Weighted Sobolev Spaces and the Nonlinear Dirichlet Problem in Unbounded Domains (*) (**).

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Sunto. – *Si dimostrano proprietà di continuità e di compattezza per una classe di operatori differenziali non lineari. In virtù di tali proprietà ed usando metodi di monotonia, si provano teoremi di esistenza di soluzioni per alcuni problemi al contorno non lineari in domini non limitati.*

Introduction.

It is well known that elliptic boundary value problems in unbounded domains present difficulties which are in general more severe than those encountered in the study of similar problems in bounded domains.

Some of these difficulties are due to the lack of compact embedding theorems such as that of Rellich-Kondrachov for bounded domains. Some authors have overcome this difficulty by using suitable classes of weighted Sobolev spaces (for linear problems cf., among others, [1], [3], [8], [11]).

In the last years also nonlinear boundary value problems have been studied whether in the framework of weighted-Sobolev spaces (cf. [6]. [7]) or by other tools ([4], [9]).

Another difficulty in the study of the above problems is due to the fact that the Poincaré-inequality does not hold in the ordinary Sobolev spaces $W_0^{1,p}(\Omega)$, if Ω is a general unbounded domain.

In this paper we prove continuity, compactness and coercitity properties for a class of non linear differential operators between weighted spaces.

In this framework it is then possible, by applying monotonicity methods, to prove existence theorems for the solutions of non linear Dirichlet problems in unbounded domains. The present paper is organized as follows.

In § 1 we study some embedding properties for a class of weighted Sobolev spaces. In § 2 we prove a necessary and sufficient condition in order that the Nemytskii

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operator F associated to a function $f\colon \Omega\times R^k \to R$ be continuous between weighted spaces.

This condition is utilized to prove the continuity, the compactness and the complete continuity of some nonlinear differential operators. These results include the previous ones ([4], [6], [7]), also between the ordinary Sobolev spaces.

In § 3 we consider some boundary value problems for which the tools developed in the previous paragraphs can be used: Problem I is an easy application of the results contained in § 2. Problem II and problem III (which is linear) represent situations in which the Hardy inequality

$$\int_\Omega |x|^s |u(x)|^2\,dx \leqslant \text{const}\int_\Omega |x|^{s+2}\,|\operatorname{grad} u(x)|^2\,dx$$

(where $s\in R$, $s\neq -n$, $u\in C_0^\infty(\Omega)$ and $0\notin\bar\Omega$) is the main tool.

At last (problem IV) we deal with a noncompact perturbation of the operator $-\Delta u+\lambda u$ $(\lambda>0)$. This problem has more general assumptions than problems considered in [9] in the case in which the leading term is the Laplacian.

1. – Some preliminary results.

Let Ω be an open set in R^n with boundary $\partial\Omega$; let $p\in[1,+\infty]$ and m an integer, nonnegative number.

We shall consider the following function spaces (1): $\mathcal{D}(\Omega)$ is the space of the infinitely differentiable functions on Ω and with compact support in Ω, equipped with the inductive limit topology of L. Schwartz.

If ϱ is a positive, (Lebesgue) measurable function on Ω, we denote by $L^p(\Omega,\varrho)$ the space of (equivalence class of) functions u on Ω, which are (Lebesgue) measurable and satisfy

$$\|u\|_{L^p(\Omega,\varrho)}=\begin{cases}\left(\int_\Omega |u(x)|^p\varrho(x)\,dx\right)^{1/p}<+\infty & \text{for } p\in[1,+\infty[\\ \sup\operatorname{ess}\varrho(x)|u(x)|<+\infty & \text{for } p=+\infty\end{cases}$$

equipped with the norm $\|\cdot\|_{L^p(\Omega,\varrho)}$.

As usual, we set $L^p(\Omega,1)=L^p(\Omega)$.

If $\varrho_0,\dots,\varrho_m$ are positive, measurable functions on Ω we denote by $\Gamma^{m,p}(\Omega,\varrho_0,\dots,\varrho_m)$ the space of distributions u on Ω such that (2)

$$D^\alpha u\in L^p(\Omega,\varrho_{|\alpha|})\,,\qquad \forall\alpha\in N^n,\ |\alpha|\leqslant m$$

(1) If $\alpha=(\alpha_1,\dots,\alpha_n)$ is an n-tuple of nonnegative integers we set $|\alpha|=\alpha_1+\dots+\alpha_n$ and $D^\alpha=\partial^\alpha/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$.

(2) We shall denote by R, R+, N respectively the set of real numbers, the set of positive real numbers, the set of nonnegative integers.

normed by

$$\|u\|_{\Gamma^{m,p}(\Omega,\varrho_0,\dots,\varrho_m)}=\begin{cases}\Big(\sum_{|\alpha|\leqslant m}\|D^\alpha u\|^p_{L^p(\Omega,\varrho_{|\alpha|})}\Big)^{1/p} & \text{for } p\in[1,+\infty[\\ \max_{|\alpha|\leqslant m}\|D^\alpha u\|_{L^\infty(\Omega,\varrho_{|\alpha|})} & \text{for } p=+\infty\,.\end{cases}$$

As usual, we set $\Gamma^{m,p}(\Omega,1,\dots,1)=W^{m,p}(\Omega)$.

Let U and V be two Banach spaces: a map $f\colon U\to V$ is called compact if it is continuous and maps bounded sets of U into relatively compact sets of V; f is called (sequentially) completely continuous if [3]

$$(u_n\rightharpoonup u_0 \text{ in } U)\Rightarrow\big(f(u_n)\to f(u_0) \text{ in } V\big)\,.$$

Let us now recall the following well known result (Hardy inequality).

THEOREM 1.1. – *If $p>1$, $s,t\in R$, $s\neq -n$ with $t=s+p$ and $0\notin\bar\Omega$ then there exists $c>0$ such that for each $u\in\mathcal{D}(\Omega)$*

$$\int_\Omega |x|^s|u(x)|^p\,dx\leqslant c\int_\Omega|x|^t|\operatorname{grad}u(x)|^p\,dx\,. \tag{1.1}$$

We remark that, if $s=-n$, (1.1) is true with $t>s+p$.

LEMMA 1.2. – *Let σ_1, σ_2 be two positive measurable functions on Ω and $p\in]1,+\infty[$, $p'=p/(p-1)$; then there exists an isometric isomorphism $\Lambda_{\sigma_1,\sigma_2}$ between $(L^p(\Omega,\sigma_1))'$ and $L^{p'}(\Omega,\sigma_2)$ defined as follows* [4] $\Lambda_{\sigma_1,\sigma_2}(T)=g\in L^{p'}(\Omega,\sigma_2)$ s.t. $\forall\varphi\in L^p(\Omega,\sigma_1)$:

$$\langle T,\varphi\rangle=\int_\Omega g(x)\varphi(x)\big(\sigma_1(x)\big)^{1/p}\big(\sigma_2(x)\big)^{1/p'}dx\,.$$

PROOF. – Let us observe that the map

$$\Phi_{\sigma_1,\sigma_2}\colon L^{p'}(\Omega,\sigma_1)\to L^{p'}(\Omega,\sigma_2)\,,\qquad \Phi_{\sigma_1,\sigma_2}(f)=\Big(\frac{\sigma_1}{\sigma_2}\Big)^{1/p'}f\qquad \forall f\in L^{p'}(\Omega,\sigma_1)$$

is an isometric isomorphism.

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(3) We denote by « $u_n\rightharpoonup u_0$ » and « $u_n\to u_0$ » respectively the weak and the strong convergence of the sequence $\{u_n\}$ to the element u_0.

(4) If X a topological vector space, X' denotes its topological dual and $\langle\cdot,\cdot\rangle$ the canonical pairing.

On the other hand the map

$$\Lambda_{\sigma_1}\colon (L^p(\Omega, \sigma_1))' \to L^{p'}(\Omega, \sigma_1)\,, \qquad \Lambda_{\sigma_1}(T) = g \ \text{s.t.}$$
$$\forall \varphi \in L^p(\Omega, \sigma_1)\, \langle T, \varphi\rangle = \int_\Omega g(x)\varphi(x)\sigma_1(x)\, dx$$

is an isometric isomorphism. It is obvious that $\Lambda_{\sigma_1,\sigma_2} = \Phi_{\sigma_1,\sigma_2}\circ\Lambda_{\sigma_1}$ satisfies the required properties. Q.E.D.

By Lemma 2.1 it is easily deduced that $L^p(\Omega, \sigma)$ $(p > 1)$ is reflexive.

THEOREM 1.3. – *Let* $p \in]1, +\infty[$, $m \in N$ *and assume that for each* $\Omega_0 \subset \Omega$ (Ω_0 *bounded and open*) $\inf_{x\in\Omega_0} \operatorname{ess} \varrho_{|\alpha|}(x) > 0$ $(|\alpha| \leqslant m)$. *Then* $\Gamma^{m,p}(\Omega, p_0, \ldots, p_m)$ *is a reflexive, separable Banach space.*

PROOF. – Let $E = \prod_{|\alpha|\leqslant m} L^p(\Omega, \varrho_{|\alpha|})$ equipped with the canonical norm: E is a separable, reflexive space. Let us consider the map

$$P\colon u \to (D^\alpha u)_{|\alpha|\leqslant m}\,, \qquad u \in \Gamma^{m,p}(\Omega, \varrho_0, \ldots, \varrho_m)\,.$$

It is now easy to verify that P is an isometric isomorphism of $\Gamma^{m,p}(\Omega, \varrho_0, \ldots, \varrho_m)$ into a closed subspace $F \subset E$. Q.E.D.

Let $\varrho \in C^\infty(R^n)$ be a positive function satisfying the following properties

(1.2) $$\varrho(x) \to +\infty \ \text{for} \ |x| \to +\infty$$

(1.3) $$\forall r \in R \ \text{and} \ \alpha \in N^n,\ \exists c \in R_+ \ \text{s.t.}$$
$$|(D^\alpha \varrho^r)(x)| \leqslant c(\varrho(x))^r\,, \qquad \forall x \in R^n \ (^5)\,.$$

If $m \in N$, $p \in [1, +\infty[$ we set

$$U_s^{m,p}(\Omega) = \Gamma^{m,p}(\Omega, \varrho^{sp}, \ldots, \varrho^{sp})\,, \qquad \mathring{U}_s^{m,p}(\Omega) = \overline{\mathfrak{D}(\Omega)} \ (\text{closure in } U_s^{m,p}(\Omega))$$

and denote by

$C_{b,s}^m(\Omega)$ the space of functions u on Ω which possess continuous partial derivatives up to order m and such that

$$\|u\|_{C_{b,s}^m(\Omega)} = \max_{|\alpha|\leqslant m}\Big(\sup_{x\in\Omega} \varrho^s(x)|D^\alpha u(x)|\Big) < +\infty$$

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(5) We set ϱ^r the map $x \in \Omega \to (\varrho(x))^r$. The function $\varrho(x) = (1 + |x|^2)^{\frac{1}{2}}$ satisfies the property (1.2).

THEOREM 1.4. – *Let l, m be two nonnegative integers with $m<l$ there if Ω has the cone property, the following embeddings are continuous* (6):

$$U_s^{l,p}(\Omega) \hookrightarrow U_s^{m,q}(\Omega)\,, \qquad \forall q \in [p, p^*[\ (^7)\,, \tag{1.4}$$

$$U_s^{l,p}(\Omega) \hookrightarrow C_{b,s}^{m}(\Omega) \qquad \text{if } (l-m)p > n\,. \tag{1.5}$$

PROOF. – If $q \in [p, p^*[$, the embedding

$$i\colon W^{l,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$$

is continuous. On the other hand by virtue of (1.3), it is easy to verify that the maps

$$\Phi_s\colon \ u \mapsto \varrho^s u$$

$$\Phi_{-s}\colon u \mapsto \varrho^{-s} u$$

are topological isomorphisms respectively of $U_s^{l,p}(\Omega)$ onto $W^{l,p}(\Omega)$ and of $W^{m,q}(\Omega)$ onto $U_s^{m,q}(\Omega)$.

Therefore the embedding $j = \Phi_{-s} \circ i \circ \Phi_s$ of $U_s^{l,p}(\Omega)$ into $U_s^{m,q}(\Omega)$ is continuous.

The continuity of the embedding (1.5) can be proved in an analogous manner.

Q.E.D.

By theorem 1.4 and by following analogous arguments as in proving th. 2.8 of [2], it can be deduced the following result

COROLLARY 1.5. – *Under the same hypotheses and notations of Theorem* 1.4, *if $\varepsilon \in R_+$ and $p \in]1, +\infty[$, the embedding $U_s^{l,p}(\Omega) \hookrightarrow U_{s-\varepsilon}^{m,q}(\Omega)$ is compact for each $q \in [p, p^*[$; moreover, if $(l-m)p > n$, $U_s^{l,p}(\Omega)$ is compactly embedded into $C_{b,s-\varepsilon}^{m}(\Omega)$.*

By following standard arguments it can be proved that the topological dual $(\mathring{U}_s^{m,p}(\Omega))' = U_{-s}^{-m,p'}(\Omega)$ $(p \in]1, +\infty[, p' = p/(p-1))$ of the space $\mathring{U}_s^{m,p}(\Omega)$ can be «identified» with the space of distributions u in Ω which are equal to a finite sum of derivatives of order $\leqslant m$ of functions belonging to $U_{-s}^{0,p'}(\Omega)$.

2. – On the continuity and compactness of some differential operators between weighted Sobolev spaces.

Let us consider a map

$$f\colon \Omega \times R^k \to R \qquad (k \in N)$$

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(6) $p^* = \begin{cases} np/n-(l-m)p & \text{if } n > (l-m)p\,, \\ +\infty & \text{if } n \leqslant (l-m)p\,. \end{cases}$

(7) If $p^* < +\infty$ embedding (1.4) holds $\forall q \in [p, p^*]$.

satisfying the Caratheodory conditions:

(C_1) for almost every $x\in\Omega$, $f(x,\cdot)$ is continuous in R^k,

(C_2) for every $(y_1,\dots,y_k)\in R^k$, $f(\cdot,y_1,\dots,y_k)$ is measurable in Ω.

Denote by F the Nemytskii operator associated to f, i.e.

$$(F(u))(x)=f(x,u(x))\,,\qquad u(x)=\big(u_1(x),\dots,u_k(x)\big)\,,\qquad x\in\Omega\,.$$

Let us consider $k+1$ continuous and positive functions

$$\sigma_i\colon \Omega\to R_+\,,\qquad i\in\{0,\dots,k\}$$

then the following theorem holds:

THEOREM 2.1. – *If* $p_1,\dots,p_k$, $q\in[1,+\infty[$ *then following statements are equivalent*:

(*a*) *there exist* $g\in L^q(\Omega,\sigma_0)$, $b\in R_+$ *such that* $\forall(x,y_1,\dots,y_k)\in\Omega\times R^k$,

$$|f(x,y_1,\dots,y_k)|\leqslant g(x)+b\sum_{i=1}^{k}\big(\sigma_i(x)/\sigma_0(x)\big)^{1/q}|y_i|^{p_i/q}\,,$$

(*b*) $F\Big(\prod_{i=1}^{k}L^{p_i}(\Omega,\sigma_i)\Big)\subset L^q(\Omega,\sigma_0)\,,$

(*c*) F *is continuous from* $\prod_{i=1}^{k}L^{p_i}(\Omega,\sigma_i)$ *into* $L^q(\Omega,\sigma_0)$.

PROOF. – It is obvious that $(a)\Rightarrow(b)$. Let us prove that $(b)\Rightarrow(c)$. Let us consider the map $\tilde f\colon \Omega\times R^k\to R$ such that

$$\tilde f(x,y_1,\dots,y_k)=\big(\sigma_0(x)\big)^{1/q}f\left(x,\frac{y_1}{\big(\sigma_1(x)\big)^{1/p_1}},\dots,\frac{y_k}{\big(\sigma_k(x)\big)^{1/p_k}}\right)$$

$\tilde f$ satisfies the Caratheodory conditions (C_1), (C_2), and denote by $\tilde F$ the Nemytskii operator associated to $\tilde f$.

Let us consider the isometric isomorphisms:

$$\Phi_{\sigma_i,1}\colon L^{p_i}(\Omega,\sigma_i)\to L^{p_i}(\Omega)\,,\qquad \Phi(f)=(\sigma_i)^{1/p_i}f\,,\qquad i\in\{1,\dots,k\}\,,$$

$$\Phi_{\sigma_0,1}\colon L^q(\Omega,\sigma_0)\to L^q(\Omega)\,,\qquad \Phi(f)=(\sigma_0)^{1/q}f\,.$$

By virtue of (*b*) we have

$$\tilde F=\Phi_{\sigma_0,1}\circ F\circ\prod_{i=1}^{k}\Phi^{-1}_{\sigma_i,1},\qquad \tilde F\Big(\prod_{i=1}^{k}L^{p_i}(\Omega)\Big)\subset L^q(\Omega)\tag{2.1}$$

then, by virtue of a well known Theorem [12], F is continuous from $\prod_{i=1}^{k} L^{p_i}(\Omega)$ into $L^q(\Omega)$.

Therefore

$$F = \Phi_{\sigma_0,1}^{-1} \circ \tilde{F} \circ \prod_{i=1}^{k} \Phi_{\sigma_i,1}$$

is continuous from $\prod_{i=1}^{k} L^{p_i}(\Omega, \sigma_i)$ into $L^q(\Omega, \sigma_0)$.

Let us prove now that $(c) \Rightarrow (a)$:

If F is continuous from $\prod_{i=1}^{k} L^{p_i}(\Omega, \sigma_i)$ into $L^q(\Omega, \sigma_0)$, we have, by virtue of (2.1), that $\tilde{F}$ is continuous from $\prod_{i=1}^{k} L^{p_i}(\Omega)$ into $L^q(\Omega)$; therefore, by virtue a well known result [12], there exist $\tilde{g} \in L^q(\Omega)$, $b \in R_+$ s.t. for each $(x, y_1, \dots, y_k) \in \Omega \times R^k$:

$$|\tilde{f}(x, y_1, \dots, y_k)| \leqslant \tilde{g}(x) + b \sum_{i=1}^{k} |y_i|^{p_i/q};$$

then, if we set $g(x) = \tilde{g}(x) \cdot (\sigma_0(x))^{-1/q}$ and $z_i = y_i \cdot (\sigma_i(x))^{-1/p_i}$, we have

$$|f(x, z_1, \dots, z_k)| \leqslant g(x) + b \sum_{i=1}^{k} \left(\frac{\sigma_i(x)}{\sigma_0(x)}\right)^{1/q} |z_i|^{p_i/q} \qquad \text{Q.E.D.}$$

Henceforth we shall suppose that Ω has the cone property.

Let $p \in]1, +\infty[$, $m \in N$ and $|\alpha| \leqslant m$, we set

$$p_{|\alpha|} = \begin{cases} np/\big(n - (m - |\alpha|)\, p\big) & \text{if } (m - |\alpha|)\, p < n \\ \text{any number in } [p, +\infty[& \text{if } (m - |\alpha|)\, p \geqslant n\,. \end{cases} \tag{2.2}$$

If $t \in]1, +\infty[$ we set $t' = t/t - 1$.

Let us denote by n_1 and n_2 the numbers of the n-tuples such that respectively $|\alpha| \leqslant m$ and $|\alpha| \leqslant m - 1$. If $k \in N$, $k \leqslant m$ we set

$$D^k u = \{D^\alpha u\}_{|\alpha| = k}\,.$$

Let us consider the differential operators formally defined by:

$$Au = \sum_{|\gamma| \leqslant m} (-1)^{|\gamma|} D^\gamma a_\gamma(x, u, \dots, D^m u) \tag{2.3}$$

$$Bu = \sum_{|\gamma| \leqslant m-1} (-1)^{|\gamma|} D^\gamma b_\gamma(x, u, \dots, D^m u) \tag{2.4}$$

$$Cu = \sum_{|\gamma| \leqslant m} (-1)^{|\gamma|} D^\gamma c_\gamma(x, u, \dots, D^{m-1} u) \tag{2.5}$$

where $a_\gamma(|\gamma|\leqslant m)$, $b_\gamma(|\gamma|\leqslant m-1)$ are real functions defined on $\Omega\times R^{n_1}$, and $c_\gamma(|\gamma|\leqslant m)$ are real functions defined on $\Omega\times R^{n_2}$.

We shall give conditions on the functions a_γ, b_γ, c_γ which are sufficient to ensure that A, B, C induce respectively continuous, compact, completely continuous maps from $\mathring{U}_s^{m,p}(\Omega)$ to its dual $U_{-s}^{-m,p'}(\Omega)$.

In the following we shall suppose that a_γ, b_γ, c_γ, satisfy the Caratheodory conditions (C_1), (C_2).

THEOREM 2.2. – *Let us suppose that*

(2.6) $\qquad \forall|\gamma|\leqslant m,\ \forall(x,\xi)\in\Omega\times R^{n_1}$:

$$|a_\gamma(x,\xi)|\leqslant h_\gamma(x)+c_1\sum_{|\alpha|\leqslant m}\big(\varrho(x)\big)^{s\theta(\alpha,\gamma)}\cdot|\xi_\alpha|^{p_{|\alpha|}/p'_{|\gamma|}}$$

where

$$s\in R,\ c_1\in R_+,\ h_\gamma\in U_{-s}^{0,p'_{|\gamma|}}(\Omega)\qquad and\qquad \theta(\alpha,\gamma)=(p_{|\alpha|}+p'_{|\gamma|})/p'_{|\gamma|}\,.$$

With such hypotheses the operator A defined by

$$\langle Au,\varphi\rangle=\sum_{|\gamma|\leqslant m}\int_\Omega a_\gamma(x,u(x),\dots,D^m u(x))\,D^\gamma\varphi(x)\,dx\,,\qquad \varphi\in\mathcal{D}(\Omega)$$

is continuous and bounded from $\mathring{U}_s^{m,p}(\Omega)$ to $U_{-s}^{-m,p'}(\Omega)$.

PROOF. – By virtue of Theorem 1.4 the map

$$d\colon u\mapsto(u,\dots,D^m u)$$

is continuous from $\mathring{U}_s^{m,p}(\Omega)$ to $\prod_{|\alpha|\leqslant m}U_s^{0,p_{|\alpha|}}(\Omega)$.

On the other hand, by virtue of (2.6) and Theorem 2.1, the Nemytskii operator A_γ associated to a_γ is continuous and bounded from $\prod_{|\alpha|\leqslant m}U_s^{0,p_{|\alpha|}}(\Omega)$ to $U_{-s}^{0,p'_{|\gamma|}}(\Omega)$. At last by Theorem 1.4 it can be easily deduced that the map $\partial^\gamma\colon u\mapsto D^\gamma u$ is continuous from $\mathring{U}_{-s}^{0,p'_{|\gamma|}}(\Omega)$ to $U_{-s}^{-m,p'}(\Omega)$.

Therefore $A=\sum_{|\gamma|\leqslant m}\partial^\gamma\circ A_\gamma\circ d$ is continuous and bounded from $\mathring{U}_s^{m,p}(\Omega)$ to $U_{-s}^{-m,p'}(\Omega)$.
Q.E.D.

THEOREM 2.3. – *Let us suppose that*

(2.7) $\qquad \forall|\gamma|\leqslant m-1,\ \forall(x,\xi)\in\Omega\times R^{n_1}$:

$$|b_\gamma(x,\xi)|\leqslant h_\gamma(x)+c_2\sum_{|\alpha|\leqslant m}\big(\varrho(x)\big)^{(s-\varepsilon)\theta_1(\alpha,\gamma)}|\xi_\alpha|^{p_{|\alpha|}/q'_{|\gamma|}}$$

where

$$\varepsilon\in R_+,\ q_{|\gamma|}\in[p,p_{|\gamma|}[;\ h_\gamma\in U^{0,q'_{|\gamma|}}_{-s+\varepsilon}(\Omega)\,,\qquad \theta_1(\alpha,\gamma)=(p_{|\alpha|}+q'_{|\gamma|})/q'_{|\gamma|}\,,\qquad c_2\in R_+$$

then the map B defined by

$$\langle Bu,\varphi\rangle=\sum_{|\gamma|\leqslant m-1}\int_\Omega b_\gamma(x,u,\dots,D^m u)D^\gamma\varphi\,dx\,,\qquad \varphi\in\mathcal{D}(\Omega)$$

is compact from $\mathring{U}^{m,p}_s(\Omega)$ to $U^{-m,p'}_{-s}(\Omega)$.

PROOF. – Obviously $B=\sum_{|\gamma|\leqslant m-1}\partial^\gamma\circ B_\gamma\circ d$, where $d\colon u\mapsto(u,\dots,D^m u)$ is continuous from $\mathring{U}^{m,p}_s(\Omega)$ to $\prod_{|\alpha|\leqslant m}U^{0,p_{|\alpha|}}_{s-\varepsilon}(\Omega)$; B_γ (the Nemytskii operator associated with b_γ) is continuous, by virtue of (2.7), from $\prod_{|\alpha|\leqslant m}U^{0,p_{|\alpha|}}_{s-\varepsilon}(\Omega)$ to $U^{0,q'_{|\gamma|}}_{-s+\varepsilon}(\Omega)$ and $\partial^\gamma\colon u\mapsto D^\gamma u$ is compact, by virtue of Corollary 1.5, from $U^{0,q'_{|\gamma|}}_{-s+\varepsilon}(\Omega)$ to $U^{-m,p'}_{-s}(\Omega)$. Q.E.D.

THEOREM 2.4. – *Let us suppose that*

(2.8) $\quad\forall|\gamma|\leqslant m,\ \forall(x,\xi)\in\Omega\times R^{n_2}$:

$$|c_\gamma(x,\xi)|\leqslant h_\gamma(x)+c_3\sum_{|\alpha|\leqslant m-1}(\varrho(x))^{(s-\varepsilon)\theta_2(\alpha,\gamma)}|\xi_\alpha|^{q_{|\alpha|}/p'_{|\gamma|}}$$

where

$$\varepsilon\in R_+\,,\qquad q_{|\alpha|}\in[p,p_{|\alpha|}[\,,\qquad h_\gamma\in U^{0,p'_{|\gamma|}}_{-s}(\Omega)\,,\qquad \theta_2(\alpha,\gamma)=(q_{|\alpha|}+p'_{|\gamma|})/p'_{|\gamma|}\,,\qquad c_3\in R_+$$

then the map C defined by

$$\langle Cu,\varphi\rangle=\sum_{|\gamma|\leqslant m}\int_\Omega c_\gamma(x,u,\dots,D^{m-1}u)D^\gamma\varphi\,dx\,,\qquad \varphi\in\mathcal{D}(\Omega)$$

is completely continuous from $\mathring{U}^{m,p}_s(\Omega)$ to $U^{-m,p'}_{-s}(\Omega)$.

PROOF. – In this case $C=\sum_{|\gamma|\leqslant m}\partial^\gamma\circ C_\gamma\circ d$, where

$$d\colon u\in\mathring{U}^{m,p}_s(\Omega)\mapsto(u,\dots,D^{m-1}u)\in\prod_{|\alpha|\leqslant m-1}U^{0,q_{|\alpha|}}_{s-\varepsilon}(\Omega)$$

is compact; C_γ (the Nemytskii operator associated to c_γ) is continuous from $\prod_{|\alpha|\leqslant m-1}U^{0,q_{|\alpha|}}_{s-\varepsilon}(\Omega)$ to $U^{0,p'_{|\gamma|}}_{-s}(\Omega)$ and $\partial^\gamma\colon u\in U^{0,p'_{|\gamma|}}_{-s}(\Omega)\mapsto D^\gamma u\in U^{-m,p'}_{-s}(\Omega)$ in continuous. Q.E.D.

EXAMPLE 2.5. – If $n>2$ and $g\colon \Omega\times R\to R$, $f\colon \Omega\times R^{n+1}\to R$ satisfy the Caratheodory conditions, the Nemytskii operators

$$u\mapsto g(\cdot, u(\cdot))\,, \qquad u\mapsto f(\cdot, u(\cdot), \operatorname{grad} u(\cdot))$$

are respectively completely continuous and compact from $\mathring{U}_s^{1,2}(\Omega)$ to $U_{-s}^{-1,2}(\Omega)$ if

$$|g(x,y)|\leqslant h(x)+(\varrho(x))^\alpha|y|^\beta\,, \qquad x\in\Omega,\ y\in R\,, \tag{2.9}$$

$$|f(x,y,z)|\leqslant h(x)+(\varrho(x))^\alpha|y|^\beta+(\varrho(x))^\gamma|z|^\delta\,, \qquad (x,y,z)\in\Omega\times R\times R^n \tag{2.10}$$

where

$$h\in U^{0,q'}_{-s+\varepsilon}(\Omega)\,, \qquad (\varepsilon\in R_+,\, q\in[2, 2n/(n-2)[)\,, \qquad \alpha<\frac{2n}{n-2}\,s\,,$$

$$\beta\in[1,(n+2)/(n-2)[\,, \qquad \gamma<\frac{2(n+1)}{n}\,s\,, \qquad \delta\in[1,(n+2)/n[\,.$$

For example, the Nemytskii operator associated to the function

$$g(x,u)=(\varrho(x))^{-\varepsilon}|u|^\beta\sin|u|^\eta x \qquad (\varepsilon,\eta\in R_+ \text{ and } \beta\in[1,5[)$$

is completely continuous from $W_0^{1,2}(R^3)$ to $W^{-1,2}(R^3)$.

3. – The nonlinear Dirichlet problem.

We shall apply the results of the previous paragraphs to some Dirichlet problems.

PROBLEM I. – Let us consider the following nonlinear Dirichlet problem: to find u such that

$$\begin{cases} u\in\mathring{U}_s^{m,p}(\Omega) \\ Eu=Au+Bu+Cu=f \end{cases} \tag{3.1}$$

where A, B, C are defined by (2.3), (2.4), (2.5) respectively and $f\in U_{-s}^{-m,p'}(\Omega)$.

The following theorem holds.

THEOREM 3.1. – *Let A, B, C satisfy the hypotheses of Theorems* 2.2, 2.3, 2.4 *moreover we suppose that*

$$\langle Au-Av, u-v\rangle\geqslant\theta(\|u-v\|_{U_s^{m,p}(\Omega)})\,, \qquad \forall_v^u\in\mathring{U}_s^{m,p}(\Omega) \tag{3.2}$$

where θ is a continuous, increasing function, such that $\theta(0)=0$ and

$$\langle Eu, u\rangle / \|u\|_{\mathring{U}_s^{m,p}(\Omega)} \to +\infty \quad \text{for} \quad \|u\|_{\mathring{U}_s^{m,p}(\Omega)} \to +\infty . \tag{3.3}$$

Then for each $f \in U_{-s}^{-m,p'}(\Omega)$ there exists $u \in \mathring{U}_s^{m,p}(\Omega)$ such that $Eu = f$.

Moreover, if $B=0$, we can replace the assumption (3.2) *with the following one*

$$\langle Au - Av, u - v\rangle \geq 0 , \quad \forall_v^u \in \mathring{U}_s^{m,p}(\Omega) \tag{3.4}$$

and the same conclusion holds.

PROOF. – By virtue of Theorems 2.2, 2.3, 2.4 and (3.2) (or (3.4) if $B=0$) E is pseudomonotone (cf. [5], [10]). Therefore the conclusion follows from (3.3) and well-known theorems on pseudomonotone operators. Q.E.D.

EXAMPLE 3.6. – The operator

$$E(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[a(x)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right] + b(x)|u|^{p-2}u - c(x)|u|^{p-2-\delta}u$$

satisfies the hypotheses of Theorem 3.1 in $\mathring{U}_s^{1,p}(\Omega)$ if $c_1\varrho^{sp}(x) \geq {}^{a(x)}_{b(x)} \geq c_2\varrho^{sp}(x)$ and $c(x) \leq \leq c_3\varrho^{sp-\varepsilon}(x)$ where c_1, c_2, c_3 are positive constants and ${}^{\varepsilon}_{\delta} \in R_+$.

EXAMPLE 3.3. – Let us consider the operator

$$Eu = -\Delta u + \lambda u + f(x, u, \operatorname{grad} u)$$

where $f\colon \Omega \times R \times R^n \to R$. If f satisfies the hypotheses (2.10), $\lambda \in R_+$, and E satisfies (3.3) on $\mathring{U}_0^{1,2}(\Omega) = W_0^{1,2}(\Omega)$ the problem $E(u)=f$ has a solution $u \in W_0^{1,2}(\Omega)$ for each $f \in W^{-1,2}(\Omega)$.

PROBLEM II. – Let Ω be an unbounded domain such that $0 \notin \bar{\Omega}$. Let a_i $(i=1, \ldots, n)$ and f be real functions defined respectively in $\Omega \times R^n$ and $\Omega \times R \times R^n$ and satisfying the Caratheodory conditions. We want to find u s.t.

$$\begin{cases} Eu = \sum_i \dfrac{\partial}{\partial x_i} a_i(x, \operatorname{grad} u) + f(x, u, \operatorname{grad} u) = g(x) \quad \text{a.e. in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.5}$$

and verifying asymptotic conditions which will be specified later (i.e. u will be required to belong to a suitable weighted Sobolev space).

Let us suppose that

H_1) $\quad |a_i(x,\xi)| \leqslant h(x) + b|x|^{2s}\sum_{j=1}^{n}|\xi_j|\,, \quad \forall x\in\Omega,\ \xi\in R^n,\ h\in L^2(\Omega, |x|^{-2s}),\ b\in R_+,\ s\in R.$

H_2) $\quad \forall x\in\Omega,\ {\xi \atop \eta}\in R^n,\ \sum_{j=1}^{n}\big(a_j(x,\xi)-a_j(x,\eta)\big)(\xi_j-\eta_j)\geqslant m|x|^{2s}|\xi-\eta|^2$

where m is a positive constant.

H_3) $\quad |f(x,y,\xi)|\leqslant h(x) + |x|^\alpha|y|^\beta + |x|^\gamma|\xi|^\delta\,, \quad \forall(x,y,\xi)\in\Omega\times R\times R^n$

where

$$h\in U^{0,q'}_{-s+1+\varepsilon/2}(\Omega)\,, \quad \big(q\in[2, 2n/(2n-2)[,\ \varepsilon>0\big)\,, \quad \alpha<\frac{2n}{n-2}(s-1)\,,$$

$$\beta\in[1,(n+2)/(n-2)[\,, \quad \gamma<\frac{2(n+1)}{n}(s-1)\,, \quad \delta\in[1,(n+2)/n[\,.$$

H_4) $\quad f(x,y,\xi)y\geqslant -c|x|^\eta|y|^\theta\,, \quad \forall(x,y,\xi)\in\Omega\times R\times R^n$

where c is a positive constant and $\theta\in]0,2[$, $\eta<-n(2-\theta)/2+\theta(s-1)/2$.

Let W be the completion of $\mathfrak{D}(\Omega)$ with respect to the norm

$$\|u\|_W = \left(\sum_i\int|x|^{2s}\left|\frac{\partial u}{\partial x_i}(x)\right|^2\right)^{\frac12}.$$

By virtue of Theorem 1.1 we have the continuous embedding

$$W\hookrightarrow \mathring{\Gamma}^{1,2}(\Omega, |x|^{2s-2-\varepsilon}, |x|^{2s})\,, \quad \varepsilon>0 \tag{3.6}$$

where $\mathring{\Gamma}^{1,2}(\Omega, |x|^{2s-2-\varepsilon}, |x|^{2s}) = \overline{C_0^\infty(\Omega)}$ (closure in $\Gamma^{1,2}(\Omega, |x|^{2s-2-\varepsilon}, |x|^{2s})$.

The following theorem holds:

THEOREM 3.4. – *Under the hypotheses H_1), ..., H_4), for each $g\in W'$ (and by virtue of (3.6) for each $g\in L^2(\Omega, |x|^{-2s+2+\varepsilon})$) there exists $u\in W$ solution of (3.5).*

PROOF. – By virtue of (3.6), the embeddings

$$i\colon W\hookrightarrow \mathring{U}^{1,2}_{s-1-\varepsilon/2}(\Omega)\,, \qquad i^*\colon U^{-1,2}_{-s+1+\varepsilon/2}(\Omega)\hookrightarrow W'$$

are continuous. On the other hand, by virtue of Theorem 2.3 and H_3) (cf. (2.10)) the Nemytskii operator B, associated to f, is compact between $\mathring{U}^{1,2}_{s-1-\varepsilon/2}(\Omega)$ and $U^{-1,2}_{-s+1+\varepsilon/2}(\Omega)$. Therefore $F=i^*\circ B\circ i$ is completely continuous. By virtue of H_1)

and with the same arguments used in Theorem 2.2, it easily follows that the operator $Au = \sum_{i=1}^{n} (\partial/\partial x_i) a_i(x, \operatorname{grad} u)$ is continuous from W into W'. Let us now prove that $E = A + F$ is coercive on W, i.e.:

$$\langle Eu, u\rangle / \|u\|_W \to +\infty \quad \text{for} \quad \|u\|_W \to +\infty . \tag{3.7}$$

By virtue of H_4) we have

$$\int_\Omega |x|^\eta |u(x)|^\theta dx \leqslant \Big(\int_\Omega |x|^{\mu(\eta - s'\theta)} dx\Big)^{1/\mu} \Big(\int_\Omega (|x|^{s'\theta} |u(x)|^\theta)^{2/\theta} dx\Big)^{\theta/2}$$

where

$$\mu = 2/(2-\theta) , \quad s' = s - 1 - \varrho , \quad \varrho > 0$$

sufficiently large, therefore

$$\int_\Omega |x|^\eta |u(x)|^\theta dx \leqslant c_1 \Big(\int_\Omega |x|^{2s'} |u(x)|^2 dx\Big)^{\theta/2} \leqslant c_1 \|u\|_W^\theta \tag{3.8}$$

where

$$c_1 = \Big(\int_\Omega |x|^{\mu(\eta - s'\theta)} dx\Big)^{1/\mu} .$$

By virtue of H_2), H_4) we have

$$\langle Eu, u\rangle = \langle Au, u\rangle + \langle Fu, u\rangle \geqslant c_2 \|u\|_W^2 - c\int_\Omega |x|^\eta |u(x)|^\theta dx , \qquad c_2 \in R_+ . \tag{3.9}$$

Because $\theta \in]0, 2[$, (3.7) follows from (3.8) and (3.9). Therefore, by virtue of well-known theorems on pseudomonotone operators (cf. [3], [10]), the conclusion follows.
Q.E.D.

EXAMPLE 6.5. – If $\Omega \subset R^3$ and $h(x) \in \mathcal{S}(\Omega)$

$$Eu = -\Delta u + h(x)|u|^{\theta - 1} , \qquad (\theta \in [1, 2[)$$

satisfies the assumptions of Theorem 3.4 on the space $\mathring{I}^{1,2}(\Omega, |x|^{-2-\varepsilon}, 1)$.

PROBLEM III. – Let Ω be an unbounded domain such that $0 \notin \bar{\Omega}$, and g be a measurable, real function on Ω. We consider the following scalar product in $\mathcal{D}(\Omega)$

$$[u, v] = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x)\, dx$$

and we denote by V the Hilbert space, completion of $\mathcal{D}(\Omega)$ with respect to the scalar product $[\cdot, \cdot]$.

In the following we study the linear problem:

$$\begin{cases} u \in V \\ -\Delta u + \lambda g(x) u = f \end{cases} \tag{3.10}$$

where λ is a real parameter.

THEOREM 3.6. – *Let us suppose that*

$$\sup_{x\in\Omega} \operatorname{ess} g(x)^2 |x|^{4+2\varepsilon} < +\infty \tag{3.11}$$

then there exists a discrete set Λ of real numbers bounded from below such that:

I) *for $\lambda \notin \Lambda$ problem (6.10) has only one solution u for each $f \in V'$ (and in parcular for each $f \in L^2(\Omega, |x|^{2+\varepsilon})$, $\varepsilon > 0$),*

II) *for $\lambda \in \Lambda$ the homogeneous problem*

$$\begin{cases} u \in V \\ -\Delta u + \lambda g(x) u = 0 \end{cases}$$

has a finite number of linearly independent solutions. Moreover for each λ the Theorem of the Fredholm alternative holds in V.

PROOF. – Let us consider the canonical isomorphism

$$H: u \mapsto [\cdot, u], \quad u \in V, \; H(u) \in V'.$$

By virtue of (3.11), the map

$$G_1: u \mapsto gu$$

is continuous from $L^2(\Omega, |x|^{-2-\varepsilon})$ into $L^2(\Omega, |x|^{2+\varepsilon})$; on the other hand, by virtue of Theorem 1.1 and Corollary 2.9 of [2] the embedding

$$i: V \hookrightarrow L^2(\Omega, |x|^{-2-\varepsilon})$$

is compact, therefore also the embedding

$$i^* \; L^2(\Omega, |x|^{2+\varepsilon}) \hookrightarrow V'$$

is compact. We deduce that $G = i^* \circ G_1 \circ i$ is compact from V into V'. On the other hand problem (3.10) is equivalent to the following functional equation

$$u + \lambda(H^{-1} \circ G)(u) = H^{-1}(f)$$

where $H^{-1} \circ G\colon V \to V$ is compact and $H^{-1}(f) \in V$. Q.E.D.

PROBLEM IV. – Let Ω be an unbounded domain in R^n. Let us consider the following problem

$$\left\{ \begin{aligned} & -\Delta u + \lambda u + f(x, u, \operatorname{grad} u) = g(x) \\ & u \in W_0^{1,2}(\Omega) \end{aligned} \right. \tag{3.12}$$

where $\lambda \in R_+$, $g \in W^{-1,2}(\Omega)$ and f is a real function defined in $\Omega \times R \times R^n$, satisfying the Caratheodory conditions and the following properties:

$$\forall (x, t, \xi) \in \Omega \times R \times R^n, \qquad |f(x, t, \xi)| \leqslant h(x) + b_1 |t|^\beta + b_2 |\xi|^\delta \tag{3.13}$$

where

$$h \in L^{q'}(\Omega), \qquad q \in [2, 2n/(n-2)[\,, \qquad \beta \in [1, (n+2)/(n-2)[\,,$$

$$\delta \in [1, (n+2)/n[\,, \qquad b_1, b_2 \in R_+$$

$$\forall (x, t, \xi) \in \Omega \times R \times R^n, \qquad f(x, t, \xi)\, t \geqslant -\delta |\xi|^2 - r(x) \tag{3.14}$$

where $\delta < \min\{1, \lambda\} = M_1$, $r \in L^1(\Omega)$.

Let us observe that, by virtue of (3.13), the Nemytskii operator F associated to f is continuous from $W_0^{1,2}(\Omega)$ to $W_0^{-1,2}(\Omega)$ but it is not, in general compact; therefore we cannot directly use the Theorem 3.1. However the following theorem holds:

THEOREM 3.7. – *Under the hypotheses* (3.13), (3.14), *the problem* (3.12) *has at least one solution* $u \in W_0^{1,2}(\Omega)$ *for each* $g \in L^{q'}(\Omega)$.

The proof of Theorem 3.7 easily follows from the following lemma

LEMMA 3.8. – *For each* $\varepsilon \in R_+$ *the problem*

$$\left\{ \begin{aligned} & u_\varepsilon \in W_0^{1,2}(\Omega) \\ & E_\varepsilon u_\varepsilon = -\Delta u_\varepsilon + \lambda u_\varepsilon + \varrho^{-\varepsilon} f(x, u_\varepsilon, \operatorname{grad} u_\varepsilon) = g \end{aligned} \right. \tag{3.15}$$

has a solution u_ε. *Moreover the sequence of solutions* $\{u_\varepsilon\}$ *is bounded in* $W_0^{1,2}(\Omega)$ *and we can select a subsequence* $\{u_{\varepsilon_n}\}$ *with the properties*: $\varepsilon_n \to 0$ *for* $n \to +\infty$ *and there exists* $u_0 \in W_0^{1,2}(\Omega)$ s.t.

$$u_{\varepsilon_n} \rightharpoonup u_0 \ \textit{in}\ W_0^{1,2}(\Omega) \qquad \textit{and} \qquad u_{\varepsilon_n} \to u_0 \ \textit{in}\ U_s^{1,2}(\Omega)\,, \qquad \forall s < 0\,.$$

PROOF. – For each $\varepsilon \in R_+$ the Nemytskii operator F_ε, associated to the function $\varrho^{-\varepsilon} f$, is compact from $W_0^{1,2}(\Omega)$ into $W^{-1,2}(\Omega)$. On the other hand, by virtue of (3.14), we have

$$\langle E_\varepsilon v, v\rangle / \|v\|_{W_0^{1,2}(\Omega)} \to +\infty \quad \text{for} \quad \|v\|_{W_0^{1,2}(\Omega)} \to +\infty .$$

Therefore, by virtue of Theorem 3.1, for each $\varepsilon \in R_+$ the problem (3.15) has at least one solution $u_\varepsilon \in W_0^{1,2}(\Omega)$. Moreover, by virtue of (3.14), we have:

$$\begin{aligned}\|u_\varepsilon\|^2_{W_0^{1,2}(\Omega)} &\leqslant \frac{1}{M_1}\langle(-\Delta+\lambda)u_\varepsilon, u_\varepsilon\rangle = \frac{1}{M_1}\langle g, u_\varepsilon\rangle - \frac{1}{M_1}\langle F_\varepsilon(u_\varepsilon), u_\varepsilon\rangle \leqslant \\ &\leqslant \frac{1}{M_1}\|g\|_{W^{-1,2}(\Omega)}\|u_\varepsilon\|^2_{W_0^{1,2}(\Omega)} + \frac{\delta}{M_1}\int_\Omega \varrho^{-\varepsilon}(x)|\operatorname{grad} u_\varepsilon(x)|^2 dx + \frac{1}{M_1}\|r\|_{L^1(\Omega)} \leqslant \\ &\leqslant c_1\|u_\varepsilon\|_{W_0^{1,2}(\Omega)} + \frac{\delta}{M_1}\|u_\varepsilon\|_{W_0^{1,2}(\Omega)} + c_2\end{aligned}$$

where

$$c_1 = \frac{1}{M_1}\|g\|_{W^{-1,2}(\Omega)} \quad \text{and} \quad c_2 = \frac{1}{M_1}\|r\|_{L^1(\Omega)} .$$

Then it is easily deduced that $\{u_\varepsilon\}$ is bounded in $W_0^{1,2}(\Omega)$. Therefore we can select a subsequence $\{u_{\varepsilon_n}\}$ such that

$$\varepsilon_n \to 0 \ \text{ for } \ n \to +\infty \quad \text{and} \quad u_{\varepsilon_n} \rightharpoonup u_0 \ \text{ in } \ W_0^{1,2}(\Omega) .$$

Let us now observe that, by virtue of Corollary 1.5

$$u_{\varepsilon_n} \to u_0 \ \text{ in } \ U_\theta^{0,p}(\Omega) , \quad \forall \theta < 0, \ \forall p \in [2, 2n/(n-2)[\tag{3.16}$$

u_{ε_n} are solutions of (3.15), therefore

$$-\Delta u_{\varepsilon_n} + \lambda u_{\varepsilon_n} = g - F_{\varepsilon_n}(u_{\varepsilon_n}) . \tag{3.17}$$

From which, if $s < 0$, we have

$$\begin{aligned}|\langle(-\Delta+\lambda)u_{\varepsilon_n}, \varrho^{2s}(u_{\varepsilon_n}-u_0)\rangle| &= |\langle g, \varrho^{2s}(u_{\varepsilon_n}-u_0)\rangle - \langle F_{\varepsilon_n}(u_{\varepsilon_n}), \varrho^{2s}(u_{\varepsilon_n}-u_0)\rangle| \leqslant \\ &\leqslant \|g\|_{L^{q'}(\Omega)}\cdot\|u_{\varepsilon_n}-u_0\|_{U^{0,q}_{2s/q}(\Omega)} + \|F_{\varepsilon_n}(u_{\varepsilon_n})\|_{L^{q'}(\Omega)}\cdot\|u_{\varepsilon_n}-u_0\|_{U^{0,q}_{2s/q}(\Omega)} .\end{aligned} \tag{3.18}$$

By virtue of (3.13), $\{F_{\varepsilon_n}(u_{\varepsilon_n})\}$ is bounded in $L^{q'}(\Omega)$, therefore from (3.16) and (3.18) it follows that

$$\langle(-\Delta+\lambda)u_{\varepsilon_n}, \varrho^{2s}(u_{\varepsilon_n}-u_0)\rangle \to 0 \quad \text{for } n \to +\infty . \tag{3.19}$$

Now, by virtue of (3.17), $\{(-\Delta+\lambda)u_{\varepsilon_n}\}$ is bounded in $L^{q'}(\Omega)$ therefore we may suppose (if necessary by passing to a further subsequence) that

$$(3.20) \qquad (-\Delta+\lambda)u_{\varepsilon_n} \rightharpoonup \chi_0 \quad \text{in } L^{q'}(\Omega)\,.$$

On the other hand the operator $v \mapsto (-\Delta+\lambda)v$ is weakly continuous from $W_0^{1,2}(\Omega)$ into $W^{-1,2}(\Omega)$, thus

$$(3.21) \qquad (-\Delta+\lambda)u_{\varepsilon_n} \rightharpoonup (-\Delta+\lambda)u_0 \quad \text{in } W^{-1,2}(\Omega)\,.$$

From (3.20), (3.21) we deduce that

$$(-\Delta+\lambda)u_0 \in L^{q'}(\Omega)\,.$$

Therefore we have also

$$(3.22) \qquad \langle(-\Delta+\lambda)u_0,\, \varrho^{2s}(u_{\varepsilon_n}-u_0)\rangle \to 0 \quad \text{for } n\to+\infty\,.$$

From (3.19) and (3.22) it follows that

$$(3.23) \qquad \langle(-\Delta+\lambda)(u_0-u_{\varepsilon_n}),\, \varrho^{2s}(u_0-u_{\varepsilon_n})\rangle \to 0 \quad \text{for } n\to+\infty\,,$$

i.e.

$$(3.24) \qquad \sum_{i=1}^{n}\int_\Omega \varrho^{2s}(x)\left|\frac{\partial}{\partial x_i}\big(u_0(x)-u_{\varepsilon_n}(x)\big)\right|^2 dx + \lambda\int_\Omega \varrho^{2s}(x)\,|u_0(x)-u_{\varepsilon_n}(x)|^2\,dx +$$
$$+\sum_{i=1}^{n}\int_\Omega\left[\frac{\partial\varrho^{2s}}{\partial x_i}(x)\,\frac{\partial}{\partial x_i}\big(u_0(x)-u_{\varepsilon_n}(x)\big)\right]\big(u_0(x)-u_{\varepsilon_n}(x)\big)\,dx \to 0 \qquad \text{for } n\to+\infty$$

On the other hand

$$(3.25) \qquad I_n = \left|\int_\Omega\left[\frac{\partial\varrho^{2s}}{\partial x_i}(x)\,\frac{\partial}{\partial x_i}\big(u_0(x)-u_{\varepsilon_n}(x)\big)\right]\big(u_0(x)-u_{\varepsilon_n}(x)\big)\,dx\right| \leqslant$$
$$\leqslant c_1\left(\int_\Omega \varrho^{2(2s-1)}(x)\cdot|u_0(x)-u_{\varepsilon_n}(x)|^2\,dx\right)^{\frac12}\left(\int_\Omega\left|\frac{\partial}{\partial x_i}\big(u_0(x)-u_{\varepsilon_n}(x)\big)\right|^2 dx\right)^{\frac12} \leqslant$$
$$\leqslant c_1\|u_0-u_{\varepsilon_n}\|_{U^{0,2}_{2s-1}(\Omega)}\|u_0-u_{\varepsilon_n}\|^2_{W_0^{1,2}(\Omega)}\,, \qquad c_1 = \sup_{x\in\Omega}\left|\frac{\partial\varrho}{\partial x_i}(x)\right|.$$

Therefore, by virtue of (3.16) and the boundedness of $\{\|u_0-u_{\varepsilon_n}\|^2_{W_0^{1,2}(\Omega)}\}$, we have

$$(3.26) \qquad I_n \to 0 \quad \text{for } n\to+\infty\,.$$

Finally from (3.24) and (3.26) it follows that

$$\|u_0-u_{\varepsilon_n}\|_{U_s^{0,2}(\Omega)} \to 0 \quad \text{for } n\to+\infty\,. \qquad \text{Q.E.D.}$$

PROOF OF THE THEOREM 3.7. – We preserve the notations introduced in proving Lemma 3.8. It is easy to see that the map $v \mapsto F(v)$ is continuous from $W^{1,2}_{\mathrm{loc}}(\Omega)$ into $L^{q'}_{\mathrm{loc}}(\Omega)$, therefore, by virtue of Lemma 3.8, we have

$$F_{\varepsilon_n}(u_{\varepsilon_n}) \to F(u_0) \quad \text{in } L^{q'}_{\mathrm{loc}}(\Omega)\,. \tag{3.27}$$

On the other hand $\{F_{\varepsilon_n}(u_{\varepsilon_n})\}$ is bounded in $L^q(\Omega)$, thus we may suppose (if necessary by passing to a further subsequence) that

$$F_{\varepsilon_n}(u_{\varepsilon_n}) \rightharpoonup \chi \quad \text{in } L^{q'}(\Omega)\,. \tag{3.28}$$

From (3.27) and (3.28) it follows that

$$F_{\varepsilon_n}(u_{\varepsilon_n}) \rightharpoonup F(u_0) \quad \text{in } L^{q'}(\Omega)\,.$$

Thus, passing to the weak limit in the equation $-\Delta u_{\varepsilon_n} + \lambda u_{\varepsilon_n} + F_{\varepsilon_n}(u_{\varepsilon_n}) = g$, the conclusion easily follows. Q.E.D.

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