On Partial Stability of Differential Equations with Time Delay (*).

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Summary. – *We investigate stability, uniform stability and equi-asymptotic stability with respect to the x-components and y-components of a differential equation with time delay. We also obtain necessary and sufficient conditions for thé generalized asymptotic stability of the exponential type with respect to the components which generalizes the work of Corduneanu* [3]. *We make use of Lyapunov functionals and differential inequalities in our study.*

1. – Introduction.

Lyapunov's theorems give sufficient conditions for the stability, asymptotic stability and instability of systems of ordinary differential equations. In recent years considerable attention has been paid to the generalization of these stability concepts in several directions. In particular, the concept of partial stability or stability with respect to a part of the variables has been studied by several authors for ordinary differential equations. In these investigations, Lyapunov's second method together with the theory of differential inequalities, have been widely used. Notable among these investigations are those of CORDUNEANU [1, 2], LADDE and LEELA [4], LAKSHMIKANTHAM and LEELA [5] and RUMIANTSEV [8]. Recently, CORDUNEANU [3] investigated some problems of partial stability related to linear differential equations with delay:

$$(*)\qquad \begin{aligned} \dot{x}(t) &= A(t, x_t) + B(t, y_t) \\ \dot{y}(t) &= C(t, x_t) + D(t, y_t) \end{aligned}$$

where $x \in R^n$, $y \in R^m$, $t \in R^+ = [0, \infty)$ and the subscript t indicates the restriction of the corresponding function to the interval $[t-h, t]$, with $h > 0$ fixed: $x_t(\theta) = x(t+\theta)$, $-h \leqslant \theta \leqslant 0$. In his investigation of partial stability of the exponential type, only the x-components of the unknown solution of $(*)$ was under consideration with respect to their behaviour. However, partial stability is useful from the practical point of view since in certain situations, one may be interested only in the behaviour of some part of the variables. It would therefore be interesting to search simultaneously for information on both the x-components and the y-components of

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the solution, as observed in [3]. As far as the author is aware this consideration is yet to be investigated, for delay equations.

In this paper, we shall consider a more general system of equations with delay and would be interested in searching for informations on both the x-components and the y-components of the solutions of our differential equations. We investigate stability, uniform stability, equi-asymptotic stability with respect to the x-components and the y-components. We also obtain necessary and sufficient conditions for the generalized asymptotic stability of the exponential type with respect to the components, which generalizes the work of CORDUNEANU [3]. We approach the problem by making use of two Lyapunov functionals and the theory of differential inequalities.

2. – Preliminaries.

Let R^n denote the n-dimensional Euclidean space with convenient norm $\|\cdot\|$. Denote by R^+, the non-negative real numbers. For $h>0$, let $l^n = C([-h, 0], R^n)$ denote the space of continuous functions with domain $[-h, 0]$ and range in R^n. For $\varphi \in l^n$, we define $\|\varphi\|_0 = \sup\limits_{-h\leqslant s\leqslant 0} \|\varphi(s)\|$. Suppose that $x \in C([-h, 0], R^n)$ and for $t \in R^+$, x_t denotes a translation of the restriction of x to the interval $[t-h, t]$, then x_t is an element of l^n defined by $x_t(s) = x(t+s)$, $-h \leqslant s \leqslant 0$. Consider the functional differential system

$$\dot{x}(t) = f(t, x_t, y_t)$$
$$\dot{y}(t) = g(t, x_t, y_t) \tag{1}$$

where $t \in R^+$, $x \in R^n$, $y \in R^m$ and f, g are continuous functions from $R^+ \times C([-h, 0], R^n) \times \times C([-h, 0], R^m)$ into R^n and R^m respectively. Also $f(t, 0, 0) = 0$ and $g(t, 0, 0) = 0$ for $t \in R^+$. Let (t_0, φ, ψ) belong to $R^+ \times l^n \times l^m$, we denote by $x = x(t; t_0, \varphi, \psi)$ and $y = y(t; t_0, \varphi, \psi)$ the solution of (1) such that $x_{t_0} = \varphi$ and $y_{t_0} = \psi$. For any $t \geqslant t_0$ we denote by $x_t(t_0, \varphi, \psi)$ and $y_t(t_0, \varphi, \psi)$ the corresponding elements of $C([-h, 0], R^n)$ and $C([-h, 0], R^m)$ respectively such that $x_{t_0}(t_0, \varphi, \psi) = \varphi$ and $y_{t_0}(t_0, \varphi, \psi) = \psi$. If we assume that f and g are locally Lipschitzian in (φ, ψ), then solutions of (1) are uniquely determined to the right by their initial values. Moreover each solution can be continued to the right for as long as the solution remains in a compact subset of the domain. For any $V \in C\big(R^+ \times C([-h, 0], R^n) \times C((-h, 0], R^m), R^n\big)$ define,

$$\begin{aligned} D^+ V\big(t, x_t(t_0, \varphi_0), y_t(t_0, \psi_0)\big) &= \\ &= \limsup_{\delta \to 0^+} \frac{1}{\delta} \big[V\big(t+\delta, x_{t+\delta}(t_0, \varphi_0), y_{t+\delta}(t_0, \psi_0)\big) - V\big(t, x_t(t_0, \varphi_0), y_t(t_0, \psi_0)\big)\big]\,. \end{aligned} \tag{2}$$

We also define,

$$D^+ V(t, \varphi, \psi) = \limsup_{\delta \to 0^+} \frac{1}{\delta} \big[V\big(t+\delta, x_{t+\delta}(t, \varphi) y_{t+\delta}(t, \psi)\big) - V(t, \varphi, \psi)\big] \tag{3}$$

where $x_t(t, \varphi)$, $y_t(t, \psi)$ is any solution of (1) with initial function φ, ψ at t. We assume the uniqueness of solutions of (1), so that (2) and (3) coincide since,

$$\varphi = x_t(t_0, \varphi_0)\,, \qquad \psi = y_t(t_0, \psi_0) \qquad \text{and} \qquad x_{t+\delta}(t_0, \varphi_0) = x_{t+\delta}(t, \varphi)\,,$$

$y_{t+\delta}(t_0, \psi_0) = y_{t+\delta}(t, \psi)$. Denote by $\mathcal{K}$ the class of functions $\sigma \in C([0, \varrho), R^+)$ such that $\sigma(0) = 0$ and $\sigma(t)$ is strictly monotone increasing in t.

DEFINITION 2.1. – (*a*) The solution $x = 0$, $y = 0$ of the system (1) is said to be stable relative to the x-component or partially stable if for any $\varepsilon > 0$, $t_0 \geqslant 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that for every $t \geqslant t_0$, and $\|\varphi\|_0 + \|\psi\|_0 < \delta$,

$$\|x_t(t_0; \varphi, \psi)\| < \varepsilon\,.$$

(*b*) The solution $x = 0$, $y = 0$ is uniformly stable relative to the x-component, if we can choose $\delta(\varepsilon)$ independent of (t_0) in (*a*).

(*c*) The trivial solution of (1) is said to be equi-asymptotically stable relative to the x-component or partially equi-asymptotically stable if there exists $\delta(t_0, \varepsilon) > 0$ and $T(t_0, \varepsilon) > 0$ such that (*a*) holds and for $t \geqslant t_0 + T$

$$\|x_t(t_0; \varphi, \psi)\| < \varepsilon \qquad \text{provided} \qquad \|\varphi\|_0 + \|\psi\|_0 < \delta\,.$$

(*d*) The trivial solution $x = 0$, $y = 0$ of (1) is partially asymptotically uniformly stable if T and δ in (*c*) are independent of t_0.

(*e*) The trivial solution $x = 0$, $y = 0$ of (1) is asymptotically exponentially stable relative to the x-component or partially asymptotically exponentially stable if there exist $M > 0$ and $\alpha > 0$ both real numbers such that

$$\|x_t(t_0; \varphi, \psi)\| \leqslant M\big(\|\varphi\|_0 + \|\psi\|_0\big) \exp[-\alpha(t - t_0)] \qquad \text{for } t \geqslant t_0\,.$$

(*f*) The trivial solution of (1) is generalized exponentially stable relative to the x-component or generalized partially asymptotically exponentially stable if there exist a continuous function $K(t) > 0$, for $t \in R^+$ and another function $p \in \mathcal{K}$ for $t \in R^+$ with $p(t) \to \infty$ as $t \to \infty$ such that

$$\|x_t(t_0; \varphi, \psi)\| \leqslant K(t)\big(\|\varphi\|_0 + \|\psi\|_0\big) \exp[p(t_0) - p(t)]$$

for $t \geqslant t_0$. In particular, if $K(t) = K > 0$ and $p(t) = \alpha t$, $\alpha > 0$, then definition (*f*) reduces to (*e*).

(*g*) The solution $x = 0$, $y = 0$ of the system (1) is said to be stable relative to the y-component or s-partially stable if for any $\varepsilon > 0$, $t_0 \geqslant 0$ there exists $\delta(t_0, \varepsilon) > 0$ such that for every $t \geqslant t_0$

$$\|y_t(t_0; \varphi, \psi)\| < \varepsilon$$

provided $\|\varphi\|_0 + \|\psi\|_0 < \delta$.

Definitions similar to (*g*) can be given for (*a*), (*b*), (*c*), (*d*), and (*e*) in terms of the y-components.

(*h*) The trivial solution $x = 0$, $y = 0$ of (1) is said to be strictly partially stable or stable with respect to the x-component and unstable with respect to the y-component (or vice-versa) if given $\varepsilon > 0$, $t_0 \in R^+$, there exists $\delta(t_0, \varepsilon) > 0$ such that provided $\|\varphi\|_0 + \|\psi\|_0 < \delta$

$$\|x_t(t_0; \varphi, \psi)\| < \varepsilon \quad \text{and} \quad \|y_t(t_0; \varphi, \psi)\| \geqslant \varepsilon \qquad \text{for } t \geqslant t_0 .$$

Definitions of strict partial stability corresponding to (*b*), (*c*), (*d*), (*e*), (*f*) and (*g*) can be given similarly.

(i) The trivial solution of $x = 0$, $y = 0$ of (1) is said to be stable with respect to both the x-component and the y-component simultaneously or simultaneously stable, if given $\varepsilon > 0$, $\varepsilon_1 > 0$, $t_0 \in R^+$, there exists $\delta(t_0, \varepsilon, \varepsilon_1) > 0$ such that provided $\|\varphi\|_0 + \|\psi\|_0 < \delta$,

$$\|x_t(t_0; \varphi, \psi)\| < \varepsilon \quad \text{and} \quad \|y_t(t_0; \varphi, \psi)\| < \varepsilon_1 \qquad \text{for } t \geqslant t_0 .$$

Similar definitions can be given for other types of stability.

DEFINITION 2.2. – A functional $V(t, \varphi, \psi)$ is said to be positive definite with respect to φ, if there exists a positive definite function $c(\varphi)$ not depending explicitly on t such that $c(\varphi) \leqslant V(t, \varphi, \psi)$ for all $t \geqslant 0$.

DEFINITION 2.3. A functional $V(t, \varphi, \psi)$ admits an infinitesimal upper bound in φ if there exists a function $b \in \mathcal{K}$ such that $V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0)$. It admits a strict infinitesimal upper bound in (φ, ψ) if $V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0, \|\psi\|_0)$ for $t \geqslant 0$.

LEMMA 2.4. – A functional $V(t, \varphi, \psi)$ is positive definite with respect to φ if and only if there exists $a \in \mathcal{K}$ such that

$$a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \qquad \text{for } t \geqslant 0 .$$

3. – Stability relative to components.

We now present several theorems which give sufficient conditions for the stability of system (1) with respect to the x-component and the y-component in terms of the existence of Lyapunov functionals.

THEOREM 3.1. – Suppose there exists a functional $V(t, \varphi, \psi)$ with the following properties:

(i) V is continuous and satisfies Lipschitz condition in φ and ψ and $V(t, 0, 0) = 0$;

(ii) there exists $a \in \mathcal{K}$ such that

$$a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \quad \text{for } (t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m$$

where $C_\varrho = \{\varphi \in l^n : \|\varphi\|_0 < \varrho\}$;

(iii) for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m$,

$$D^+ V(t, \varphi, \psi) \leqslant 0 .$$

Then the solution $x = 0$, $y = 0$ of the system (1) is stable with respect to the x-component or partially stable.

PROOF. – For $\varepsilon > 0$, $t_0 \geqslant 0$, $\exists \delta(t_0, \varepsilon) > 0$ such that $\|\varphi_0\|_0 + \|\psi_0\|_0 < \delta$ implies $V(t_0, \varphi_0, \psi_0) < a(\varepsilon)$. Let $x(t), y(t)$ be a solution of (1) with $\|\varphi_0\|_0 + \|\psi_0\|_0 < \delta$, then by (iii) V is non-increasing with respect to t and so

$$V(t, \varphi, \psi) \leqslant V(t_0, \varphi_0, \psi_0) \quad \text{for } t \geqslant t_0 .$$

Thus, for $t \geqslant t_0$, $\|\varphi\|_0 + \|\psi\|_0 < \delta$ implies,

$$a(\|x_t(t_0; \varphi, \psi)\|) = a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \leqslant V(t_0, \varphi_0, \psi_0) < a(\varepsilon) .$$

It follows that $\|x_t(t_0; \varphi, \psi)\| < \varepsilon$ for $t \geqslant t_0$.

THEOREM 3.2. – Suppose that there exists a functional $W(t, \varphi, \psi)$ with the following properties:

(i) W is continuous and satisfies the Lipschitz condition in φ and ψ and $W(t, 0, 0) = 0$;

(ii) there exists $a \in \mathcal{K}$ such that

$$a(\|\psi\|_0) \leqslant V(t, \varphi, \psi) \quad \text{for } (t, \varphi, \psi) \in R^+ \times l^n \times C_\tau$$

where $C_\tau = \{\psi \in l^m : \|\psi\|_0 < \tau\}$;

(iii) for $(t, \varphi, \psi) \in R^+ \times l^n \times C_\tau^1$,

$$D^+ W(t, \varphi, \psi) \leqslant 0 .$$

Then the solution $x = 0$, $y = 0$ of the system (1) is stable with respect to the y-component or s-partially stable.

DEFINITION 3.2. – Let $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$ be two Lyapunov functionals. $W(t, \varphi, \psi)$ is said to be positive definite in the region $V(t, \varphi, \psi) > 0$ if given $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that for every point $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$ satisfying $V(t, \varphi, \psi) \geqslant \varepsilon$ the inequality $W(t, \varphi, \psi) \geqslant \delta$ is satisfied.

THEOREM 3.3. – Suppose there exists two Lyapunov functionals $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$ with the following properties.

(i) V and W are continuous and locally Lipschitzian in φ, ψ and $V(t, 0, 0) = W(t, 0, 0) = 0$;

(ii) there exists $a \in \mathcal{K}$ such that

$$a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \qquad \text{for } (t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m ,$$

(iii) there exists $b \in \mathcal{K}$ such that for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m$, $W(t, \varphi, \psi) \leqslant b(\|\psi\|_0)$, and $W(t, \varphi, \psi)$ is bounded in the region $V(t, \varphi, \psi) > 0$ existing for $t \geqslant 0$ and for $\|\varphi\|_0 + \|\psi\|_0 < \eta$;

(iv) $D^+ W(t, \varphi, \psi)$ is a positive-definite function in the region $V(t, \varphi, \psi) > 0$.

Then the solution $x = 0$, $y = 0$ is stable with respect to the x – component and unstable with respect to the y-component or strictly partially stable.

PROOF. – Hypothesis (i), (ii), and (iii) imply that given $\varepsilon > 0$, $\exists \delta(\varepsilon, t_0) > 0$ such that $\|x_t(t_0, \varphi, \psi)\| < \varepsilon$ provided $\|\varphi\|_0 + \|\psi\|_0 < \delta$. Now let $0 < \varepsilon < \tau$ and $t_0 \in R^+$, then for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

$$W(t, \varphi, \psi) = W(t_0, \varphi_0, \psi_0) + \int_{t_0}^{t} D^+ W(s, \varphi, \psi)\, ds$$

where $x_{t_0}(t_0, \varphi_0, \psi_0) = \varphi_0$, $y_{t_0}(t_0; \varphi_0, \psi_0) = \psi_0$. By (iv), $\exists \delta_1(\varepsilon) > 0$ such that $V(t, \varphi, \psi) \geqslant \varepsilon$ implies $D^+ W(t, \varphi, \psi) \geqslant \delta$.

Hence

$$W(t, \varphi, \psi) \geqslant W(t_0, \varphi_0, \psi_0) + \delta_1(\varepsilon)(t - t_0) .$$

Suppose $x = 0$, $y = 0$ is stable with respect to ψ, then $\exists \delta_2(\varepsilon, t_0) > 0$ such that $\|y_t(t_0; \varphi, \psi)\| < \varepsilon$ for $t \geqslant t_0$ provided $\|\varphi_0\|_0 + \|\psi\|_0 < \delta_2$. Now set $\eta = \min\{\delta, \delta_2\}$, then by (iii) $W(t_0, \varphi_0, \psi_0) \leqslant M$ and

$$W(t_0, \varphi_0, \psi_0) + \delta_1(\varepsilon)(t - t_0) \leqslant W(t, \varphi, \psi) \leqslant b(\|y_t(t_0; \varphi, \psi)\|) < b(\varepsilon) .$$

As $t \to \infty$ we arrive at a contradiction.

The following theorem gives sufficient conditions for the stability of (1) with respect to the two components simultaneously.

THEOREM 3.4. – Let $V(t, \varphi, \psi)$ be a Lyapunov functional defined for $0 \leqslant t < \infty$, $\|\varphi\|_0 < \varrho$ and $\|\psi\|_0 < \infty$, such that the following conditions hold

(i) $V(t, \varphi, \psi)$ is locally Lipschitzian in (φ, ψ), $V(t, 0, 0) = 0$;

(ii) $a(\|\psi\|_0) \leqslant V(t, \varphi, \psi)$ for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m$, where $a(r)$ is a continuous monotone increasing function on R^+ into R^+ and $a(0) = 0$;

(iii) $D^+V(t, \varphi, \psi) \leqslant 0$ for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m$.

Suppose, for each $0 < \varepsilon < \varrho$, $W(t, \varphi, \psi)$ is another Lyapunov functional defined for $\|\psi\|_0 < \varepsilon$, $\|\varphi\|_0 < \varrho$ and $0 \leqslant t < \infty$ and satisfying the following properties:

(iv) $a_1(\|\varphi\|_0) \leqslant W(t, \varphi, \psi) \leqslant b_1(\|\varphi\|_0)$, where $a_1(r)$, $b_1(r)$ are continuous monotone increasing functions on R^+ such that $a_1(0) = 0$ and $b_1(0) = 0$;

(v) $D^+W(t, \varphi, \psi) \leqslant 0$ for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\varrho$.

Then the trivial solution $x = 0$, $y = 0$ of (1) is stable with respect to the x-component and with respect to the y-component or simultaneously stable.

PROOF. – Given $0 < \varepsilon < \varrho$, there exists $\delta(t_0, \varepsilon) > 0$ for $t_0 \in R^+$, such that $\|\varphi\|_0 + \|\psi\|_0 < \delta$ implies $V(t, \varphi, \psi) < a(\varepsilon)$. Hence if $\|\varphi_0\|_0 + \|\psi_0\|_0 < \delta$, then $\|\psi_0\|_0 < \delta$ and moreover by (ii) and (iii)

$$a(\|\psi\|_0) \leqslant V(t, \varphi, \psi) \leqslant V(t_0, \varphi_0, \psi_0) + \int_{t_0}^{t} D^+V(s, \varphi, \psi)\, ds < a(\varepsilon)\,.$$

Hence $\|y_t(t_0; \varphi, \psi)\| < \varepsilon$ for $t \geqslant t_0$ provided $\|\psi\|_0 < \delta$. Let $\varepsilon_1 > 0$ be such that $0 < \varepsilon < \varepsilon_1 < \varrho$ and choose δ_0 small enough so that $b_1(\delta_0) < a_1(\varepsilon_1)$. Set $\delta_1 = \min\{\delta_0, \delta\}$, then we claim that $\|\varphi\|_0 + \|\psi\|_0 < \delta_1$ implies $\|x_t(t_0; \varphi, \psi)\| < \varepsilon_1$ for all $t \geqslant t_0$. Suppose not, then there exist t_1, t_2 such that $t_0 < t_1 < t_2$ and

$$\|x_{t_1}(t_0; \varphi, \psi)\| = \delta_0\,, \quad \|x_{t_2}(t_0, \varphi, \psi)\| = \varepsilon_1 \quad \text{and} \quad \delta_0 \leqslant \|x_t(t_0; \varphi, \psi)\| \leqslant \varepsilon_1 \quad \text{for } t \in [t_1, t_2].$$

Hence by (iv) and (v),

$$a_1(\varepsilon_1) \leqslant W(t_2, \varphi, \psi) \leqslant W(t_1, \varphi, \psi) \leqslant b_1(\|x_{t_1}(t_0; \varphi, \psi)\|) \leqslant b_1(\delta_0) < a_1(\varepsilon_1)$$

which is a contradiction. Therefore $\|x_t(t_0; \varphi, \psi)\| < \varepsilon_1$ for $t \geqslant t_0$.

THEOREM 3.5. – Suppose $V(t, \varphi, \psi)$ is a Lyapunov functional with the properties (i) and (ii) of Theorem 3.1 and $V(t, \varphi, \psi)$ does not increase on any solution $x = x(t, t_0, \varphi, \psi)$,

$y = y(t, t_0, \varphi, \psi)$ as long as $x_t \in C_\varrho$. Then the trivial solution $x = 0$, $y = 0$ of (1) is stable with respect to the x-component.

PROOF. – By hypothesis $V(t, \varphi, \psi) \leqslant V(t_0, \varphi_0, \psi_0)$. So that the arguments of Theorem 3.1 remains valid for this case.

4. – Uniform stability with respect to components.

In this section we investigate some problems of partial uniform stability of the system (1). We characterize this concept in terms of Lyapunov functionals.

THEOREM 4.1. – Let the hypothesis (i) and (iii) of Theorem 3.1 hold. Suppose for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m$,

$$a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0)$$

where $a, b \in \mathcal{K}$, then the trivial solution $x = 0$, $y = 0$ is uniformly stable with respect to the x-component or partially uniformly stable.

PROOF. – Choose $\delta(\varepsilon) = b^{-1}(a(\varepsilon))$ independent of t_0. Then $\|\varphi_0\|_0 + \|\psi_0\|_0 < \delta(\varepsilon)$ implies

$$V(t_0, \varphi_0, \psi_0) \leqslant b(\|\varphi_0\|_0 + \|\psi_0\|_0) < b\big(b^{-1}(a(\varepsilon))\big) = a(\varepsilon)\,.$$

The result then follows.

COROLLARY 4.2. – Let the hypothesis (i) and (iii) of Theorem 3.1 hold. Suppose for $(t, \varphi, \psi) \in R^+ \times l^n \times C_\tau$,

$$a_1(\|\psi\|_0) \leqslant V(t, \varphi, \psi) \leqslant b_1(\|\varphi\|_0 + \|\psi\|_0)$$

where a_1, $b_1 \in \mathcal{K}$, then the trivial solution $x = 0$, $y = 0$ is uniformly stable with respect to the y-component.

THEOREM 4.3. – Assume that hypothesis (i) and (iii) of Theorem 3.1 are satisfied and that $\exists a$, $b \in \mathcal{K}$ such that

$$a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0)\,,$$

then for any $\varepsilon > 0$, $\exists \eta(\varepsilon) > 0$ such that for $t_0 \geqslant 0$, $\|\varphi_0\|_0 < \eta$ and $\|\psi_0\|_0 < \infty$ implies

$$\|x_t(t_0, \varphi, \psi)\| < \varepsilon \qquad \text{for all } t \geqslant t_0\,.$$

Moreover, $f(t, 0, y_t) = 0$ and $g(t, 0, y_t) = 0$.

PROOF. – Let $\varepsilon > 0$ and set $\eta(\varepsilon) = b^{-1}(a(\varepsilon))$, then for $t_0 \geqslant 0$, $\|\varphi_0\|_0 < \eta$ implies $V(t_0, \varphi_0, \psi_0) < b\big(b^{-1}(a(\varepsilon))\big) = a(\varepsilon)$ and so $\|x_t(t_0; \varphi, \psi)\| < \varepsilon$ for all $t \geqslant t_0$. Consider the solution $x = x(t; t_0, 0, \psi_0)$, $y = y(t; t_0, 0, \psi_0)$ for $t_0 > 0$ and ψ_0. $V(t_0, 0, \psi_0) = b(0) = 0$ and since $V \geqslant 0$ and $D^+V(t, \varphi, \psi) \leqslant 0$, $V(t, x_t, y_t) \leqslant V(t_0, 0, \psi_0) \leqslant b(0) = 0$. Hence $V\big(t, x(t; t_0, 0, \psi_0), y(t; t_0, 0, \psi_0)\big) \equiv 0$ so that $\|x(t; t_0, 0, \psi_0)\| = 0$ for all $t \geqslant 0$, and so $f(t, 0, y_t) = 0$ and $g(t, 0, y_t) = 0$.

REMARK. – A similar theorem in the spirit of Corollary 4.2 can be stated in terms of Lyapunov functional to obtain a uniform stability result of Theorem 4.3 with respect to the y-component.

THEOREM 4.4. – (i) Suppose $V(t, \varphi, \psi)$ is a Lyapunov functional with the properties (i) and (ii) of Theorem 3.1 and

(ii) $V(t, \varphi, \psi)$ does not increase on any solution $x = x(t, t_0, \varphi, \psi)$, $y = y(t, t_0, \varphi, \psi)$ as long as $x_t \in C_\varrho$.

(iii) V satisfies for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

$$V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0) .$$

Then the trivial solution $x = 0$, $y = 0$ is uniformly stable with respect to the x-component.

PROOF. – (ii) implies $V(t, \varphi, \psi) \leqslant V(t_0, \varphi_0, \psi_0)$ hence the arguments of Theorem 4.1 remain valid.

THEOREM 4.5. – In addition to the hypothesis of Theorem 3.4 suppose V satisfies

$$V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0) , \qquad \text{for } (t, \varphi, \psi) \in R^+ \times C_\varrho \times l^m .$$

Then the trivial solution $x = 0$, $y = 0$ is uniformly stable with respect to both the x-component and the y-component, that is, simultaneously uniformly stable.

PROOF. – By the assumption on V, we can choose $\delta(\varepsilon) = b^{-1}(a(\varepsilon))$ in Theorem 3.4 independent of t_0 and so if $\|\varphi_0\|_0 + \|\psi_0\|_0 < \delta(\varepsilon)$ in the proof of Theorem 3.4

$$V(t_0, \varphi_0, \psi_0) \leqslant b(\|\varphi_0\|_0 + \|\psi_0\|_0) < b\big(b^{-1}(a(\varepsilon))\big) = a(\varepsilon) .$$

Since $\delta(\varepsilon)$ is independent of t_0, $\delta_1(\varepsilon)$ in Theorem 3.4 can also be chosen independent of t_0 and the uniform stability of both $x = 0$ and $y = 0$ follows.

The next three theorems will be devoted to a variety of results concerning the construction of Lyapunov functionals.

THEOREM 4.6. – Suppose the trivial solution $x = 0$, $y = 0$ of (1) is uniformly stable with respect to the x-component, then there exists a Lyapunov functional $V(t, \varphi, \psi)$ such that

(i) $\|\varphi\|_0 \leqslant V(t, \varphi, \psi)$, for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

(ii) $V(t, \varphi, \psi)$ does not increase in (φ, ψ) for $\varphi \in C_\varrho$, and

(iii) $V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0)$ for $b \in \mathcal{K}$.

PROOF. – Let $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$ and define $V(t, \varphi, \psi)$ by setting

$$V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|x_{t+\sigma}(t_0; \varphi, \psi)\|\} .$$

Then clearly for $\sigma = 0$, $\|x_t(t_0; \varphi, \psi)\| \leqslant V(t, \varphi, \psi)$ and (i) is satisfied. Uniform stability with respect to the x-component implies that

$$\|x_t(t_0; \varphi, \psi)\| \leqslant \varepsilon(\|\varphi\|_0 + \|\psi\|_0) \qquad \text{for } t \geqslant t_0 .$$

Hence setting $b(\|\varphi\|_0 + \|\psi\|_0) = \varepsilon(\|\varphi\|_0 + \|\psi\|_0)$ we have

$$V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0) .$$

Since $\varphi = x_t(t_0, \varphi_0, \psi_0)$, $\psi = y_t(t_0, \varphi_0, \psi_0)$ and by uniqueness of solutions $x_t(\tau, \varphi, \psi) = x_t(\tau, \varphi_0, \psi_0)$ we have

$$V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|x_{t+\sigma}(t_0, \varphi, \psi)\|\} = \sup_{\sigma \geqslant 0} \{\|x_{t+\sigma}(t_0, \varphi_0, \psi_0)\|\} ,$$

so that for $t_1 > t_2 \geqslant t_0$,

$$\begin{aligned} V(t_1, x_{t_1}(t_0, \varphi_0, \psi_0)) &= \sup_{\sigma \geqslant 0} \{\|x_{t_1+\sigma}(t_0; \varphi_0, \psi_0)\|\} \\ &\leqslant \sup_{\sigma \geqslant 0} \{\|x_{t_2+\sigma}(t_0; \varphi_0, \psi_0)\|\} \\ &= V(t_2, x_{t_2}(t_0; \varphi_0, \psi_0), y_{t_2}(t_0 : \varphi_0, \psi_0)) . \end{aligned}$$

Hence V does not increase and the proof is complete.

THEOREM 4.7. – Suppose the trivial solution $x = 0$, $y = 0$ of (1) is uniformly stable with respect to the y-component, then there exists a Lyapunov functional $W(t, \varphi, \psi)$ such that

(i) $\|\psi\|_0 \leqslant W(t, \varphi, \psi)$, for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$.

(ii) $W(t, \varphi, \psi)$ does not increase in (φ, ψ) for $\psi \in C_\tau$, and

(iii) $W(t, \varphi, \psi) \leqslant b_1(\|\varphi\|_0 + \|\psi\|_0)$ foı $b_1 \in \mathcal{K}$.

PROOF. – Let $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$, set

$$W(t, t, \varphi, \psi) = \sup_{\tau \geqslant 0} \{\|y_{t+\tau}(t_0, \varphi, \psi\|\}$$

and proceed as in the last theorem.

THEOREM 4.8. – Suppose the trivial solution $x = 0$, $y = 0$ of (1) is uniformly stable with respect to the x-component and with respect to the y-component, then there exist two Lyapunov functionals $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$ such that

(i) $\|\varphi\|_0 \leqslant V(t, \varphi, \psi) \leqslant b\,(\|\varphi\|_0 + \|\psi\|_0)$ for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$, $b \in \mathcal{K}$,

(ii) $\|\psi\|_0 \leqslant W(t, \varphi, \psi) \leqslant b_1(\|\varphi\|_0 + \|\psi\|_0)$ for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$, $b_1 \in \mathcal{K}$,

(iii) $V(t, \varphi, \psi)$ does not increase in (φ, ψ) for any $\varphi \in C_\varrho$, and $W(t, \varphi, \psi)$ does not increase in (φ, ψ) for any $\psi \in C_\tau$.

PROOF. – Let $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$ and define two functionals $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$ as follows:

$$V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|x_{t+\tau}(t_0; \varphi, \psi)\|\}$$

and

$$W(t, \varphi, \psi) = \sup_{\delta \geqslant 0} \{\|y_{t+\delta}(t_0; \varphi, \psi)\|\} .$$

It is easy to check that (i), (ii) and (iii) hold for V and W in view of Theorems 4.6 and 4.7.

THEOREM 4.9. – Assume that $\exists$ two Lyapunov functionals V and W such that

(i) V is locally Lipschitzian in φ and ψ and $V(t, 0, 0) = 0$.

(ii) $a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0)$ for $a, b \in \mathcal{K}$ and $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

(iii) $D^+V(t, \varphi, \psi) \leqslant 0$ for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

(iv) $W(t, \varphi, \psi) \leqslant b_1(\|\psi\|_0)$, $b_1 \in \mathcal{K}$ and $W(t, \varphi, \psi)$ is bounded in the region $V(t, \varphi, \psi) > 0$ existing for $t \geqslant 0$ and for $\|\varphi\|_0 + \|\psi\|_0 < \eta$.

(v) $D^+W(t, \varphi, \psi)$ is a positive definite function in the region $V(t, \varphi, \psi) > 0$.

Then the trivial solution $x = 0$, $y = 0$ is uniformly stable with respect to the x-component and unstable with respect to the y-component.

PROOF. – (i), (ii) and (iii) imply the uniform stability with respect to the x-component by Theorem 4.1, and the rest of the proof is similar to Theorem 3.3.

THEOREM 4.10. – Assume that

(i) $V(t, \varphi, \psi)$ is a locally Lipschitzian functional such that $V(t, 0, 0) = 0$,

(ii) $a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0)$ for $a, b \in \mathcal{K}$ and $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

(iii) $D^+ V(t, \varphi, \psi) \leqslant 0$ for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$.

(iv) The trivial solution $y = 0$ of

$$\dot{y}(t) = g(t, 0, y_t)$$

is uniformly asymptotically stable,

(v) $g(t, \varphi, \psi)$ is locally Lipschitzian in φ and ψ for a constant $K > 0$.

Then the trivial solution $x = 0$, $y = 0$ of (1) is uniformly stable with respect to the x-component and y-component.

PROOF. – Arguments parallel to Theorem 3.11.2 of [5] can be used to prove the Theorem. We omit details.

THEOREM 4.11. – If hypothesis (iv) and (v) of the last theorem hold and the trivial solution of (1) is uniformly asymptotically stable with respect to the x-component, then the trivial solution $x = 0$, $y = 0$ is uniformly asymptotically stable.

PROOF. – In view of Theorem 7.1.4 of [6], the proof is analogous to that of Theorem 3.11.3 of [5].

REMARK. – Theorems 4.10 and 4.11 are the corresponding partial stability results obtained for ordinary differential equations in [5].

The following is another version of Theorem 4.10.

THEOREM 4.12. – Assume that (i) and (iii) of Theorem 4.10 hold, and that

(*a*) There exist $a, b \in \mathcal{K}$ such that for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

$$a(\|\psi\|_0) \leqslant V(t, \varphi, \psi) \leqslant b(\|\varphi\|_0 + \|\psi\|_0)$$

(*b*) The trivial solution $x = 0$ of

$$\dot{x}(t) = f(t, x_t, 0)$$

is uniformly asymptotically stable,

(v) $f(t, \varphi, \psi)$ is locally Lipschitzian in φ and ψ for a constant $M > 0$.

Then the trivial solution $x = 0$, $y = 0$ of (1) is uniformly stable with respect to the x-component and y-component.

5. – Equi-asymptotic stability with respect to components.

In this section we characterize the concept of equi-asymptotic stability with respect to the x-component or the y-component or both in terms of the existence of Lyapunov functionals.

THEOREM 5.1. Let $V(t, \varphi, \psi)$ be a Lyapunov functional such that

(i) $a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi)$ for $a \in \mathcal{K}$ and $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

(ii) $D^+ V(t, \varphi, \psi) \leqslant - C(\|\varphi\|_0)$ for $C \in \mathcal{K}$ and $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$.

Then for $0 < \varepsilon < \varrho$, $t_0 \geqslant 0$ there exist $\delta(t_0) > 0$ and $T(t_0, \varepsilon) > 0$ such that for $x_t(t_0, \varphi, \psi)$, $y_t(t_0, \varphi, \psi)$ with $\|\varphi\|_0 + \|\psi\|_0 < \delta$ there exists $t_* \in (t_0, t_0 + T)$ such that $\|x_{t_*}(t_0, \varphi, \psi)\| < \varepsilon$.

PROOF. – By Theorem 3.1, for $t_0 \geqslant 0$, $\exists \delta(t_0) > 0$ such that $\|\varphi\|_0 + \|\psi\|_0 < \delta$ implies $\|x_t(t_0, \varphi, \psi)\| < \varepsilon$ for $t \geqslant t_0$. Now let $\lambda(t_0) = \sup \{V(t_0, \varphi, \psi) \colon \|\varphi\|_0 + \|\psi\|_0 < \delta\}$ and set $T(t_0, \varepsilon) = \lambda(t_0)/C(\varepsilon)$. Then $\exists t_* \in (t_0, t_0 + T)$ such that $\|x_{t_*}(t_0, \varphi, \psi)\| < \varepsilon$. Suppose not, then $\varepsilon \leqslant \|x_t(t_0, \varphi, \psi)\| < \varrho$ for $t \in (t_0, t_0 + T)$ and so

$$a(\varepsilon) \leqslant V(t_0 + T, \varphi, \psi) \leqslant V(t_0, \varphi, \psi) - C(\varepsilon) T \leqslant 0 ,$$

which is impossible. Hence the required result.

THEOREM 5.2. – Let $V(t, \varphi, \psi)$ be a Lyapunov functional such that

(i) $a(\|\psi\|_0) \leqslant V(t, \varphi, \psi)$ for $a \in \mathcal{K}$ and $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

(ii) $D^+ V(t, \varphi, \psi) \leqslant - C(\|\psi\|_0)$ for $C \in \mathcal{K}$ and

$$(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau .$$

Then the trivial solution $x = 0$, $y = 0$ of (1) is equi-asymptotically stable with respect to the y-component.

PROOF. – The proof is similar to that of the last theorem.

THEOREM 5.3. – Assume that there exist two functionals $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$ such that

(i) V and W are continuous and locally Lipschitzian in φ, and ψ,

$$V(t, 0, 0) = 0 \quad \text{and} \quad W(t, 0, 0) = 0 ,$$

(ii) $\exists a \in \mathcal{K}$ such that

$$a(\|\varphi\|_0) \leqslant V(t, \varphi, \psi) \qquad \text{for } (t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau;$$

(iii) $D^+ V(t, \varphi, \psi) \leqslant -C(\|\varphi\|_0)$ for $C \in \mathcal{K}$ and

$$(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau;$$

(iv) $\exists b \in \mathcal{K}$ such that

$$W(t, \varphi, \psi) \leqslant b(\|\psi\|_0)\,;\ (t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau\,.$$

(v) $W(t, \varphi, \psi)$ is bounded in the region H where

$$H = \{(t, \varphi, \psi): t \geqslant 0,\ V(t, \varphi, \psi) > 0 \text{ and } \|\varphi\|_0 + \|\psi\|_0 < \eta\};$$

(vi) $D^+ W(t, \varphi, \psi)$ is positive definite in the region H.

Then the trivial solution $x = 0$, $y = 0$ is equi-asymptotically stable with respect to the x-component and unstable with respect to the y-component.

PROOF. – Theorem 5.1 implies the equi-asymptotic stability with respect to the x-component and an argument similar to that of Theorem 3.3 shows that $x = 0$, $y = 0$ is unstable with respect to the y-component.

REMARK. – A similar theorem can be stated for the instability with respect to the x-component and equi-asymptotic stability with respect to the y-component.

THEOREM 5.4. Let $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$ be two functional defined for $0 \leqslant t < \infty$ such that hypothesis (i), (ii) and (iv) of Theorem 3.4 hold. Assume that the following also hold

(i) $\exists C \in \mathcal{K}$ such that for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

$$D^+ V(t, \varphi, \psi) \leqslant -C(\|\psi\|_0)$$

and

(ii) there exists $C^* \in \mathcal{K}$ such that for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

$$D^+ W(t, \varphi, \psi) \leqslant -C^*(\|\varphi\|_0)\,.$$

Then the trivial solution $x = 0$, $y = 0$ is equi-asymptotic stable with respect to the x-component and with respect to the y-component.

PROOF. – By Theorem 3.4 the solution $x=0$, $y=0$ is stable with respect to the x-component and the y-component. Hence given $0<\varepsilon<\tau$, $\exists\delta(t_0,\varepsilon)>0$ such that $\|\varphi\|_0+\|\psi\|_0<\delta$ implies $\|y_t(t_0;\varphi,\psi)\|<\varepsilon$. Let

$$\lambda(t_0)=\sup\{V(t_0,\varphi,\psi):\|\varphi\|_0+\|\psi\|_0<\delta\}\quad\text{and set}\quad T_0(t_0,\varepsilon)=\lambda(t_0)/C(\varepsilon)\,.$$

Suppose $\varepsilon\leqslant\|y_t(t_0;\varphi,\psi)\|<\tau$ for $t\in(t_0,t_0+T)$, then as in Theorem 5.1,

$$a(\varepsilon)\leqslant V(t_0+T_0,\varphi,\psi)\leqslant V(t_0,\varphi,\psi)-\lambda(t_0)\leqslant 0$$

which is impossible. Hence

$$\|y_t(t_0;\varphi,\psi)\|<\varepsilon\qquad\text{for all } t\geqslant t_0+T_0\,.$$

Now given $\varepsilon_1>0$, $\exists\delta_1$ such that $\|\varphi\|_0+\|\psi\|_0<\delta_1$ implies $\|x_t(t_0;\varphi,\psi)\|<\varepsilon_1$ for all $t\geqslant t_0$. Define $H(t_0)=\sup\{W(t_0,\varphi,\psi):\|\varphi\|_0+\|\psi\|_0<\delta_1\}$ and set $T_1(t_0,\varepsilon)=H(t_0)/C^*(\varepsilon_1)$. Choose $T=\max\{T_1,T_0\}$, and suppose $\varepsilon_1\leqslant\|x_t(t_0;\varphi,\psi)\|<\varrho$ for $t\in(t_0,t_0+T)$ then

$$a_1(\varepsilon_1)\leqslant a_1\big(\|x_t(t_0,\varphi,\psi)\big)\leqslant W(t_0+T,\varphi,\psi)\leqslant W(t_0,\varphi,\psi)-C^*(\varepsilon_1)T_1\leqslant 0\,,$$

which is impossible. Hence $\|x_{t^*}(t_0,\varphi,\psi)\|<\varepsilon_1$ for $t^*\in(t_0,t_0+T)$. So given $\varepsilon>0$, $0<\varepsilon<\tau$, $\varepsilon_1>0$, $0<\varepsilon_1<\varrho$, and choose $\hat{\delta}=\min\{\delta,\delta_1\}$

$$\|x_t(t_0;\varphi,\psi)\|<\varepsilon_1\qquad\text{and}\qquad\|y_t(t_0;\varphi,\psi)\|<\varepsilon$$

for all $t\geqslant t_0+T$ provided $\|\varphi\|_0+\|\psi\|_0<\hat{\delta}$, and the proof is complete.

6. – Generalized exponential stability with respect to components.

In this section we give necessary and sufficient conditions for the concept of stability of the generalized exponential type with respect to both components in terms of Lyapunov functionals. We then find conditions for this type of stability property to be preserved under certain perturbations.

THEOREM 6.1. – Let $f(t,\varphi,\psi)$ be linear in φ, ψ and $p(t)$ be a continuously differentiable function on R^+. Then the solution $x=0$, $y=0$ of the system (1) is generalized asymptotically exponentially stable with respect to the x-component if and only if there exists a continuous Lyapunov functional $V(t,\varphi,\psi)$ such that for $(t,\varphi,\psi)\in R^+\times C_\varrho\times C_\tau$,

(i) $\|\varphi\|_0\leqslant V(t,\varphi,\psi)\leqslant K(t)\big(\|\varphi\|_0+\|\psi\|_0\big)$,

(ii) $|V(t,\varphi_1,\psi_1)-V(t,\varphi_2,\psi_2)|\leqslant K(t)\big(\|\varphi_1-\varphi_2\|_0+\|\psi_1-\psi_2\|_0\big)$ and

(iii) $D^+V(t,\varphi,\psi')\leqslant -p'(t)V(t,\varphi,\psi)$.

PROOF – The sufficiency can be proved by integrating (iii) which yields

$$\|x_t(t_0;\varphi,\psi)\| \leqslant V(t,\varphi,\psi) \leqslant V(t_0,\varphi_0,\psi_0)\exp[p(t_0)-p(t)] \leqslant K(t)(\|\varphi_0\|_0+\|\psi_0\|_0)\exp[p(t_0)-p(t)]\,.$$

To prove the necessity, define a Lyapunov functional as follows: for $(t,\varphi,\psi)\in R^+\times C_\varrho\times C_\tau$,

$$V(t,\varphi,\psi)=\sup_{\sigma\geqslant 0}\{\|x_{t+\sigma}(t_0;\varphi,\psi)\|\exp[p(t+\sigma)-p(t)]\,.$$

At $\sigma=0$

$$\|x_t(t_0,\varphi,\psi)\|\leqslant V(t,\varphi,\psi)$$

and by assumption and the definition of $V(t,\varphi,\psi)$

$$V(t,\varphi,\psi)\leqslant K(t)(\|\varphi\|_0+\|\psi\|_0)\,.$$

Now let (φ,ψ); $(\varphi_0,\psi_0)\in C_\varrho\times C_\tau$, then using the linearity of f in (φ,ψ), we obtain

$$\begin{aligned}|V(t,\varphi,\psi)-V(t,\varphi_0,\psi)| &= |\sup_{\sigma\geqslant 0}\{\|x_{t+\sigma}(t_0;\varphi,\psi)\|\exp[p(t+\sigma)-p(t)]\} \\ &\qquad -\sup_{\sigma\geqslant 0}\{\|x_{t+\delta}(t_0;\varphi_0,\psi_0)\|\exp[p(t+\sigma)-p(t)]\}| \\ &\leqslant \sup_{\sigma\geqslant 0}\{\|x_{t+\sigma}(t_0;\varphi,\psi)-x_{t+\sigma}(t_0;\varphi_0,\psi_0)\|\exp[p(t+\sigma)-p(t)]\} \\ &= \sup_{\sigma\geqslant 0}\{\|x_{t+\sigma}(t_0;\varphi-\varphi_0,\psi-\psi_0)\|\exp[p(t+\sigma)-p(t)]\} \\ &\leqslant K(t)(\|\varphi-\varphi_0\|_0+\|\psi-\psi_0\|_0)\,.\end{aligned}$$

Moreover, by uniqueness of solutions, we have

$$x_{t+h+\sigma}(t_0+h,x_{t+h}(t_0;\varphi,\psi),y_{t+h}(t_0,\varphi,\psi))=x_{t+h+\sigma}(t_0;\varphi,\psi)$$

and so

$$\begin{aligned}&V(t+h,x_{t+h}(t,\varphi,\psi),y_{t+h}(t;\varphi,\psi))= \\ &\quad=\sup_{\sigma\geqslant 0}\{\|x_{t+h+\sigma}(t+h;x_{t+h}(t;\varphi,\psi),y_{t+h}(t;\varphi,\psi)\|\exp[p(t+h+\sigma)-p(t+h)]\} \\ &\quad=\sup_{\sigma\geqslant 0}\{\|x_{t+h+\sigma}(t;\varphi,\psi)\|\exp[p(t+h+\sigma)-p(t+h)]\} \\ &\quad=\sup_{\sigma\geqslant h}\{\|x_{t+\sigma}(t;\varphi,\psi)\|\exp[p(t+\sigma)-p(t)\}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&V(t+h,x_{tt+h}(t_0;\varphi,\psi),y_{+h}(t_0;\varphi,\psi))-V(t,\varphi,\psi) \\ &\qquad=\sup_{\sigma\geqslant h}\{\|x_{t+\sigma}(t_0;\varphi,\psi)\|\exp[p(t+\sigma)-p(t)]\} \\ &\qquad-\sup_{\sigma\geqslant 0}\{\|x_{t+\sigma}(t_0;\varphi,\psi)\|\exp[p(t+\sigma)-p(t)]\}\,,\end{aligned}$$

and

$$\limsup_{h\to 0^+}\frac{1}{h}\left[V\big(t+h,x_{t+h}(t_0;\varphi,\psi),y_{t+h}(t_0;\varphi,\psi)\big)-V(t,\varphi,\psi)\right]$$
$$\leqslant \limsup_{h\to 0^+}\frac{1}{h}\Big[\sup_{\sigma\geqslant 0}\big\{\|x_{t+\sigma}(t_0;\varphi,\psi)\|\exp[p(t+\sigma)-p(t)]\big\{\exp[p(t)-p(t+h)]-1\big\}\Big]$$
$$=V(t,\varphi,\psi)\limsup_{h\to 0^+}\frac{1}{h}\big\{\exp[p(t)-p(t+h)]-1\big\}.$$

Thus

$$D^+V(t,\varphi,\psi)\leqslant -p'(t)V(t,\varphi,\psi).$$

We now show that $V(t,\varphi,\psi)$ is continuous.

$$\begin{aligned}
|V(t+h,\varphi,\psi)-V(t,\varphi_0,\psi_0)| &\leqslant |V(t+h,\varphi,\psi)-V(t+h,\varphi_0,\psi_0)|\\
&\quad+|V(t+h,\varphi_0,\psi_0)-V\big(t+h,x_{t+h}(t_0;\varphi_0,\psi_0),y_{t+h}(t_0;\varphi_0,\psi_0)\big)|\\
&\quad+|V\big(t+h,x_{t+h}(t_0;\varphi_0,\psi_0),y_{t+h}(t_0;\varphi_0,\psi_0)\big)-V(t,\varphi_0,\psi_0)\big)|.\\
\leqslant K(t+h)\big(\|\varphi-\varphi_0\|_0+\|\psi-\psi_0\|_0\big)+&\\
&\quad+|V(t+h,\varphi_0,\psi_0)-V\big(t+h,x_{t+h}(t_0;\varphi_0,\psi_0),y_{t+h}(t_0;\varphi_0,\psi_0)\big)|\\
&\quad+|V\big(t+h,x_{t+h}(t_0;\varphi_0,\psi_0),y_{t+h}(t_0;\varphi_0,\psi_0)\big)-V(t,\varphi_0,\psi_0)|.
\end{aligned}$$

The first and second terms on the R.H.S. of the last inequality are small for h by the definitions of x_t and y_t and as $h\to 0$ the last term tends to zero, so that $V(t,\varphi,\psi)$ is continuous.

REMARK. – Theorem 6.1 is a generalization of Theorem 1 of [3].

The next theorem gives an analogue of the last theorem in terms of stability with respect to the y-component.

THEOREM 6.2. – Let $g(t,\varphi,\psi)$ be linear in (φ,ψ) and $p(t)$ be a continuously differentiable function on R^+. Then the solution $x=0$, $y=0$ of the system (1) is generalized exponentially asymptotically stable with respect to the y-component if and only if there exists a continuous Lyapunov functional $W(t,\varphi,\psi)$ such that for $(t,\varphi,\psi)\in R^+\times C_\varrho\times C_\tau$

(i) $\|\psi\|_0\leqslant W(t,\varphi,\psi)\leqslant M(t)\big(\|\varphi\|_0+\|\psi\|_0\big)$;

(ii) $|W(t,\varphi_1,\psi_1)-W(t,\varphi_2,\psi_2)|\leqslant M(t)\big(\|\varphi_1-\varphi_2\|_0+\|\psi_1-\psi_2\|_0\big)$

and

(iii) $D^+W(t,\varphi,\psi)\leqslant -p'(t)W(t,\varphi,\psi)$.

PROOF. – The sufficiently follows from a similar arguments as in Theorem 6.1. For the necessity, define

$$W(t, \varphi, \psi) = \sup_{\alpha \geqslant 0} \{\|y_{t+\alpha}(t_0; \varphi, \psi)\| \exp [p(t+\alpha) - p(t)]\}$$

for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$ and proceed as in Theorem 6.1.

THEOREM 6.3. – Assume that f, g and p satisfy the conditions of Theorems 6.1 and 6.2. Then the solution $x = 0$, $y = 0$ of the system (1) is generalized exponential asymptotically stable with respect to the x-component and with respect to the y-component simultaneously if and only if there exist two continuous Lyapunov functionals $V(t, \varphi, \psi)$ and $W(t, \varphi, \psi)$, satisfying the following properties.

(i) $\|\varphi\|_0 \leqslant V(t, \varphi, \psi) \leqslant K(t)(\|\varphi\|_0 + \|\psi\|_0)$;

(ii) $\|\psi\|_0 \leqslant W(t, \varphi, \psi) \leqslant M(t)(\|\varphi\|_0 + \|\psi\|_0)$;

(iii) $|V(t, \varphi, \psi) - V(t, \varphi_0, \psi_0)| \leqslant K(t)(\|\varphi - \varphi_0\|_0 + \|\psi - \psi_0\|_0)$;

(iv) $|W(t, \varphi, \psi) - W(t, \varphi_0, \psi_0)| \leqslant M(t)(\|\varphi - \varphi_0\|_0 + \|\psi - \psi_0\|_0)$;

(v) $D^+V(t, \varphi, \psi) \leqslant -p'(t)V(t, \varphi, \psi)$;

(vi) $D^+W(t, \varphi, \psi) \leqslant -p'(t)W(t, \varphi, \psi)$;

where $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$.

PROOF. – The same type of arguments in Theorem 3.4 modified along the lines of Theorem 6.1 proves the sufficiency. To prove the necessity we define V and W as follows: for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$,

$$V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|x_{t+\sigma}(t_0; \varphi, \psi)\| \exp [p(t+\sigma) - p(t)]\}$$

and

$$W(t, \varphi, \psi) = \sup_{\alpha \geqslant 0} \{\|y_{t+\alpha}(t_0; \varphi, \psi)\| \exp [p(t+\alpha) - p(t)]\} .$$

Proceeding with the rest of the proof with arguments parallel to that of Theorem 6.1 we have the result.

REMARK. – If in place of the linearity of f and g in (φ, ψ) we assume only Lipstichiz continuity the results of Theorems 6.1, 6.2 and 6.3 still remain valid. The following theorem is one of such results.

THEOREM 6.4. – Assume that $p(t)$ is continuously differentiable for $t \in R^+$ and $f(t, \varphi, \psi)$ and $g(t, \varphi, \psi)$ satisfy Lipschitz condition in φ and ψ respectively with constant $L > 0$. Let $K(t)$ be bounded and for some q with $0 < q < 1$, let there exist $T > 0$ such that

$$K(t) \exp[-q(p(t+T) - p(t))] \leqslant 1 \qquad \text{for } t \in R^+ .$$

Then the solution $x = 0$, $y = 0$ of the system (1) is generalized exponentially asymptotically stable with respect to the x-component if and only if there exists a continuous Lyapunov functional $V(t, \varphi, \psi)$ satisfying: for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$:

(i) $\|\varphi\|_0 \leqslant V(t, \varphi, \psi) \leqslant K(t)(\|\varphi\|_0 + \|\psi\|_0)$

(ii) $|V(t, \varphi, \psi) - V(t, \varphi_0, \psi_0)| \leqslant \exp[LT] \sup\limits_{0 \leqslant \sigma \leqslant T} \{\exp\ [(1-q)(p(t+\sigma)] - p(t))\} \cdot (\|\varphi - \varphi_0\|_0 + \|\psi - \psi_0\|_0)$

and

(iii) $D^+ V(t, \varphi, \psi) \leqslant -(1-q)p'(t) V(t, \varphi, \psi)$.

PROOF. – The sufficiency follows easily. Let q, T be as given and define for $(t, \varphi, \psi) \in R^+ \times C_\varrho \times C_\tau$.

$$V(t, \varphi, \psi) = \sup_{\sigma \geqslant 0} \{\|x_{t+\sigma}(t, \varphi, \psi)\| \exp[(1-q)(p(t+\sigma) - p(t))]\}$$

$K(t)$ is bounded, hence $\exists M = \sup\limits_{t \in R^+} K(t) < \infty$: Set $\varrho_0 = \varrho/M$, then $V \in C(R^+ \times C_{\varrho_0} \times \times C_{\varrho_0}, R^+)$, and (i) and (iii) thus follow using the arguments parallel to that of Theorem 6.1. We now establish (ii). Note that

$$\|x_{t+\sigma}(t; \varphi, \psi)\| \exp\ [(1-q)\{p(t+\sigma) - p(t)\}] \leqslant K(t)(\|\varphi\|_0 + \|\psi\|_0) \exp\ [-q(p(t+\sigma) - p(t))] ,$$

so that the assumption implies,

$$V(t, \varphi, \psi) = \sup_{0 \leqslant \sigma \leqslant T} \{\|x_{t+\sigma}(t_0; \varphi, \psi)\| \exp[(1-q)(\mathrm{p}(t+\sigma) - p(t))]\} .$$

Hence for $(\varphi_0, \psi_0) \in C_{\varrho_0} \times C_{\varrho_0}$ and $t \in R^+$,

$$|V(t, \varphi, \psi) - V(t, \varphi, \psi)| \leqslant \sup_{0 \leqslant \sigma \leqslant T} \{\|x_{t+\sigma}(t; \varphi, \psi) - x_{t+\sigma}(t; \varphi_0, \psi_0)\| \exp[(1-q)(p(t+\sigma) - p(t))]\} .$$

Define

$$m(t) = \|x_t(t; \varphi, \psi) - x_t(t; \varphi_0, \psi_0)\| + \|y_t(t; \varphi, \psi) - y_t(t; \varphi_0, \psi_0)\|$$

then

$$\underline{D}m(t) = \liminf_{h\to 0^-} \frac{m(t+h) - m(t)}{h}$$
$$\leqslant \|x_t'(t, \varphi, \psi) - x_t'(t; \varphi_0, \psi_0)\| + \|y_t'(t; \varphi, \psi) - y_t'(t; \varphi_0, \psi_0)\|$$
$$\leqslant \|x_t'(t; \varphi, \psi) - f(t, \varphi, \psi) - x_t'(t; \varphi_0, \psi_0) + f(t, \psi_0, \psi_0) + f(t, \varphi, \psi) - f(t, \varphi_0, \psi_0)\|$$
$$+ \|y_t'(t; \varphi, \psi) - g(t, \varphi, \psi) - y_t'(t; \varphi_0, \psi_0) + g(t, \varphi_0, \psi_0) + g(t, \varphi, \psi) - g(t, \varphi_0, \psi_0)\|$$
$$\leqslant \|f(t, \varphi, \psi) - f(t, \varphi_0, \psi_0)\| + \|g(t, \varphi, \psi) - g(t, \varphi_0, \psi_0)\|$$
$$\leqslant L\|\varphi - \varphi_0\|_0 + L\|\psi - \psi_0\|_0 = L[\|\varphi - \varphi_0\|_0 + \|\psi - \psi_0\|_0] = L|m_t|_0 .$$

Hence Lemma 6.1.1 of [6] implies

$$\|x_t(t; \varphi, \psi) - x_t(t; \varphi_0, \psi_0)\| \leqslant \|x_t(t; \varphi, \psi) - x_t(t; \varphi_0, \psi_0)\| + \|y_t(t; \varphi, \psi) - y_t(t; \varphi_0, \psi_0)\|$$
$$\leqslant \exp[L(t-t_0)](\|\varphi - \varphi_0\|_0 + \|\psi - \psi_0\|_0) .$$

Thus,

$$|V(t, \varphi, \psi) - V(t, \varphi_0, \psi_0)|$$
$$\leqslant \sup_{0\leqslant\sigma\leqslant T} [\|x_{t+\sigma}(t; \varphi, \psi) - x_{t+\sigma}(t; \varphi_0, \psi_0)\| \exp[(1-q)\{p(t+\sigma) - p(t)\}]$$
$$\leqslant \exp[LT] \sup_{0\leqslant\sigma\leqslant T} \exp[(1-q)(p(t+\sigma) - p(t))](\|\varphi - \varphi_0\|_0 + \|\psi - \psi_0\|_0) .$$

We now consider the perturbed system,

$$\begin{aligned} \dot{x}(t) &= f(t, x_t, y_t) + G(t, x_t, y_t) \\ \dot{y}(t) &= g(t, x_t, y_t) + H(t, x_t, y_t) \end{aligned} \tag{4}$$

where $G(t, \varphi, \psi)$ and $H(t, \varphi, \psi)$ are continuous mappings from $R^+ \times C([-h, 0], R^n) \times \times C([-h, 0], R^m)$ into R^n and R^m respectively. We assume that G and H satisfy locally a Lipschitz condition with respect to φ and ψ and such that

$$\|G(t, \varphi, \psi)\| + \|H(t, \varphi, \psi)\| \leqslant \omega(t, \|\varphi\|_0)$$

where $\omega(t, u)$ is a continuous scalar function for $t \geqslant 0$, $u \geqslant 0$, satisfying a locally Lipschitz condition and nondecreasing in u, with $\omega(t, 0) = 0$. We have the following generalization of Theorem 2 of [3].

THEOREM 6.5. – Assume that the solution $x = 0$, $y = 0$ of the system (1) is generalized exponential asymptotically stable with respect to the x-component or with respect to the y-component. Let f and g be linear in φ, ψ respectively. Then the trivial solution $x = 0$, $y = 0$ of (4) has the same stability property with respect to the x-component or with respect to the y-component as the solution $u = 0$ of the differential equation.

$$\dot{u} = -p'(t)u + K\omega(t, u), \qquad u(t_0) = u_0. \tag{5}$$

PROOF. – Let $x_t(t_0; \varphi_0, \psi_0)$ and $y_t(t_0; \varphi_0, \psi_0)$ be any solution of (4) such that $\|\varphi_0\|_0 + \|\psi_0\|_0 < \varrho/K(t_0)$. Setting $\varphi = x_t(t_0; \varphi_0, \psi_0)$ we have $x_{t+\delta}(t_0; \varphi_0, \psi_0) = x_{t+\delta}(t; \varphi, \psi)$, $\delta \geqslant 0$ by uniqueness. Suppose $z_{t+\delta}(t; \varphi, \psi)$, $w_{t+\delta}(t; \varphi, \psi)$, $\delta \geqslant 0$ is any solution of (1) through (t, φ, ψ), and let $\|x_t(t_0; \varphi_0, \psi_0)\| < \varrho$, $\|y_t(t_0; \varphi_0, \psi_0\| < \varrho$ for $t \geqslant t_0$, then it is easy to show that

$$D^+ V(t, \varphi, \psi) \leqslant -p'(t) V(t, \varphi, \psi) + K\omega\big(t, V(t, \varphi, \psi)\big).$$

Setting $m(t) = V(t, \varphi, \psi)$, then $V(t_0, \varphi_0, \psi_0) \leqslant u_0$ implies by Theorem 1.4.1 of [5] that

$$V\big(t, x_t(t_0; \varphi_0, \psi_0), y_t(t_0; \psi_0, \psi_0\big) \leqslant r(t, t_0, u_0) \qquad \text{for } t \geqslant t_0,$$

where $r(t, t_0, u_0)$ is the maximal solution of (5) existing for $t \geqslant t_0$.

If we choose $K(t_0)\big(\|\varphi_0\|_0 + \|\psi\|_0\big) = u_0$, then $u_0 < \varrho$ and by the assumption on the solutions of (1),

$$\|x_t(t_0; \varphi_0, \psi_0)\| \leqslant V\big(t; x_t(t_0, \varphi_0, \psi_0), y_t(t_0; \varphi_0, \psi_0)\big) \leqslant r(t, t_0, u_0).$$

The result then follows by the choices of u_0, for the stability with respect to the x-components. The same arguments establish stability with respect to the y-components.

COROLLARY 6.6. – The solution $x = 0$, $y = 0$ of (4) is asymptotically stable with respect to the x-component or the y-component if $w(t, u) = \lambda(t)u$ where

$$p(t_0) - p(t) + \int_{t_0}^{t} K(s)\lambda(s)\, ds \to -\infty \qquad \text{as } t \to \infty.$$

PROOF. – The general solution of (5) is

$$u(t, t_0, u_0) = u_0 \exp\left[p(t_0) - p(t) + \int_{t_0}^{t} K(s)\lambda(s)\right] \qquad t \geqslant t_0.$$

The result then follows from the last theorem.

REMARK. – If $p(t) = \alpha t$, $\alpha > 0$ and $K(t) = K > 0$ in Corollary 6.6, then the last Corollary reduces to a result of [3].

THEOREM 6.7. – Assume that the solution $x = 0$, $y = 0$ of the system (1) is generalized exponential asymptotically stable with respect to the x-component and with respect to the y-component simultaneously. Let f and g be linear in φ and ψ respectively. Then the zero solution of (4) has the same stability property with respect to the x-component and with respect to the y-component as the solution $u = 0$ of the differential equation (5).

PROOF. – Using arguments parallel to Theorem 6.6 (modified along Theorem 3.4) together with Theorem 6.3 establish the result. We omit details.

We now state analogous result to Theorem 6.6 and 6.7 for the case in which f, g satisfy the Lipschitz condition with respect to (φ, ψ).

THEOREM 6.8. – Assume that the hypothesis of Theorem 6.4 are satisfied. Assume that the solution $x = 0$, $y = 0$ of the system (1) is generalized exponential asymptotically stable with respect to the x-component or with respect to the y-component as the solution $u = 0$ of (5).

PROOF. – Using arguments parallel to that of Theorem 6.5 with obvious modifications, the result follows.

THEOREM 6.9. – Assume that the hypothesis of Theorem 6.4 are satisfied. Assume that the solution $x = 0$, $y = 0$ of the system (1) is generalized exponential asymptotically stable with respect to the x-component and with respect to the y-component simultaneously. Then the zero solution of (4) has the same stability property with respect to the x-component and with respect to the y-component as the solution $u = 0$ of the differential equation (5).

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