in E, b \in B)$. Obviously $\mu(B) = 0$ and, moreover $\mu(a, b) = \mu((a - c, 0) + (c, b)) = \eta(a - c) \neq 0$. Now, by extending μ to a character of E^n (Lemma 2.1.), we can conclude.

2) $b \notin \pi_2(B)$. By the inductive hypothesis, there exists a character of E^{n-1} such that $\eta(\pi_2(B)) = 0$ and $\eta(b) \neq 0$.

Define $\mu \colon E^1 \oplus E^{n-1} \to E$ by putting $\mu|_{E^1} = 0$ and $\mu|_{E^{n-1}} = \eta$. μ is zero on $E^1 + B$, hence $\mu(B) = 0$ and moreover $\mu(a, b) = \eta(b) \neq 0$. //

We can now prove the main result of this paragraph.

THEOREM 2.7. – *Let $E \in CMR$ be a faithful module having properties P_1), P_2), P_3). The following conditions are equivalent:*

(*a*) *E is s.q.i.*

(*b*) *Δ_1 is a duality between $\mathfrak{D}(E)$ and $\mathfrak{C}(E)$.*

PROOF. – By Prop. 2.4., every E-discrete module is reflexive. By [0] Theor. 3.6. and by Prop. 2.2., Δ_1 is a duality iff E^n is s.q.i., hence by Lemma 2.5., iff E is s.q.i. //

As we shall see later on, the fact that E is s.q.i., is indeed a consequence of properties P_1), P_2), P_3) and of the Peter-Weyl theorem.

3. – The Pontryagin duality.

Let us consider the compact topological group $\boldsymbol{K}$ of complex numbers of modulo 1.

It is well known that $\boldsymbol{K}$, as a $\boldsymbol{Z}$-module, has properties $P_1)$, $P_2)$, $P_3)$ and that $\boldsymbol{K}$ is s.q.i.

Theorem 2.7. ensures that $\Delta_{\boldsymbol{K}}$ is a duality.

In this case $\mathfrak{D}(\boldsymbol{K}) = \text{Mod-}\boldsymbol{Z}$, $\mathcal{C}(\boldsymbol{K})$ is the category of $\boldsymbol{K}$-compact groups. By the Peter-Weyl theorem, every compact group is $\boldsymbol{K}$-compact, therefore $\Delta_{\boldsymbol{K}}$ is the Pontryagin duality between discrete and compact groups.

4. – The R-module P.

In this paragraph we first prove the equivalence of (*b*) and (*c*) of the conditions stated in the main theorem. Next, we prove the equalities 1) and 2) of the same theorem and finally the implication $(b) \Rightarrow (a)$.

We state in advance the following important remarks whose proofs are based on results of Pontryagin's theory on duality.

OBSERVATION 4.1. – We recall that Mod-R denotes the category of discrete R-modules and CMR the category of compact and Hausdorff R-modules. For every $M \in \text{Mod-}R$ or $M \in CMR$, the Pontryagin dual $M^* = \text{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{K})$ of M, endowed with the compact-open topology, is an object of CMR or Mod-R respectively.

In fact, M^* is an R-module via the multiplication:

$$r \in R\,, \qquad \alpha \in M^*\; (r\alpha)(x) = \alpha(rx) \qquad \text{for all } x \in M\,.$$

Moreover, if $f \in \text{Chom}_R(M, N)$ is a morphism in Mod-R or in CMR, its transpose $f^*\colon N^* \to M^*$ defined by $f^*(\beta) = \beta \circ f$, $\forall \beta \in N^*$, is an R-morphism; therefore $f^* \in \text{Chom}_R(N^*, M^*)$ and so it is a morphism in CMR or in Mod-R respectively.

Thus we can state that the functor:

$$\Gamma\colon \text{Mod-}R \to CMR\,, \qquad M \rightarrowtail M^* = \text{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{K})\,,$$

is a duality and hence, that, for every pair M, N of objects of Mod-R or CMR, $\text{Chom}_R(M, N)$ is canonically isomorphic to $\text{Chom}_R(N^*, M^*)$.

Moreover, Γ sends exact sequences into exact sequences.

Obviously, $\mathcal{C}(E)$ is a full subcategory of CMR, so for every $M \in \mathcal{C}(E)$ we can construct the Pontryagin dual M^*.

From now on, it is assumed that $E \in CMR$ is a faithful module.

Let $P = E^*$ be the Pontryagin dual of E. By the previous remark $P \in \text{Mod-}R$ and $P^* = E^{**}$ is topologically isomorphic to the compact R-module E. We can, thus, identify E and P^*.

OBSERVATION 4.2. – The map that to each closed submodule B of E associates its orthogonal $B^\perp = \{f \in P: f(B) = 0\}$ gives an antiisomorphism between the lattice of closed submodules of E and the lattice of R-submodules of P, whose inverse is given by $L \mapsto L^\perp$, where L is an R-submodule of P and $L^\perp = \{x \in E: x(L) = 0\}$.

OBSERVATION 4.3. – The topology on E coincides with the finite topology of $\mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{K})$. Therefore, a basis of neighbourhoods of zero in E, is given by the subsets of the type $W(F; U)$, where F is a finite subset of P, U is a neighbourhood of zero in $\mathbf{K}$ and $W(F; U) = \{x \in E: x(F) \subseteq U\}$.

We note that, if U is a small neighbourhood of zero in $\mathbf{K}$, then $W(F; U)$ contains a maximal submodule and precisely the R-module $\langle F\rangle_R^\perp$ where $\langle F\rangle_R$ is the R-submodule of P generated by F.

In fact, let $V \neq 0$ be a submodule of E contained in $W(F; U)$ and let $0 \neq x \in V$.

Then, for each $f \in F$ and for each $r \in R$, $(rx)(f) = x(rf) \subseteq U$; therefore $x(Rf) \subseteq U$ and, because U is small, $x(Rf) = 0$, that is, $x \in \langle F\rangle_R^\perp$.

PROPOSITION 4.4. – *Let* $P = E^* = \mathrm{Chom}_{\mathbf{Z}}(E, \mathbf{K})$. *Then:*

(*a*) *E has property P_1) iff* $\mathrm{End}_R(P) \cong R$.

(*b*) *E has property P_2) iff P is a finitely generated R-module.*

(*c*) *E has property P_3) iff P is a quasi-projective module.* (That is, every R-morphism of P into a homomorphic image of P can be lifted to P.)

PROOF. – (*a*) By Obs. 4.1., $\mathrm{Cend}_R(E) \cong \mathrm{Hom}_R(E^*, E^*)$, so $\mathrm{Cend}_R(E) \cong \mathrm{End}_R(P)$.
(*b*) « $\Rightarrow$ » Let W be a small neighbourhood of zero in E; By Obs. 4.3., $W \supseteq W(F; U)$ where F is a finite subset of P and U is a neighbourhood of zero in $\mathbf{K}$. Since $\langle F\rangle_R^\perp$ is an R-submodule of E contained in W, $\langle F\rangle_R^\perp = 0$ and so $P = \langle F\rangle_R$.

« $\Leftarrow$ » Let F be a finite system of generators of P, and U a small neighbourhood of zero in $\mathbf{K}$. Then $0 = P^\perp = \langle F\rangle_R^\perp$ and hence, by Obs. 4.3., $W(F; U)$ is a small neighbourhood of zero in E.

(*c*) By Obs. 4.1., every exact sequence in $\mathcal{C}(E)$ of the type

$$\begin{array}{c} 0 \to B \to E \\ \quad\ \ \downarrow f \\ \quad\ \ E \end{array}$$

with B a closed submodule of E, gives the exact sequence in Mod-R:

$$\begin{array}{c} P \to B^* \to 0 \\ \ \ \uparrow f^* \\ \ \ P \end{array}$$

Since Γ is a duality, f is exentible iff f^* can be lifted. Now $B^* \cong P/B^{\perp}$, hence by Obs. 4.2., E is topologically quasi-injective iff P is a quasi-projective module. //

PROPOSITION 4.5. – *Let P be an R-module such that*:

(*a*) $\mathrm{End}_R(P) \simeq R$.

(*b*) *P is finitely generated.*

(*c*) *P is quasi-projective.*

Then P is a projective generator of Mod-R.

PROOF. – By [F_2] Coroll. 3.2, (*b*) and (*c*) imply that P is Σ-quasi-projective, that is $P^{(X)}$ is quasi-projective for every set X. Now, by (*b*) there exists an exact sequence $R^n \to P \to 0$; by applying $\mathrm{Hom}_R(-, P)$ we get the exact sequence $0 \to \mathrm{Hom}_R(P, P) \to \to P^n$, that is $0 \to R \to P^n$. Evidently P^n is Σ-quasi-projective and, since it contains a copy of R, it is projective by [F_1] Lemma 4.1., and hence, also P is projective.

Now, by [AF] Theor. 17.18 and by (*a*), P is a generator of Mod-R. //

Summing up the results obtained so far, we have the following:

THEOREM 4.6. – *The R-module E has properties P_1), P_2), P_3) iff the R-module $P = \mathrm{Chom}_{\mathbf{Z}}(E, \mathbf{K})$ is projective, finitely generated, and with endomorphism ring isomorphic to R.*

This Theorem has the following important Corollaries.

Let E have properties P_1), P_2), P_3). Then:

COROLLARY 4.7. – *E is an injective cogenerator of* Mod-R.

PROOF. – We first prove that E is injective.

By [*AF*], Lemma 19.14, E is injective iff P is flat, so, by the preceeding Theorem, we get the conclusion.

Let $\mathcal{M}$ be a maximal ideal of R; we prove that there exists an injection $R/\mathcal{M} \hookrightarrow E$ or, equivalently, that $E[\mathcal{M}] = \{x \in E : x\mathcal{M} = 0\} \neq 0$. Now $E[\mathcal{M}] = \mathrm{Hom}(P/\mathcal{M}P, \mathbf{K})$. The right hand side is zero iff $P/\mathcal{M}P$ is zero. By [AMD] Coroll. 2.5. (pag. 21), if $P = \mathcal{M}P$, there exists $a \in R$, such that $a \equiv 1 \bmod \mathcal{M}$ and $aP = 0$; but, since P is a faithful module, we get a contradiction. //

COROLLARY 4.8. – *$\mathcal{C}(E)$ coincides with the category CMR of compact and Hausdorff R-modules.*

PROOF. – Obviously $\mathcal{C}(E) \subseteq CMR$. Let $M \in CMR$ and $M^* = \mathrm{Chom}_{\mathbf{Z}}(M, \mathbf{K})$. By Prop. 4.5., there exists an exact sequence in Mod-R: $P^{(X)} \xrightarrow{\pi} M^* \to 0$. By Pontryagin's dualization we get $0 \to M \xrightarrow{\pi^*} E^X$, where π^* is a topological embedding, hence $M \in \mathcal{C}(E)$. //

Theorem 4.9. – *If E has properties P_1), P_2), P_3), then E is s.q.i.*

Proof. – E is q.i., therefore it is s.q.i. iff, for every closed submodule B of E, the characters of the R-module E/B endowed with the quotient topology τ, separate points of E/B. Now, since $(E/B, \tau)$ is compact and E is Hausdorff, the latter fact is equivalent to the condition that $(E/B, \tau)$ belongs to $\mathcal{C}(E)$. Then it suffices to apply Coroll. 4.5. //

We note that the preceeding Theorem and Theor. 2.7., prove the implication $(b) \Rightarrow (a)$ of the main theorem.

5. – The functor Δ_E and the module P.

The aim of this second part of the paper is to prove the implication $(a) \Rightarrow (b)$ of the conditions stated in the main theorem and points 3) and 4) of the same theorem.

We recall that, for every $M \in \mathcal{D}(E)$, $\Delta_1(M)$ is the R-module $\mathrm{Hom}_R(M, E)$ equipped with the finite topology and for every $M \in \mathcal{C}(E)$, $\Delta_2(M)$ is $\mathrm{Chom}_R(M, E)$ with the discrete topology.

Moreover Δ_E is a duality iff $\Delta_1 \circ \Delta_2$ is equivalent to the identity of $\mathcal{C}(E)$ and $\Delta_2 \circ \Delta_1$ to the identity of $\mathcal{D}(E)$.

First we note the following:

Proposition 5.1. – *Let Δ_E be a duality. Then* $\mathrm{Cend}_R(E) \cong R$.

Proof. – Since E is a faithful module, $R \in \mathcal{D}(E)$; $\Delta_1(R) = \mathrm{Hom}_R(R, E)$ is topologically isomorphic to E, hence $\mathrm{Chend}_R(E) \cong \mathrm{End}_R(R) \cong R$. //

We state now some facts about $\mathcal{C}(E)$ and $\mathcal{D}(E)$.

For every closed submodule B of E, E/B with the quotient topology τ is an object of CMR. As we noted in the proof of Theor. 4.6., $(E/B, \tau) \in \mathcal{C}(E)$ iff the characters of E/B separate points.

Therefore, by Prop. 5.1., $(E/B, \tau) \in \mathcal{C}(E)$ iff for every $x_0 \in E \setminus B$, there exists an $r \in R$ such that: $aB = 0$ and $ax_0 \neq 0$.

Lemma 5.2. – *Let B be a closed submodule of E.*
If $I = \mathrm{Ann}_R(B)$, then R/I belongs to $\mathcal{D}(E)$.

Proof. – Let $a \in R \setminus I$; there exists an element $b \in B$ such that $ab \neq 0$; then the morphism $g: R/I \to E$, defined by $g(1 + I) = b$ is well defined, since I annihilates b, and $g(1 + I) = ab \neq 0$. //

Let $P = E^* = \mathrm{Chom}_Z(P, \boldsymbol{K})$. As in I 4, we shall identify E and P^*.

We note that, by Obs. 4.1. and Prop. 5.1., $\mathrm{End}_R(P) \cong R$.

LEMMA 5.3. – *Let* $P = E^*$.

(*a*) *For every submodule* L *of* P, $\mathrm{Ann}_R(P/L) = \mathrm{Ann}_R(L^\perp)$.

(*b*) $E/B \in \mathcal{C}(E)$, *for every closed submodule* B *of* E *iff* $L = \mathrm{Ann}_R(P/L)P$, *for every* R*-submodule* L *of* P.

PROOF. – (*a*) is obvious since $(P/L)^* \simeq L^\perp$ and, for each $r \in R$, the multiplication by r on P/L and on $L^\perp$ are morphisms which are transposed of each other.

(*b*) « $\Rightarrow$ » Let $I = \mathrm{Ann}_R(P/L)$; clearly $IP \subseteq L$ and $\mathrm{Ann}_R(P/IP) = I$.
Therefore, by (*a*), $I = \mathrm{Ann}_R(L^\perp) = \mathrm{Ann}_R\big((IP)^\perp\big)$.
If $IP \subsetneq L$, then $L^\perp \subsetneq (IP)^\perp$. Let $x_0 \in (IP)^\perp \setminus L^\perp$ and $aL^\perp = 0$ $(a \in R)$, then $a \in I$ and hence $ax_0 = 0$, contrarily to the hypothesis on E.
« $\Leftarrow$ » Let $I = \mathrm{Ann}_R(B)$ and $x_0 \in E \setminus B$. If $Ix_0 = 0$, then $I = \mathrm{Ann}_R(B + Rx_0)$ and, again by (*a*), $\mathrm{Ann}_R(P/B^\perp) = \mathrm{Ann}_R\big(P/(B + Rx_0)^\perp\big)$. Thus, by the hypothesis on P, we get the contradiction $(B + Rx_0)^\perp = B^\perp$, that is, $B = B + Rx_0$. //

We pass, now, to analyzing the functor Δ_E and the conditions that P has to satisfy in case Δ_E is a duality.
We consider the following functors:

$$T\colon \text{Mod-}R \to \text{Mod-}R \qquad M \mapsto M \otimes_R P$$
$$H\colon \text{Mod-}R \to \text{Mod-}R \qquad M \mapsto \mathrm{Hom}_R(P, M)$$

and the functors $\Delta_1\colon \mathcal{D}(E) \to \mathcal{C}(E)$, $\Delta_2\colon \mathcal{C}(E) \to \mathcal{D}(E)$, $\Gamma\colon \text{Mod-}R \to CMR$ already defined.
We set

$$\mathrm{Im}\,H = \{M \in \text{Mod-}R\colon M \simeq \mathrm{Hom}_R(P, N) \text{ for some } N \in \text{Mod-}R\},$$
$$\mathrm{Im}\,T = \{M \in \text{Mod-}R\colon M \simeq N \otimes_R P \text{ for some } N \in \text{Mod-}R\}.$$

PROPOSITION 5.4. – Δ_1 *and* Δ_2 *are naturally equivalent to* $\Gamma \circ T$ *and* $H \circ \Gamma$, *respectively.* $\Delta_2 \circ \Delta_1 \sim 1_{\mathcal{D}(E)}$ *iff* $T\colon \mathcal{D}(E) \to T(\mathcal{D}(E))$ *is a category equivalence whose inverse is given by the functor* H.

PROOF. – Let $M \in \mathcal{D}(E)$; $\Delta_1(M) = \mathrm{Hom}_R\big(M, \mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{K})\big)$ which is canonically isomorphic to $\mathrm{Hom}_{\mathbf{Z}}(M \otimes_R P, \mathbf{K})$ ([B] Alg. I 4 n. 1 pag. 105) that is to $\Gamma \circ T(M)$.
Now, both $\Delta_1(M)$ and $\Gamma \circ T(M)$ have the finite topology and it can be easily seen that the algebraic isomorphism between these two modules is indeed topological.
Let $M \in \mathcal{C}(E)$, $\Delta_2(M) = \mathrm{Chom}_R(M, E)$, which is, by Obs. 4.1., canonically isomorphic to $\mathrm{Hom}_R(P, M)$, moreover both are discrete, hence $\Delta_1(M)$ is topologically isomorphic to $H \circ \Gamma(M)$.

The second statement of the Proposition is a consequence of the first one, since $\Delta_2 \circ \Delta_1 \sim H \circ \Gamma \circ \Gamma \circ T$ and, by the fact that Γ is a duality, we get $\Delta_2 \circ \Delta_1 \sim H \circ T$. //

PROPOSITION 5.5. – *Let* $\Delta_1 \circ \Delta_2 \sim 1\mathfrak{D}(E)$, *then* $\mathfrak{D}(E) = \mathrm{Im}\, H$, $T\colon \mathrm{Im}\, H \to \mathrm{Im}\, T$ *and* $H\colon \mathrm{Im}\, T \to \mathrm{Im}\, H$ *are mutually inverse equivalences.*

Moreover, $T(\mathfrak{D}(E)) = \mathrm{Im}\, T$.

PROOF. – Let $M \in \mathrm{Im}\, H$, $M \cong \mathrm{Hom}_R(P, N) \cong$ (Obs. 4.1.) $\mathrm{Chom}_R(N^*, E) \leq E^{N^*}$, therefore $M \in \mathfrak{D}(E)$.

Viceversa, let $M \in \mathfrak{D}(E)$ and $\Delta_2 \circ \Delta_1 \sim 1_{\mathfrak{D}(E)}$; by Prop. 5.4., $M \cong \mathrm{Hom}_R(P, M \otimes_R P)$ and so $M \in \mathrm{Im}\, H$.

Now, $\mathfrak{D}(E) = \mathrm{Im}\, H$ and Prop. 5.4., imply that for every $N \in$ Mod-R, $HTH(N)$ is canonically isomorphic to $H(N)$ and hence condition 5) of [S] Theor. 1.3. holds. Thus, condition 1) of the same theorem states that $T\colon \mathrm{Im}\, H \to \mathrm{Im}\, T$ is an equivalence and that $H\colon \mathrm{Im}\, T \to \mathrm{Im}\, H$ is its inverse.

In regard to the equality $\mathrm{Im}\, T = T(\mathfrak{D}(E))$, one inclusion is clear. For the other, let $N \in \mathrm{Im}\, T$; we proved just now that $N \cong TH(N)$ and so

$$N \in T(\mathrm{Im}\, H) = T(\mathfrak{D}(E)) . \quad //$$

OBSERVATION 5.6. – By Prop. 5.5., every $M \in \mathrm{Im}\, H$ is isomorphic to $HT(M)$ and every $N \in \mathrm{Im}\, T$ is isomorphic to $TH(N)$. By [S] Theor. 1.3. the isomorphisms in question may be assumed to be the following:

$$\psi_M\colon M \to \mathrm{Hom}_R\left(P, M \otimes_R P\right), \qquad m \mapsto [-, m]\colon p \mapsto m \otimes p \ (m \in M;\, p \in P)$$
$$\varphi_N\colon \mathrm{Hom}_R(P, N) \otimes_R P \to N, \qquad \alpha \otimes p \mapsto \alpha(p) \ \left(\alpha \in \mathrm{Hom}_R(P, N),\, p \in P\right).$$

We set Gen (P) to be the full subcategory of Mod-R consisting of R-modules generated by P, that is Gen $(P) = \{M \in$ Mod-R: there exists an exact sequence $P^{(X)} \to M \to 0$ for some set $X\}$.

$\overline{\mathrm{Gen}}$ (P) denotes the smallest subcategory of Mod-R containing Gen (P) and closed under taking submodules, homomorphic images and direct sums.

LEMMA 5.7. – $\Gamma(\mathcal{C}(E)) = \mathrm{Gen}\,(P)$.

PROOF. – Let $N \in \mathcal{C}(E)$; there exists a topological embedding $0 \to N \to E^X$. By applying the functor Γ, we get the exact sequence $P^{(X)} \to \Gamma(N) \to 0$ and hence $\Gamma(N) \in \mathrm{Gen}\,(P)$.

For the converse we proceed analogously by dualizing, for every $M \in \mathrm{Gen}\,(P)$, the exact sequence $P^{(X)} \to M \to 0$. //

LEMMA 5.8. – *Let* Δ_E *be a duality. Then* $\mathrm{Im}\, T = \mathrm{Gen}\,(P)$, $\mathrm{Gen}\,(P) = \overline{\mathrm{Gen}}\,(P)$ *and* P *is a flat* R*-module.*

PROOF. – If Δ_E is a duality it is $\Delta_1(\mathfrak{D}(E)) = \mathcal{C}(E)$ and hence, by Prop. 5.4. and Obs. 4.1., $T(\mathfrak{D}(E)) = \Gamma(\mathcal{C}(E))$; now, by Prop. 5.5. and 5.7., we get $\operatorname{Im} T = \operatorname{Gen}(P)$.

Therefore, every $M \in \operatorname{Gen}(P)$ is canonically isomorphic to $\operatorname{Hom}_R(P, M) \otimes_R P$. Then by [S] Lemma 2.1., $\operatorname{Gen}(P) = \overline{\operatorname{Gen}}(P)$ and P is flat. //

6. – In view of the results obtained up to now, we can prove that if Δ_E is a duality, then E has properties P_1), P_2), P_3).

We state first the following Lemmas.

LEMMA 6.1. – *Let Δ_E be a duality. Then for every submodule L of $P = E^*$, $L = \operatorname{Ann}_R(P/L)P$. If I is any ideal of R, $I = \operatorname{Ann}_R(P/IP)$.*

PROOF. – Let $L \leq P$, then $L \in \overline{\operatorname{Gen}}(P)$. By Lemma 5.8. and Prop. 5.5., L is isomorphic to $\operatorname{Hom}_R(P, L) \otimes_R P$. By Obs. 5.6., the above isomorphism is given by $\varphi_L\colon \alpha \otimes p \mapsto \alpha(p)$ where $\alpha \in \operatorname{Hom}_R(P, L)$ and $p \in P$.

Now, $\operatorname{Hom}_R(P, L)$ is clearly isomorphic to $\operatorname{Ann}_R(P/L)$ and hence the surjectivity of φ_L implies that $L = \operatorname{Ann}_R(P/L)P$.

Let I be an ideal of R; since $I \in \mathfrak{D}(E)$, $I \cong \operatorname{Hom}_R(P, I \otimes_R P)$, by Prop. 5.4., and by the flatness of P, $I \otimes_R P \cong IP$.

Moreover, $\operatorname{Hom}_R(P, IP) \cong \operatorname{Ann}_R(P/IP)$ and by the nature of the isomorphisms considered, $I = \operatorname{Ann}_R(P/IP)$. //

LEMMA 6.2. – *For every ideal I of R, R/I belongs to $\mathfrak{D}(E)$.*

PROOF. – By Lemma 6.1., $I = \operatorname{Ann}_R(P/IP)$ which coincides with $\operatorname{Ann}_R((IP)^{\perp})$ (Lemma 5.3. (*a*)) and so, by Prop. 5.2., $R/I \in \mathfrak{D}(E)$. //

PROPOSITION 6.3. – *Let Δ_E be a duality, then E has property P_2).*

PROOF. – We prove that $P = E^*$ is a finitely generated R-module and then apply Prop. 4.4.

Let $\mathcal{F} = \{L_\alpha \leqslant P : L_\alpha$ is finitely generated$\}$ and, for each α, $I_\alpha = \operatorname{Ann}_R(P/L_\alpha)$. Then, by Lemma 6.1., $L_\alpha = I_\alpha P$. If $S = \sum_\alpha I_\alpha$, $SP = \sum_\alpha (I_\alpha P) = \sum_\alpha L_\alpha = P$.

We want to prove that $S = R$, because from that we get $1 \in I_{\alpha_1} + \ldots + I_{\alpha_n}$ and so $P = \left(\sum_{i=1}^{n} I_{\alpha_i}\right)P = \sum_{i=1}^{n} L_{\alpha_i}$ is finitely generated.

By Lemma 6.2., $R/S \in \mathfrak{D}(E)$, hence, by Prop. 5.4., $R/S \cong \operatorname{Hom}_R\left(P, R/S \otimes_R P\right)$, but $R/S \otimes_R P \cong P/SP = 0$, so $R/S = 0$. //

PROPOSITION 6.4. – *Let Δ_E be a duality. Then E is topologically quasi-injective.*

PROOF. – We prove that P is quasi-projective and then apply Prop. 4.4. Let L be a submodule of P and $I = \operatorname{Ann}_R(P/L)$. By Lemma 6.1., $L = IP$ and therefore

$P/L = P/IP \cong R/I \otimes_R P$. Now $R/I \in \mathfrak{D}(E)$ by Lemma 6.2, hence $\mathrm{Hom}_R(P, P/IP) \cong$ $\cong R/I$ and this means that every morphism of P into P/L can be lifted to P. //

We have so proved the implication $(a) \Rightarrow (b)$ of the main theorem. Points 3) and 4) of the theorem follow from Prop. 5.4. and from the results of § 4.

7. – Some examples.

If R is a commutative ring with $1 \neq 0$, we say that a duality between Mod-R and CMR is a *character duality* if it is induced by a faithful module of CMR in the way defined in § 1.

Every commutative ring, viewed as an R-module over itself, satisfies the conditions of Prop. 4.4; hence, by the main theorem Δ_{R^*} is a duality between Mod-R and CMR. ($R^* = \Gamma(R)$).

But, $\Delta_{R^*} \sim \Gamma_0 \otimes_R R$; so Δ_{R^*} is naturally equivalent to the Pontryagin duality.

There exist some rings, for which the duality considered above is the only duality existing, up to equivalences. In fact we have the following:

PROPOSITION 7.1. – *If R is a principal ideal domain (P.I.D.) or a local ring, there exists—up to topological isomorphisms—a unique faithful module $E \in CMR$ having properties P_1), P_2), P_3). E is topologically isomorphic to R^* and hence, every character duality between* Mod-R *and CMR is equivalent to the Pontryagin duality.*

PROOF. – If E is an R-module satisfying the required hypothesis, then by Prop. 4.4. and 4.5., $P = E^*$ is a projective module.

Now, it is well known that, if R is a P.I.D. or a local ring, every projective module is free. (The second case is a theorem of Kaplansky—[K] Theor. 2 n. 4—).

Then P is free and, since $\mathrm{End}_R(P) \cong R$ (Prop. 4.4.), we get $P \cong R$, that is $E \simeq R^*$.

The second statement of the proposition is the preceeding remark. //

We give, now, an example of a ring R over which there exists a module E satisfying the hypothesis of the previous proposition, but such that $E \not\simeq R^*$.

PROPOSITION 7.2. – *Let R be a Dedekind domain with a non principal ideal I. Then I is a projective finitely generated R-module with endomorphism ring isomorphic to R.*

Therefore Δ_{I^*} is a character duality between Mod-R *and CMR not equivalent to the Pontryagin duality.*

PROOF. – Obviously I is finitely generated and projective (cfr. e.g. [AB]). We prove now, that $\mathrm{End}_R(I) \simeq R$.

Let $\varphi \in \mathrm{End}_R(I)$. For every prime ideal $\mathfrak{I}$ of R, let $I_{\mathfrak{I}}$ be the localization of I at $\mathfrak{I}$ and $\varphi_{\mathfrak{I}}: I_{\mathfrak{I}} \to I_{\mathfrak{I}}$ defined by $\varphi_{\mathfrak{I}}(x/t) = \varphi(x)/t$ ($x \in I$, $t \in R \setminus \mathfrak{I}$). $\varphi_{\mathfrak{I}}$ is an $R_{\mathfrak{I}}$ morphism such that $\varphi_{\mathfrak{I}}|_I = \varphi$.

For every prime $\mathfrak{P}$, $I_{\mathfrak{P}}$ is an ideal of the discrete valuation ring $R_{\mathfrak{P}}$, therefore $\mathrm{End}_R(I) \cong R_{\mathfrak{P}}$, that is $\varphi_{\mathfrak{P}}$ is the multiplication by an element of $R_{\mathfrak{P}}$. Thus φ is the multiplication by an element of $\bigcap_{\mathfrak{P}\ \mathrm{prime}} R_{\mathfrak{P}}$ which coincides with R because R is Dedekind.

So Δ_{I^*} is a duality, by the main theorem, and $\Delta_{I^*} \not\approx \Gamma$ because otherwise, $R^* \cong \cong \Delta_{I^*}(R) \cong I^*$, that is $I \cong R$ contrarily to the hypothesis on I. //

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