On the Coefficients of Functions Analytic in the Unit Disc Having Fast Rates of Growth (*).

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Summary. – *The present paper is concerned with functions analytic in the unit disc having rapidly increasing maximum moduli. To study the precise rates of growth of such functions the concept of index is introduced. Several growth parameters in terms of the index are defined for a function analytic in the unit disc and their characterizations in terms of the Taylor-series development of the function are obtained. The results in the present paper improve and refine the earlier results of* SONS (J. Math. Anal. Appl., **24** (1968), pp. 296-306), MACLANE (*Asymptotic values of holomorphic functions*, Rice University Studies, Houston, 1963), *and* KAPOOR *and* JUNEJA (Indian J. Pura Appl. Kath., **7** (3) (1976), pp. 241-248).

1. – Introduction.

Let $f(z)=\sum_{n=0}^{\infty}a_n z^{\lambda_n}$ be analytic in the unit disc $D=\{z\colon |z|<1\}$, $\lambda_0=0$ and $\{\lambda_n\}_{n=0}^{\infty}$ be the strictly increasing sequence of positive integers such that $a_n\neq 0$ for $n=1,2,\dots$. The rate of growth of $f(z)$ as measured by its maximum modulus $M(r)\equiv M(r,f)=\max_{|z|=r}|f(z)|$, $0<r<1$, is studied in terms of the order ϱ and lower order λ of $f(z)$ defined as $\varrho(\lambda)=\limsup_{r\to 1}(\inf)\{\log^+\log^+M(r)/-\log(1-r)\}$, where $\log^+x=\max(\log x, 0)$, $0\leqslant x\leqslant\infty$. A characterization of ϱ in terms of the coefficients a_n when $\lambda_n=n$ was given independently by BEUERMANN [1] and MACLANE [5, p. 47]. In case $f(z)$ is defined by a gap Taylor series the methods of Beuermann and Maclane can be easily adopted to prove $\varrho/(1+\varrho)=\limsup_{n\to\infty}(\log^+\log^+|a_n|/\log\lambda_n)$. Coefficient equivalents of λ can be found in [4, 3, p. 137].

Further, SONS [6] proved that for every function $f(z)$ analytic in D and having order ϱ $(0<\varrho<\infty)$, $1+\lambda\leqslant(1+\varrho)\liminf(\log\lambda_{n-1}/\log\lambda_n)$. But, for the function $f(z)=\sum_{j=0}^{\infty}\exp(k_j)z^{k_j^3}$ where k_j is an increasing sequence of positive integers such that $k_0>1$ and $k_{j+1}=k_j^2$, it can easily be seen that $\lambda=\frac{1}{2}$ and $\varrho=1$ so that the above result is not true in general.

SONS [6] also obtained a decomposition theorem for functions $f(z)$ analytic in D having $\varrho>\lambda$.

However, these results do not give any specific information about the growth of $f(z)$ if $M(r)$ is increasing so rapidly that the order of $f(z)$ is infinite. In the

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present paper we extend and improve the above results for the functions having rapidly increasing maximum modulus.

For a function $f(z)$, analytic in D, set

$$\varrho(q) = \limsup_{r\to 1^-} \frac{\log^{[q]} M(r)}{-\log(1-r)}, \tag{1.1}$$

where $\log^{[0]} M(r) = M(r)$ and $\log^{[q]} M(r) = \log(\log^{[q-1]} M(r))$, $q = 1, 2, \ldots$. To avoid the trivial cases we shall assume throughout that $M(r) \to \infty$ as $r \to 1^-$. We have the following definitions:

DEFINITION 1. – A function $f(z)$ analytic in D, is said to have the index q if $\varrho(q) < \infty$ and $\varrho(q-1) = \infty$, $q = 1, 2, \ldots$. If q is the index of $f(z)$, then $\varrho(q)$ is called the *q-order* of $f(z)$.

DEFINITION 2. – A function $f(z)$ analytic in D and having the index q is said to have *lower q-order* $\lambda(q)$ if

$$\lambda(q) = \liminf_{r\to 1^-} \frac{\log^{[q]} M(r)}{-\log(1-r)}, \qquad q = 1, 2, \ldots. \tag{1.2}$$

DEFINITION 3. – A function $f(z)$ analytic in D and having the index q is said to be of *regular q-growth* if $\varrho(q) = \lambda(q)$, $q = 1, 2, \ldots$. $f(z)$ is said to be of *irregular q-growth* if $\varrho(q) > \lambda(q)$, $q = 1, 2, \ldots$.

In the following sections coefficient characterizations of $\varrho(q)$ and $\lambda(q)$ for a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in the unit disc, are obtained. Furthermore, a decomposition theorem for functions of irregular q-growth is proved. For $q = 2$, these results have previously been the subject of the papers by SONS [6], KAPOOR [4], BEUERMANN [1] and MACLANE [5].

We observe that there is no loss of generality in restricting our study to the functions analytic in the unit disc D, since if $g(z)$ is analytic in $D_R \equiv \{z: |z| < R\}$, $0 < R < \infty$, then by a trivial transformation of the variable z we can construct a function analytic in D which, in view of Lemma 1 given below, has the same q-order and lower q-order as those of $g(z)$.

2. – Coefficient characterization of $\varrho(q)$.

THEOREM 1. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in* D *having index* q *and q-order* $\varrho(q)$, *then*

$$\varrho(q) + A(q) = \limsup_{n\to\infty} \frac{\log^{[q-1]} \lambda_n}{\log \lambda_n - \log^+ \log^+ |a_n|}, \qquad q = 2, 3, \ldots, \tag{2.1}$$

where, $A(q) = 1$ *if* $q = 2$ *and* $A(q) = 0$ *if* $q = 3, 4, \ldots$.

PROOF. – Let

$$\liminf_{n\to\infty} \frac{\log \lambda_n - \log^+ \log^+ |a_n|}{\log^{[q-1]} n} = \theta . \tag{2.2}$$

First assume $\theta < \infty$. For every $\varepsilon > 0$, (2.2) implies there exists a sequence $\{n_k\}$ of natural numbers such that

$$\log |a_{n_k}| > \lambda_{n_k} (\log^{[q-2]} \lambda_{n_k})^{-(\theta+\varepsilon)} .$$

Using Cauchy's estimate, this inequality gives for all $r (0 < r < 1)$ and all $k = 1, 2, \dots$

$$\log M(r) \geqslant \log |a_{n_k}| + \lambda_{n_k} \log r > \lambda_{n_k} (\log^{[q-2]} \lambda_{n_k})^{-(\theta+\varepsilon)} + \log r_k . \tag{2.4}$$

For the sequence r_k defined by

$$\log \frac{1}{r_k} = \frac{1}{e} (\log^{[q-2]} \lambda_{n_k})^{-(\theta+\varepsilon)}, \qquad k = 1, 2, \dots ,$$

(2.4) gives

$$\log M(r_k) > (1 - e) \lambda_{n_k} (\log^{[q-2]} \lambda_{n_k})^{-(\theta+\varepsilon)} = (e-1) \left\{ \exp^{[q-2]} \left(e \log \frac{1}{r_k} \right)^{-1/(\theta+\varepsilon)} \right\} \left(\log \frac{1}{r_k} \right),$$

from which a simple calculation would yield $\varrho(q) + \Lambda(q) \geqslant 1/\theta$. This inequality is trivially true if $\theta = \infty$.

To prove the reverse inequality let lim sup in (2.1) be β and assume that $\beta < \infty$, since there is nothing to prove if $\beta = \infty$. For given $\varepsilon > 0$ and for all $n > n_0 = n_0(\varepsilon)$, it follows that

$$\log^+ |a_n| < \lambda_n (\log^{[q-2]} \lambda_n)^{-1/\alpha} , \qquad \alpha = \beta + \varepsilon .$$

Thus,

$$M(r) < B + \sum_{n=n_0}^{\infty} \exp \{\lambda_n (\log^{[q-2]} \lambda_n)^{-1/\alpha}\} r^{\lambda_n} ,$$

where B is a positive constant. Choose

$$N = \left[\exp^{[q-2]} \left(\frac{1}{2} \log \frac{1}{r} \right)^{-\alpha} \right] .$$

If $n > N$, then the above estimate of $M(r)$ becomes

$$M(r) < B + N H(r) + \sum_{N+1}^{\infty} r^{n/2} \tag{2.5}$$

where $H(r) = \max_n \{\exp(\lambda_n(\log^{[q-2]}\lambda_n)^{-1/\alpha})\, r^{\lambda_n}\}$ and B is a positive constant. Since

$$\frac{r^{(N+1)/2}}{1-r^{\frac{1}{2}}} \to -\infty \text{ as } r \to 1\,,$$

we have $\sum_{N+1}^{\infty} r^{n/2} = o(1)$. Therefore, by (2.5),

$$\text{(2.6)} \qquad \log M(r) < \log N + \log H(r) + o(1)\,,$$
$$< \exp^{[q-3]}\left(\frac{1}{2}\log\frac{1}{r}\right)^{-\alpha} + \log H(r) + o(1)\,, \qquad \text{as } r \to 1^-.$$

Now,

$$\log H(r) \leqslant \log F(r) = \max_{0\leqslant x \leqslant \infty}\left\{x(\log^{[q-2]} x)^{-1/\alpha} - x\log\frac{1}{r}\right\},$$

and the maximum on the right hand side occurs at the point $x = x_0$ satisfying

$$(\log^{[q-2]} x)^{-1/\alpha} - \frac{x(\log^{[q-2]} x)^{-1/\alpha}}{\alpha \prod_{i=0}^{q-3} \log^{[i]} x} = \log\frac{1}{r}\,,$$

where the product occuring in the denominator is interpretd to be 1 if $q = 2$. The point x_0 is uniquely determined by the above equation since the left hand side is an increasing function for large values of x. Thus,

$$\log F(r) = \max_{0\leqslant x\leqslant\infty}\left\{\frac{x^2(\log^{[q-2]} x)^{-1/\alpha}}{\alpha\prod_{i=0}^{q-2}\log^{[i]} x}\right\}.$$

Since, $(\log^{[q-2]}x)^{-1/\alpha} = C(q)\log 1/r$, where $C(q) = \alpha/(\alpha-1)$ if $q = 2$ and $C(q) = 1$ if $q = 3, 4, \ldots$, it follows that for all r sufficiently close to 1,

$$\log F(r) = \frac{\{\exp^{[q-2]}(C(q)\log(1/r))^{-\alpha}\}(C(q)\log(1/r))}{\alpha\prod_{i=0}^{q-3}\exp^{[i]}(C(q)\log(1/r))^{-\alpha}}\,.$$

It is easily seen that

$$\frac{\left(\prod_{i=0}^{q-3}\exp^{[i]}(C(q)\log(1/r))^{-\alpha}\right)\exp^{[q-3]}(\frac{1}{2}\log 1/r)^{-\alpha}}{\{\exp^{[q-2]}(C(q)\log(1/r))^{-\alpha}\}(C(q)\log(1/r))} \to 0 \qquad \text{as } r \to 1\,.$$

Therefore, by (2.6), for all r sufficiently close to 1,

$$\log M(r) < \frac{\{\exp^{[q-2]}(C(q) \log (1/r))^{-\alpha}\}(C)q) \log (1/r))}{\alpha \prod_{i=0}^{q-3} \exp^{[i]}(C(q) \log (1/r))^{-\alpha}} \left(1 - o(1)\right) .$$

From the above inequality, simple calculations now yield that $\varrho(q) + \Lambda(q) \leqslant \beta + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\varrho(q) + \Lambda(q) \leqslant \beta$. This completes the proof of the theorem.

3. – Coefficient characterization of $\lambda(q)$.

We need the following lemmas. Lemma 1 follows on the lines of the proof of a corresponding results of SONS [6] for $q = 2$ and Lemma 3 follows on the lines of the proof of an earlier result in [2], hence we state them without proof.

LEMMA 1. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in* D *having index* q, $q = 2, 3, \ldots$, q*-order* $\varrho(q) > 0$ *and lower* q*-order* $\lambda(q)$, *then*

$$(3.1) \qquad \begin{matrix}\varrho(q)\\ \lambda(q)\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[q]} \mu(r)}{-\log (1-r)}$$

and

$$(3.2) \qquad \varrho(q) + \Lambda(q) = \limsup_{r\to 1} \frac{\log^{[q-1]} N(r)}{-\log (1-r)}$$

where, for $|z| = r$, $\mu(r) = \max_{n\geqslant 0} \{|a_n| r^{\lambda_n}\}$, $N(r) = \max \{\lambda_n : \mu(r) = |a_n| r^{\lambda_n}\}$, $0 < r < 1$, $\Lambda(q) = 1$ *if* $q = 2$ *and* $\Lambda(q) = 0$ *if* $q = 3, 4, \ldots$.

LEMMA 2. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in* D *having index* q, $q = 2, 3, \ldots$ *and lower* q*-order* $\lambda(q)$. *Let* $\{n_k\}$ *be an increasing sequence of positive integers. Then,*

$$(3.3) \qquad \lambda(q) + \Lambda(q) \geqslant \liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{k-1}}{\log \lambda_{n_k} - \log^+ \log^+ |a_{n_k}|} ,$$

where $\Lambda(q) = 1$ *if* $q = 2$ *and* $\Lambda(q) = 0$ *if* $q = 3, 4, \ldots$.

PROOF. – Let lim inf in (3.2) be δ. Without loss of generality we can assume $\delta > 0$. For any ε such that $\delta > \varepsilon > 0$, and for all $k > k_0 = k_0(\varepsilon)$, we have

$$\log |a_{n_k}| > \lambda_{n_k} (\log^{[q-2]} \lambda_{k-1})^{-1/(\delta-\varepsilon)} .$$

Let $r_k \leqslant r \leqslant r_{k+1}$, where

$$\log \frac{1}{r_k} = \frac{1}{e} (\log^{[q-2]} \lambda_{n_k})^{-1/(\delta-\varepsilon)} .$$

Using Cauchy's estimate, the above estimate of $\log |a_{n_k}|$ and the value of r_k, we get

$$\begin{aligned}\log M(r) &\geqslant \log |a_{n_k}| + \lambda_{n_k} \log r \\ &\geqslant \log |a_{n_k}| + \lambda_{n_k} \log r_k \\ &> \lambda_{n_k} (\log^{[q-2]} \lambda_{n_{k-1}})^{-1/(\delta-\varepsilon)} - \lambda_{n_k} \log \frac{1}{r_k} \\ &= \left(1 - \frac{1}{e}\right) \lambda_{n_k} (\log^{[q-2]} \lambda_{n_{k-1}})^{-1/(\delta-\varepsilon)} . \\ &\geqslant (e-1)\left(\log \frac{1}{r}\right) \exp^{[q-2]}\left(e \log \frac{1}{r}\right)^{-(\delta-\varepsilon)} .\end{aligned}$$

This estimate of $\log M(r)$ after some simple calculation yields $\lambda(q) + \Lambda(q) \geqslant \delta$. Hence the lemma.

From Lemma 2 and Theorem 1 it is clear that for a function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, analytic in D and having the index q, if

$$\text{(i)} \lim_{n\to\infty} \frac{\log^{[q-1]} \lambda_{n-1}}{\log^{[q-1]} \lambda_n} = 1 \quad \text{and (ii)} \ S_0 \equiv \lim_{n\to\infty} \frac{\log^{[q-1]} \lambda_n}{\log \lambda_n - \log^+ \log^+ |a_n|} \ \text{exists},$$

then $f(z)$ is of regular q-growth and $\varrho(q) = \lambda(q) = S_0 - \Lambda(q)$.

LEMMA 3. – *Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be analytic in D having inedx q, q-order $\varrho(q)$ (> 0) and lower q-order $\lambda(q)$. Further, let $\psi(n) \equiv |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ form a nondecreasing function of n for $n > n_0$ and*

$$\Lambda(q) + \lambda(q) = \liminf_{r\to 1} \frac{\log^{[q-1]} \nu(r)}{-\log(1-r)} .$$

Then,

$$\lambda(q) + \Lambda(q) = \liminf_{n\to\infty} \frac{\log^{[q-1]} \lambda_{n-1}}{\log \lambda_n - \log^+ \log^+ |a_n|} \tag{3.4}$$

where $\Lambda(q) = 1$ if $q = 2$ and $\Lambda(q) = 0$ if $q = 3, 4, \ldots$.

The following example shows that the hypothesis $\psi(n)$ is an increasing function of n for $n > n_0$ in the above lemma does not imply that $f(z)$ is of regular q-growth.

Example. - Let,

$$n_1 = 4\,, \qquad n_{k+1} = n_k^4 \ (k = 1, 2, \ldots)$$

$$r_1 = r_2 = r_3 = e\,, \qquad r_m = \exp(m^{-\frac{1}{2}}) \qquad \text{if} \qquad n_k \leqslant m < n_k^2$$

$$r_m = \exp(n_{k+1}^{-\frac{1}{2}}) + \frac{\exp(m^{-1}) - \exp(n_{k+1}^{-1})}{n_{k+1}} \qquad \text{if } n_k^2 \leqslant m < n_{k+1}$$

and let

$$f(z) = 1 + \sum_{k=1}^{\infty} (r_1 r_2 \ldots r_k) z^k\,. \tag{3.5}$$

Then $\psi(k) = |a_{k-1}/a_k| = 1/r_k$ is a decreasing function of k. Let,

$$p(k) = \frac{\log k}{-\log\log(1/r_k)}\,.$$

Then, $\lim_{k\to\infty} p(n_k^2) = 1$ and $\lim_{k\to\infty} p(n_{k+1}) = 2$. It can be easily seen that $\limsup_{k\leftarrow\infty} p(k) = 2$ and $\liminf_{k\to\infty} p(k) = 1$, so that by (3.2), $\varrho(2) = 1$ and $\lambda(2) = 0$. Thus, $f(z)$ is of irregular 2-growth.

Theorem 2. - *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in* D *having index* q, *q-order* $\varrho(q)$ (>0) *and lower q-order* $\lambda(q)$ *and*

$$\Lambda(q) + \lambda(q) = \liminf_{r\to 1} \frac{\log^{[q-1]} \nu(r)}{-\log(1-r)}\,.$$

Then,

$$\Lambda(q) + \lambda(q) = \max_{\{n_k\}} \left[\liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ |a_{n_k}|}\right], \tag{3.6}$$

where maximum in (3.6) *is taken over all increasing sequences* $\{n_k\}_{k=0}^{\infty}$ *of natural numbers, and* $\Lambda(q) = 1$ *if* $q = 2$, $\Lambda(q) = 0$ *if* $q = 3, 4, \ldots$.

Proof. - Let $S(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_{n_k}}$, $|z| < 1$, where $\{\lambda_{n_k}\}_{k=0}^{\infty}$ is the sequence of elements in the range set of $N(r)$. It is easily seen that $S(z)$ is analytic in D, and for every z in D, $f(z)$ and $S(z)$ have the same maximum term. Hence, by (3.1), the q-order and lower-q-order of $S(z)$ are the same as those of $f(z)$. Thus, $S(z)$ is of lower q-order $\lambda(q)$. Further, let $\varrho(n_k) = \max\{r\colon N(r) = \lambda_{n_k}\}$. Then, $\varrho(n_k) = \psi(n_k)$, and consequently, $\psi(n_k)$ is an increasing function of k. Therefore, $S(z)$ satisfies the hypothesis of Lemma 3 and so by (3.4) we have

$$\Lambda(q) + \lambda(q) = \liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log \lambda_{n_k} - \log^+ \log^+ |a_{n_k}|}\,. \tag{3.7}$$

But, from Lemma 2, we have

$$(3.8) \qquad \Lambda(q) + \lambda(q) \geqslant \max_{\{n_h\}} \left[\liminf_{h\to\infty} \frac{\log^{[q-1]} \lambda_{n_{h-1}}}{\log \lambda_{n_h} - \log^+ \log^+ |a_{n_h}|} \right].$$

Combining (3.7) and (3.8) we get (3.6). Hence the theorem.

Our next theorem generalizes a result of SONS [6]. The techniques employed here give an alternative proof of Sons result.

THEOREM 3. – *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ *be analytic in* D *having index* q $(q = 2, 3, \dots)$, q*-order* $\varrho(q)$ (>0) *and lower* q*-order* $\lambda(q)$ *and*

$$\Lambda(q) + \lambda(q) = \liminf_{r\to 1} \frac{\log^{[q-1]} \nu(r)}{-\log(1-r)}.$$

Then,

$$(3.9) \qquad A(q) + \Lambda(q) \leqslant (A(q) + \varrho(q)) \liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{k-1}}{\log^{[q-1]} \lambda_k}$$

where $A(q) = 1$ *if* $q = 2$ *and* $A(q) = 0$ *if* $q = 3, 4, \dots$.

PROOF. – From (3.6), we get

$$A(q) + \lambda(q) \leqslant \sup_{\{n_k\}} \left[\limsup_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_k}}{\log \lambda_{n_k} - \log^+ \log^+ |a_{n_k}|} \right] \sup_{\{n_k\}} \left[\liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log^{[q-1]} \lambda_{n_k}} \right]$$

$$= \left[\limsup_{k\to\infty} \frac{\log^{[q-1]} \lambda_k}{\log \lambda_k - \log^+ \log^+ |a_k|} \right] \sup_{\{n_k\}} \left[\liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log^{[q-1]} \lambda_{n_k}} \right]$$

which, in view of Theorem 1, gives

$$(3.10) \qquad \Lambda(q) + \lambda(q) \leqslant (A(q) + \varrho(q)) \sup_{\{n_k\}} \left[\liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log^{[q-1]} \lambda_{n_k}} \right]$$

Now, for any arbitrary sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers, let

$$\xi(m) = \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log^{[q-1]} \lambda_{n_k}} \text{ for } n_{k-1} < m \leqslant n_k, \qquad k = 2, 3, \dots$$

and

$$w(m) = \frac{\log^{[q-1]} \lambda_{m-1}}{\log^{[q-1]} \lambda_m} \qquad \text{for } m = 2, 3, \dots.$$

It is easily seen that $\xi(m) \leqslant w(m)$ for all m. Therefore it follows that

$$\liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{n_{k-1}}}{\log^{[q-1]} \lambda_{n_k}} = \liminf_{m\to\infty} \xi(m) \leqslant \liminf_{m\to\infty} w(m) = \liminf_{k\to\infty} \frac{\log^{[q-1]} \lambda_{k-1}}{\log^{[q-1]} \lambda_k}.$$

Since the sequence $\{n_k\}$ is arbitrary, this gives

$$\sup_{\{n_k\}}\left[\liminf_{k\to\infty}\frac{\log^{[q-1]}\lambda_{n_{k-1}}}{\log^{[q-1]}\lambda_{n_k}}\right]\leqslant\liminf_{k\to\infty}\frac{\log^{[q-1]}\lambda_{k-1}}{\log^{[q-1]}\lambda_k}\,.$$

In fact equality holds in this inequality since the reverse inequality is obviously true. Hence, (3.10) implies (3.9).

Theorem 3 implies that for functions of regular q-growth

$$\lim_{k\to\infty}\frac{\log^{[q-1]}\lambda_{k-1}}{\log^{[q-1]}\lambda_k}=1\,.$$

However, the converse need not to be true as is clear by the function (3.5) for which

$$\lim_{k\to\infty}\frac{\log\lambda_{k-1}}{\log\lambda_k}=1$$

but the function is irregular 2-growth.

4. – The decomposition theorem.

The following theorem concerns the functions analytic in D and having irregular q-growth.

THEOREM 4. – *Let* $f(z)=\sum_{k=0}^{\infty}a_k z^k$ *be analytic in* D *having index* q $(q=2,3,\ldots)$, *q-order* $\varrho(q)$ (>0) *and lower q-order* $\lambda(q)$, *and let* $\lambda(q)<\mu(q)<\varrho(q)$. *Then,*

$$f(z)=g(z)+h(z)$$

where $g(z)$ *has q-order less than or equal to* $\mu(q)$ *and*

$$h(z)=\sum_{k=0}^{\infty}a_{m_k}z^{m_k}$$

such that

$$\lambda(q)\geqslant\mu(q)\liminf_{k\to\infty}\frac{\log^{[q-1]}m_{k-1}}{\log^{[q-1]}m_k} \tag{4.1}$$

PROOF. – Let $g(z)=\sum_{k=0}^{\infty}a_k z^k$, where

$$\begin{aligned}a_k&=c_k\quad\text{if}\quad\log^+|c_k|<k(\log^{[q-2]}k)^{-1/(A(q)+\mu(q))}\\&=0\quad\text{otherwise}\,.\end{aligned}$$

Then $g(z)$ is of q-order $\varrho(q)$. Let

$$h(z) = f(z) - g(z) = \sum_{k=0}^{\infty} a_{m_k} z^{m_k}$$

and set $A_{m_k} = |a_{m_k}|$. Then

$$\log^+ A_{m_k} > m_k (\log^{[q-2]} m_k)^{-1/(\Lambda(q)+\mu(q))} .$$

Now, let r_k be uniquely defined by the equation

$$(4.2) \qquad (\log^{[q-2]} m_k)^{-1/(\Lambda(q)+\mu(q))} - \frac{m_k (\log^{[q-2]} m_k)^{-1/(\Lambda(q)+\mu(q))}}{(\Lambda(q) + \mu(q)) \prod_{i=0}^{q-2} \log^{[i]} m_k} = \log \frac{1}{r_k},$$

and choose r such that $r_k \leqslant r \leqslant r_{k+1}$. In view of the above estimate of $\log^+ A_{m_k}$, we have

$$\log M(r) \geqslant \log A_{m_k} - m_k \log \frac{1}{r} \geqslant \log A_{m_k} - m_k \log \frac{1}{r_k}$$

$$> m_k (\log^{[q-2]} m_k)^{-1/(\Lambda(q)+\mu(q))} - m_k \log \frac{1}{r_k} .$$

Now, using (4.2), we have, for all k,

$$(4.3) \qquad \log M(r) > \frac{m_k^2 (\log^{[q-2]} m_k)^{-1/(\Lambda(q)+\mu(q))}}{(\Lambda(q) + \mu(q)) \prod_{i=0}^{q-2} \log^{[i]} m_k} .$$

From (4.2), it follows that

$$\left(\log \frac{1}{r_k}\right)^{-1} = \frac{\mu(2)}{\mu(2)+1} m_k^{-1/(\mu(2)+1)} \qquad \text{if } q = 2$$

and

$$\log \frac{1}{r_k} = (\log^{[q-2]} m_k)^{-1/\mu(q)} + o(1) \qquad \text{as } k \to \infty \text{ if } q = 3, 4, \ldots .$$

Thus, after some simple calculation (4.3) yields

$$\varrho(q) \geqslant \mu(q) \liminf_{k\to\infty} \frac{\log^{[q-1]} m_k}{\log^{[q-1]} m_{k+1}}$$

which proves (4.1) and the proof is complete.

5. – Functions having the same q-order.

We observe that if any two functions analytic in D have the same q-order and lower q-order then the notions in Section 1 do not give a precise information about their comparative rates of growth. For this purpose, we introduce the following definitions:

DEFINITION 1. – A function $f(z)$ analytic in D and having q-order $\varrho(q)$, $(\varrho(q) > 0$, $q = 2, 3, \dots)$, is said to have *q-type* $T(q)$ and *lower q-type* $t(q)$ if

$$\frac{T(q)}{t(q)} = \lim_{r\to 1} \frac{\sup}{\inf} \frac{\log^{[q-1]} M(r)}{(1-r)^{-\varrho(q)}} .$$

DEFINITION 2. – A function $f(z)$ analytic in D is said to be of growth $(\varrho(q), T(q))$ if it is of q-order not exceeding $\varrho(q)$ and q-type not exceeding $T(q)$ if it is of q-order $\varrho(q)$.

The tecniques employed in Sections 2, 3 and 4 can be suitably modified to give the following coefficient characterizations for the q-type and lower q-type.

THEOREM 5. – *Let* $V(q) = \limsup_{n\to\infty} (\log^{[q-2]} \lambda_n)(\log^+|a_n|/\lambda_n)^{\varrho(q)+A(q)}$, $\varrho(q) > 0$, $q = 2, 3, \dots$, $A(q) = 1$ *if* $q = 2$ *and* $A(q) = 0$ *if* $q = 3, 4, \dots$. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* D. *For* $0 < V(q) < \infty$, *the function* $f(z)$ *is of q-order* $\varrho(q)$ *and q-type* $T(q)$ *if and only if* $T(q) = B(q) V(q)$ *where* $B(q) = (\varrho(2)+1)^{\varrho(2)+1}/(\varrho(2))^{\varrho(2)}$ *for* $q = 2$ *and* $B(q) = 1$ *if* $q = 3, 4, \dots$. *If* $V(q) = 0$ *or* ∞, $f(z)$ *is respectively of growth* $(\varrho(q), 0)$ *or of growth* $(\varrho(q), \infty)$, *and conversely.*

REMARK. – Theorem 5 generalizes a result in [3, p. 156], obtained for $q = 2$. Later, the same result for $q = 2$, with a trivial transformation of the variables z has again been obtained by BAJPAI, *et. al.* [8].

THEOREM 6. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* D *and have q-order* $\varrho(q)$, $(\varrho(q) > 0)$, *and lower q-type* $t(q)$. *Let* $\{n_k\}$ *be an increasing sequence of natural numbers. Then,*

$$B(q)t(q) \geqslant \liminf_{k\to\infty} [(\log^{[q-2]} \lambda_{n_{k-1}}) (\log^+ |a_{n_k}|^{1/\lambda_{n_{k-1}}})^{\varrho(q)+A(q)}]$$

where $B(q)$ *and* $A(q)$ *are as in Theorem* 5.

THEOREM 7. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in* D *and have q-order* $\varrho(q)$, $(\varrho(q) > 0)$ *and lower q-type* $t(q)$. *If* $\psi(n) \equiv |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *forms a nondecreasing sequence of* n

for $n > n_0$, *then*

$$B(q)t(q) \leqslant \liminf_{n\to\infty}\left[(\log^{[q-2]}\lambda_n)(\log^+|a_n|/\lambda_n)^{\varrho(q)+A(q)}\right]$$
$$\leqslant L(q) \times \liminf_{n\to\infty}\left[(\log^{[q-2]}\lambda_{n-1})(\log^+|a_n|/\lambda_n)^{\varrho(q)+A(q)}\right]$$

where $B(q)$ *and* $A(q)$ *are as in Theorem* 5 *and* $L(q) = \limsup_{n\to\infty}(\log^{[q-2]}\lambda_n/\log^{[q-2]}\lambda_{n-1})$. *The second inequality holds whenever the product on its irght hand side is defined.*

Theorems 6 and 7 give the following coefficient characterization of the lower q-type for a subclass of functions analytic in D.

THEOREM 8. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* D *having* q*-order* $\varrho(q)$, $(\varrho(q) > 0)$ *and lower* q*-type* $t(q)$. *Let* $\psi(n) \equiv |a_n/a_{n+1}|^{1/(\lambda_{n+1}-\lambda_n)}$ *forms a nondecreasing function of* n *for* $n > n_0$ *and* $\log^{[q-2]}\lambda_n \sim \log^{[q-2]}\lambda_{n+1}$ *as* $n \to \infty$, *then*

$$B(q)t(q) = \liminf_{n\to\infty}\left[(\log^{[q-2]}\lambda_{n-1})(\log^+|a_n|/\lambda_n)^{(\varrho(q)+A(q))}\right]$$

where $B(q)$ *and* $A(q)$ *are as in Theorem* 5.

REMARK. – For $q = 2$, Theorem 8 gives a result contained in [3] and is an improvement over a similar result for $q = 2$ obtained later by BAJPAI, *et. al.* [8] under the additional hypothesis, $\log\mu(r) \sim \log M(r)$ as $r \to \infty$.

Our next theorem is a coefficient characterization of the lower q-type for the class of functions analytic in D having a restriction on the range set of the central index $N(r)$.

THEOREM 9. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* D *having* q*-order* $\varrho(q)$, $(\varrho(q) > 0)$, *and lower* q*-type* $t(q)$. *Let* $\{\lambda_{p_k}\}$ *be the range set of the central index* $N(r)$ *of* $f(z)$ *such that* $\log^{[q-2]}\lambda_{p_k} \sim \log^{[q-2]}\lambda_{p_{k+1}}$ *as* $k \to \infty$. *Then,*

$$t(q) = \max_{\{n_k\}}\left[\liminf_{k\to\infty}\left(\frac{\log^{[q-2]}\lambda_{n_{k-1}}}{B(q)}\right)\left(\frac{\log^+|a_{n_k}|}{\lambda_{n_k}}\right)^{\varrho(q)+A(q)}\right].$$

where $B(q)$ *and* $A(q)$ *are as in Theorem* 5 *and the maximum is taken over all increasing sequences* $\{n_k\}$ *of natural numbers.*

Now, we have a decomposition theorem for functions analytic in D for which $t(q) < T(q)$.

THEOREM 10. – *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *be analytic in* D *and have* q*-order* $\varrho(q)$, $(\varrho(q) > 0)$, q*-type* $T(q)$ *and lower* q*-type* $t(q)$. *Let* $t(q) < \eta(q) < T(q)$. *Then,*

$$f(z) = g(z) + h(z)$$

where $g(z)$ is analytic in D having q-type less than or equal to $\eta(q)$ and

$$h(z)=\sum_{k=0}^{\infty} a_{m_k} z^{\lambda_{m_k}}$$

such that

$$t(q)\geqslant \eta(q)\liminf_{k\to\infty}\left(\frac{\log^{[q-2]}\lambda_{m_k}}{\log^{[q-2]}\lambda_{m_{k+1}}}\right)^{\alpha(q)},$$

where $\alpha(q)=\varrho(2)/(\varrho(2)+1)$ if $q=2$ and $\alpha(q)=1$ if $q=3,4,\ldots$.

The proofs of the results in this section would appear elsewhere.

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