On the Zeros of Univalent Functions with Univalent Derivatives (*).

SWARUPCHAND M. SHAH (Kentucky, U.S.A.) - SELDEN Y. TRIMBLE (Missouri, U.S.A.)

Summary. – *A family, E, consisting of normalised univalent functions with univalent derivatives is studied with regard to the zeros of these functions.*

1. – Introduction.

Let $\Delta = \{z\colon |z| < 1\}$. Let S be the family of functions, $z + \sum_{k=2}^{\infty} a_k z^k$, each of which is univalent in Δ. Let E be the subfamily of S such that if $f \in E$, then each $f^{(n)}$ is univalent in Δ, $n = 0, 1, 2, \ldots$. It is known that the members of E are entire functions of exponential type [8].

In this paper, we are concerned with the distribution of the zeros of functions in E. If $f \in E$, then f can vanish in Δ only at the origin. We investigate how close to the boundary of Δ other zeros of f may lie. We also give conditions on the zeros of an entire function which force the function to be in E.

2. – Statements of theorems.

THEOREM 1. – Let

$$\zeta = \inf\{|z| : f(z) = 0,\ z \neq 0, f \in E\}\,.$$

(i) Then

$$1 + \frac{1}{2\alpha}\log\left(1 + \frac{2\alpha e^{-2\alpha}}{2+\alpha}\right) \leqslant \zeta \leqslant \frac{2+\beta}{1+\beta}\,, \tag{1}$$

where $\alpha = \sup\{|f''(0)|/2 : f \in E\}$ and β is a number such that $0 < \beta < 1/2$ and $(2+\beta)/(1+\beta) \leqslant (2 - 4\beta + \beta^2)/\beta(2-\beta)$.

(ii) However, there is a function in E such that each of its derivatives has a zero on the boundary of Δ. Further, this function and each of its derivatives are close-to-convex in Δ.

(*) Entrata in Redazione il 29 marzo 1978.

REMARK. – It is known [8] that $\pi/2 \leqslant \alpha < 1.7208$. Further, there are numbers, β, satisfying the inequalities in (i), e.g., $\beta = 0.29$.

COROLLARY. – With the notation of Theorem 1, $1.0084 < \zeta < 1.7752$.

We note that the exponential type of the function in Theorem 1 (ii) is at least as great as the Whittaker constant, W. It is known that $0.7259 < W < 0.7378$ [3, 4]. Further, the exponential type is no greater than 2α.

The following theorems were proved in [7] and [10], respectively:

THEOREM A. – Let a and d be numbers such that $a > d > 1$ and such that

$$\sum_{k=2}^{\infty} (1+d)(a^{k(k-1)/2} - d)^{-1} < 1 .$$

Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that $1 \leqslant (0.276)|z_1|$ and such that

$$a^k |z_k| \leqslant |z_{k+1}|$$

for $k = 1, 2, \ldots$. If

$$f(z) = z \prod_{k=1}^{\infty} (1 - z/z_k) ,$$

then $f \in E$. In fact, each $f^{(n)}$ is close-to-convex in Δ.

THEOREM B. – Let $\beta \leqslant 0$. Let $1 < z_1 < z_2 < \ldots$ be such that

$$\sum_{k=1}^{\infty} 1/(z_k - 1) \leqslant 1 + \beta .$$

If

$$f(z) = z e^{\beta z} \prod_{k=1}^{\infty} (1 - z/z_k) ,$$

then $f \in E$. In fact, f is starlike in Δ and each $f^{(n)}$ is close-to-convex there.

The functions in Theorem A may have zeros spread over the complex plane, but their order must be 0. The functions in Theorem B may have order equal to 1 and type arbitrarily close to 1, but their zeros must lie on a ray. We now state a theorem that falls between Theorems A and B.

THEOREM 2. – Let $t > 0$. Let $\{x_k\}_{k=1}^{\infty}$ be a non-decreasing sequence of positive numbers such that $1 + t < x_1$ and such that

$$\sum_{k=1}^{\infty} 1/(x_k - 1 - t) \leqslant 1/(1+t) . \tag{2}$$

Let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that

$$|x_k - z_k| \leqslant t \tag{3}$$

for all k. If

$$f(z) = z\prod_{k=2}^{\infty}(1 - z/z_k)\,, \tag{4}$$

then $f \in E$. In fact, f is starlike in Δ and each $f^{(n)}$ is closeto-convex there.

Finally, we prove the following:

THEOREM 3. – Suppose $\beta > 0$ and $1 < z_1 \leqslant z_2 \ldots$. Let

$$f(z) = ze^{\beta z}\left(\prod_{k=1}^{N} 1 - z/z_k\right). \tag{5}$$

(We allow N to be finite or infinite. If $N = \infty$, we assume f to be well-defined.) If $f \in S$, then $\sqrt{2} \leqslant z_1$. The inequality is sharp only for $ze^{z\sqrt{2}}(1 - z/\sqrt{2})$, which is starlike in Δ.

Further, if $N = 1$ and if $\beta z_1 \leqslant 1$, then $f \in S$ if and only if $z_1 \geqslant (2 + \beta)/(1 + \beta)$. In this case, $z_1 \geqslant (1 + \sqrt{5})/2$.

COROLLARY. – If $f \in E$ and if f has the form of (5), then $\sqrt{2} < z_1$.

This is of interest because functions of the form (5) are used to establish the right side of the inequality in (1).

3. – Proofs.

Suppose

$$f_1(z) = ze^{\beta z}(1 - z/z_1)\,.$$

An induction argument shows that, for $n = 0, 1, 2, \ldots$,

$$f_1^{(n)}(z) = \left[n\beta^{n-2}\left(\beta - \frac{n-1}{z_1}\right) + \beta^{n-1}\left(\beta - \frac{2n}{z_1}\right)z - \frac{\beta^n}{z_1}z^2\right]\exp[\beta z]\,.$$

Define $P_n(z)$ by $f_1^{(n)}(z) = e^{\beta z}P_n(z)$, $n = 0, 1, \ldots$. Suppose $0 < \beta \leqslant 1/2$, $0 < z_1 < 2$, and

$$4\beta + 2\beta z_1 \leqslant \beta^2(1 + z_1) + 2\,. \tag{6}$$

(Any β sufficiently close to 0 satisfies this.) We show that, for $n \geqslant 2$ and $z \in \Delta$, $\operatorname{Re}\{P_n(z)\} < 0$.

It is enough to show that $\operatorname{Re}\{P_n(z)\}\leqslant 0$ when $|z|=1$. For $z=x+iy$ and $|z|=1$,

$$\operatorname{Re}\{P_n(z)\} = n\beta^{n-2}\left(\beta-\frac{n-1}{z_1}\right)+\frac{\beta^n}{z_1}+\beta^{n-1}\left(\beta-\frac{2n}{z_1}\right)x-\frac{2\beta^n}{z_1}x^2 .$$

If $n\geqslant 3$, using (6) and the other conditions on β and z_1, it can be shown that the discriminant of this quadratic is negative. Hence, the quadratic is negative for all x. If $n=2$, these conditions imply that the roots to the quadratic lie to the left of -1. In particular the quadratic is nonpositive for $x\in[-1,1]$. This establishes the assertion of the previous paragraph.

Now suppose $0<\beta\leqslant 1/2$ and

$$(2+\beta)/(1+\beta)\leqslant z_1<2 . \tag{7}$$

Using these, it is possible to show that $\operatorname{Re}\{P_1(z)\}>0$ for all $z\in\Delta$.

LEMMA 1. – Suppose $0<\beta\leqslant 1/2$, $z_1<2$, and

$$(2+\beta)/(1+\beta)\leqslant z_1\leqslant(2-4\beta+\beta^2)/\beta(2-\beta) . \tag{8}$$

Then f_1 and all its derivatives are univalent and close-to-convex in Δ.

PROOF OF LEMMA. – Let $g(z)=(e^{\beta z}-1)/\beta$. Then g is a convex and univalent function in Δ. Further, for $n=1,2,\dots$, $\operatorname{Re}\{f^{(n)}(z)/g'(z)\}=\operatorname{Re}\{P_n(z)\}$. The conditions, (8), on β and z_1 imply the truth of (6) and (7). So, if $n=1$, $\operatorname{Re}\{P_n(z)\}>0$ in Δ, while if $n\geqslant 2$, $\operatorname{Re}\{P_n(z)\}<0$ in Δ. Hence, for all $n\geqslant 0$, $f^{(n)}$ is univalent and close-to-convex in Δ [2].

We note that the truth of the left part of (8) is a necessary condition for the univalence of f_1 in Δ provided that $0<\beta\leqslant 1$ and that z_1 is real.

PROOF OF THEOREM 1. – (i) Suppose $f\in E$ and $f(z')=0$, where $z'=re^{i\theta}$, $r\neq 0$. From [8, 9], we have that $|f'(z)|\leqslant e^{2\alpha|z|}$ for all z. Hence,

$$|f(e^{i\theta})| = \left|\int_1^r f'(\varrho e^{i\theta})e^{i\theta}\,d\varrho\right|$$

$$\leqslant\int_1^r e^{2\alpha\varrho}\,d\varrho=(e^{2\alpha r}-e^{2\alpha})/2\alpha .$$

It is known that, if f does not assume the value, w, then $|w|\geqslant 1/(2+|a_2|)$, where $a_2=f^{(2)}(0)/2$ [5, p. 214].

Hence, $|f(e^{i\theta})|\geqslant 1/(2+\alpha)$. So,

$$\log[e^{2\alpha}+2\alpha/(2+\alpha)]\leqslant 2\alpha r$$

or,

$$1+\frac{1}{2\alpha}\log\left[1+\frac{2\alpha}{(2+\alpha)e^{2\alpha}}\right]<r\,.$$

This proves the left side of (1).

The right side of (1) is proved by using Lemma 1 to produce an appropriate example. We would like to choose β so that

$$(2+\beta)/(1+\beta)<(2-4\beta+\beta^2)/\beta(2-\beta)\,.$$

Letting $\beta=0.29$, this is satisfied. Let $z_1=(2+\beta)/(1+\beta)=1.77519\ldots$. Lemma 1 then shows that $f_1\in E$. The proof of (i) is complete.

(ii) Now we prove the last part of the theorem. Let $g_0(z)=e^z-1$. Note that, in Δ, g_0 and all its derivatives are convex and therefore, close-to-convex. Let $h_1(z)=z-\left(\int_0^z g_0(s)\,ds\right)/g_0(1)$. Then $h_1'(1)=0$ and $\operatorname{Re}\{h_1'(z)\}\geqslant 1-|g_0(z)|/g_0(1)>0$ for $z\in\Delta$. So, $h_1\in E$ and each derivative of h_1 is close-to-convex in Δ. Now suppose $h_N\in E$ such that each derivative of h_N is close-to-convex in Δ and such that, for $1\leqslant n\leqslant N$, there is some z_{N-n+1} with the properties that $|z_{N-n+1}|=1$ and $h_N^{(n)}(z_{N-n+1})=0$. Let z_{N+1} be a number such that $|z_{N+1}|=1$ and such that $|h_N(z)|<|h_N(z_{N+1})|$ for all $z\in\Delta$. Let $h_{N+1}(z)=z-\left(\int_0^z h_N(s)\,ds\right)/h_N(z_{N+1})$. Then $h_{N+1}'(z_{N+1})=0$ and $\operatorname{Re}\{h_{N+1}'(z)\}\geqslant 1-|h_N(z)/h_N(z_{N+1})|>0$ for $z\in\Delta$. So, $h_{N+1}\in E$. In fact, each derivative of h_{N+1} is close-to-convex in Δ and for $1\leqslant n\leqslant N+1$, there is some z_{N+2-n} with the properties that $|z_{N-n+2}|=1$ and $h_{N+1}^{(n)}(z_{N+2-n})=0$. Thus, we inductively obtain a sequence, $\{h_N\}_{N=1}^{\infty}$, in E such that, for each N, each derivative of h_N is close-to-convex in Δ and, for $1\leqslant n\leqslant N$, there is some z_{N+1-n} with the properties that $|z_{N-n+1}|=1$ and $h_N^{(n)}(z_{N+1-n})=0$.

Now the functions in E are uniformly bounded on compact subsets of the complex plane. So Montel's Theorem shows that E is a normal family on the whole plane. Hence, $\{h_N\}_{N=1}^{\infty}$ has a subsequence that converges uniformly on all compact subsets of the plane. Without loss of generality, suppose $\{h_N\}_{N=1}^{\infty}$ itself converges, and let h be its limit function. Since a uniform limit of close-to-convex functions is either close-to-convex or constant, each $h^{(n)}$ is either close-to-convex in Δ or constant there.

For a fixed $n\geqslant 1$, $h_N^{(n)}(z_{N+1-n})=0$ for all $N\geqslant n$. Because $\{z_{N+1-n}\}_{N=n}^{\infty}$ lies on the boundary of Δ, we may suppose $\{z_{N+1-n}\}_{N=n}^{\infty}$ converges. Let z_n' be its limit. Since $\{h_N^{(n)}\}_{N=1}^{\infty}$ converges uniformly to $h^{(n)}$ on $|z|=1$, and since $\{h_N^{(n)}\}_{N=1}^{\infty}$ is an equicontinuous family on $|z|=1$, it follows that $h^{(n)}(z_n')=0$. Since this is true for all $n\geqslant 1$, and since $h(0)=0$, the only polynomial that h could be would be $h(z)\equiv 0$. But $h'(0)=1$, so h is not a polynomial. Hence, $h\in E$. Since $|z_n'|=1$ for all $n\geqslant 1$, the theorem is proved.

To prove the corollary to Theorem 1, note that the left side of the inequality follows from (1) since $\alpha < 1.7208$. To prove the right side, let $\beta = 0.29$ and $z_1 = (2+\beta)/(1+\beta)$ in Lemma 1.

In what follows, we shall need several results. In particular, we shall use Lucas' Theorem [11, p. 218], Laguerre's Theorem, [1, p. 23] [10], and Walsh's Theorem [11, p. 219]. We state Walsh's Theorem in the form in which we need it. In what follows N is a member of $\{\infty, 0, 1, \ldots\}$:

WALSH'S THEOREM. – Let $\{x_k\}_{k=1}^N$ be a sequence of real, non-zero numbers such that $\sum_{k=1}^{N} |x_k|^{-1} < \infty$. Let $t > 0$. Let $\{z_k\}_{k=1}^N$ be a sequence of complex, non-zero numbers such that $|z_k - x_k| \leqslant t$ for all k. Define f and g by

$$g(z) = z \prod_{k=1}^{N} (1 - z/x_k)$$

and

$$f(z) = z \prod_{k=1}^{N} (1 - z/z_k)\,.$$

Let $\{x_k^{(1)}\}_{k=1}^N$ be the critical points of g. Then the critical points of f lie in the circles, C_k, where

$$C_k = \{z\colon |z - x_k^{(1)}| \leqslant t\}\,.$$

Further, if T is a set of these circles, and if $\{z\colon z \in C_k \in T\}$ does not intersect any $C_j \notin T$, then the number of critical points, counted according to multiplicity, of f in $\{z\colon z \in C_k \in T\}$ is the sum of the multiplicities of the $x_k^{(1)}$, where $x_k^{(1)} \in C_k \in T$.

We shall also need one of our results in [10], restated in a more general way. The proof is unchanged, so we do not give it.

LEMMA 2. – Let f be an entire function such that

$$f'(z) = c e^{\beta z} \prod_{k=1}^{N} (1 - z/z_k^{(1)})\,,$$

where c is a non-zero complex number, each of the $z_k^{(1)}$ is a non-zero complex number, and β is a complex number. If $N = 0$, we require that $\beta \neq 0$.

Suppose that $R > 0$ such that $R < |z_k^{(1)}|$ for each k. Suppose that $\{z_k^{(1)}\}_{k=1}^N$ can be partitioned into two sets, A and B, and that $\beta = \beta_1 + \beta_2$ such that

$$|\beta_1| R + \sum_{z_k^{(1)} \in A} \frac{R}{|z_k^{(1)}| - R} \leqslant 1$$

and, for each $|z|<R$,

$$\operatorname{Re}\left\{\exp[\beta_2 z]\prod_{z_k^{(1)}\in B}\left(1-\frac{z}{z_k^{(1)}}\right)\right\}>0\,.$$

Then f is close-to-convex and, therefore, univalent in $|z|<R$. If $\beta_2=0$ and B is empty, then f maps $|z|<R$ onto a convex set.

PROOF OF THEOREM 2. - Because of (2) and (3), we have that $\operatorname{Re}\{z_k\}>2+t$ for all k. Using this, (3), and (4), it follows that, if $z\in\Delta$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>1-\sum_{i=1}^{\infty}\frac{1}{x_k-t-1}\,.$$

Therefore, from (2),

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\frac{t}{1+t}.$$

Hence, f is starlike in Δ.

Let

$$g(z)=z\prod_{j=1}^{\infty}(1-z/x_k)\,.$$

Let $\{x_k^{(n)}\}_{k=0}^{\infty}$ be the zeros of $g^{(n)}$ arranged so that $x_0^{(n)}\leqslant x_1^{(n)}\leqslant\ldots$. Using Laguerre's Theorem and an induction argument, it follows that

$$x_k^{(n)}\leqslant x_k^{(n+1)} \tag{9}$$

for all n and k. Further, using Walsh's Theorem and another induction argument, we have that the zeros, $\{z_k^{(n)}\}_{k=0}^{\infty}$, of $f^{(n)}$ can be so ordered that

$$|x_k^{(n)}-z_k^{(n)}|\leqslant t \tag{10}$$

for all n and k.

Next, we wish to find a lower bound on $x_0^{(1)}$. As in the first part of this proof, we have that, for $z\in\Delta$,

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\}>1-\sum_{k=1}^{\infty}\frac{|z|}{x_k-|z|}\,.$$

Laguerre's Theorem shows that there is exactly one zero, $x_0^{(1)}$, of g' in $(0,x_1)$. Further, if $|z|<1+t$, then (2) shows that

$$1-\sum_{k=1}^{\infty}\frac{|z|}{x_k-|z|}\geqslant 0\,.$$

Hence, g' cannot vanish in $(0,1+t)$, i.e., $1+t\leqslant x_0^{(1)}$.

Now we use this bound on $x_0^{(1)}$ to get information about the $z_k^{(n)}$. First of all, we note that $1+t\leqslant x_k^{(1)}$ for all k. Next, we note that (10) implies that $\operatorname{Re}\{z_k^{(1)}-x_k^{(1)}\}\leqslant t$ for all k. Putting these together, it follows that $\operatorname{Re}\{z_k^{(1)}\}\geqslant 1$ for all k. This and Lucas' Theorem imply that

$$\operatorname{Re}\{z_k^{(n)}\}\geqslant 1 \tag{11}$$

for all k and for $n\geqslant 1$.

We now finish the theorem. First of all, (11) shows that, if $z\in\Delta$ and if $n\geqslant 2$, then

$$\operatorname{Re}\left\{1-\frac{z}{z_0^{(n)}}\right\}>0\,. \tag{12}$$

It follows from (2), (9), and (10) that, for $n\geqslant 2$,

$$\sum_{k=1}^{\infty}\frac{1}{|z_k^{(n)}|-1}\leqslant\frac{1}{1+t}\,.$$

Using this, (12), and Lemma 2, it follows that $f^{(n-1)}$ is close-to-convex in Δ for $n\geqslant 2$, i.e., $f^{(n)}$ is close-to-convex in Δ for $n\geqslant 1$. The proof is finished.

Finally, we prove Theorem 3. The assumptions about f imply that, if $z=1$ or $z=-1$, then $zf'(z)/f(z)\geqslant 0$. So,

$$\begin{aligned}0&\leqslant f'(1)/f(1)-f'(-1)/f(-1)\\&=2[1+\sum_{k=1}^{N}1/(1-z_k^2)]\leqslant 2[1+1/(1-z_1^2)]\,.\end{aligned}$$

Hence, $z_1\geqslant\sqrt{2}$. If $z_1=\sqrt{2}$, we must have $N=1$. This means $f(z)=ze^{\beta z}(1-z/\sqrt{2})$. However, we must also have $f'(1)=0$. This forces $\beta=\sqrt{2}$. For this f and for $|z|=1$, $\operatorname{Re}\{zf'(z)/f(z)\}=2\operatorname{Re}\{1-z^2\}/|\sqrt{2}-z|^2$. Hence, f is starlike in Δ.

Further, for this f, $f''(0)>0$ and $f''(1)<0$. So, f'' vanishes on $(0,1)$, and f' cannot be univalent in D. Hence, f is not in E. This proves the corollary.

To prove the rest of the theorem, suppose $N=1$ and $\beta z_1\leqslant 1$. Then $f(z)=$ $=z+a_2z^2+\ldots$ where, for $n\geqslant 2$, $a_n=\beta^{n-2}[n-1-\beta z_1]/z_1(n-1)!$. Hence,

$$\sum_{n=2}^{\infty}n|a_n|=1+e^{\beta}[(2+\beta)/z_1-1-\beta]\,.$$

Since, for $n\geqslant 2$, a_n is negative, it follows that $f\in S$ if and only if $\sum_{n=2}^{\infty}n|a_n|\leqslant 1$ [cf., 6]. The desired result follows. Further, since $1/\beta\geqslant z_1\geqslant(2+\beta)/(1+\beta)$, we have that $\beta^2+\beta-1\leqslant 0$, or, $\beta\leqslant(\sqrt{5}-1)/2$. Hence, $z_1\geqslant(\sqrt{5}+1)/2$.

BIBLIOGRAPHY

[1] R. P. BOAS, *Entire Functions*, Academic Press, New York, 1954.

[2] W. KAPLAN, *Close to convex schlicht functions*, Mich. Math. J., **1** (1952), pp. 169-185.

[3] S. S. MACINTYRE, *An upper bound for the Whittaker constant*, London Math. Soc. J., **22** (1947), pp. 305-311.

[4] S. S. MACINTYRE, *On the zeros of successive derivatives of integral functions*, Trans. Amer. Math. Soc., **67** (1949), pp. 241-251.

[5] Z. NEHARI, *Conformal mapping*, McGraw-Hill, New York, 1925.

[6] A. SCHILD, *On a class of functions schilcht in the unit circle*, Proc. Amer. Math. Soc., **5** (1954), pp. 115-120.

[7] S. M. SHAH, *Univalent derivatives of entire functions of slow growth*, Arch. Rational Mech. Anal., **35** (1969), pp. 259-266.

[8] S. M. SHAH - S. Y. TRIMBLE, *Univalent functions with univalent derivatives*, Bull. Amer. Math. Soc., **75** (1969), pp. 153-157.

[9] S. M. SHAH - S. Y. TRIMBLE, *Univalent functions with univalent derivatives III*, J. Math. and Mech., **19** (1969), pp. 451-460.

[10] S. M. SHAH - S. Y. TRIMBLE, *Entire functions with univalent derivatives*, J. Math. Anal. and Appl., **33** (1971), pp. 220-229.

[11] J. L. WALSH, *The location of critical points of analytic and harmonic functions*, Amer. Math. Soc. Coll., **34** (1950).

Added in proof. – The first part of Theorem 3 can be extended as follows: Let f be defined by (5), where we assume only that β and all z_k are real and $1 \leqslant |z_1| \leqslant |z_k|$. Then $f \in S$ if and only if $f'(1) \geqslant 0$ and $f'(-1) \geqslant 0$. Further if $f \in S$, then f is starlike in Δ and $\sqrt{2} \leqslant |z_1|$. If $\beta \geqslant 0$, equality occurs only for the given function.