Regularity Results for the Porous Media Equation (*).

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Sunto. – *In questo lavoro si considera il problema di Cauchy per l'equazione di filtrazione* $\partial u/\partial t = \partial^2\varphi(u)/\partial x^2$ *nella regione* $\mathbf{R}\times(0, T]$, $0 < T < \infty$. *Sotto opportune ipotesi sulla funzione* $\varphi(u)$ *si determina una stima dell'incremento temporale della soluzione* $u(x, t)$ (*intesa nel senso debole*). *Nel caso politropico* ($\varphi(u) = u^m$), *quando* $m > 2$ *si trova in particolare un comportamento hölderiano di* $u(x, t)$ *rispetto a* t *con l'esponente* $1/(m-1)$; *viene anche dimostrato che questo esponente è effettivamente assunto da una particolare soluzione, per cui la stima ottenuta è la migliore possibile.*

1. – Introduction.

The diffusion of a fluid in a porous medium is described by the equation

$$\frac{\partial}{\partial t} u = \Delta\varphi(u)$$

where Δ is the Laplace operator, u represents the distribution of density, and $\varphi(\cdot)$ is a non-negative continuously differentiable function such that

$$\varphi(0) = \varphi'(0) = 0 , \qquad \varphi'(u) \geqslant 0 . \tag{1.1}$$

In the case of polytropic flow, the $\varphi(\cdot)$ can be specified as $\varphi(u) = u^m$, $m > 1$. In the physically relevant cases we have $m \geqslant 2$. We will assume here that $\varphi(\cdot)$ behaves like u^m. In particular $\varphi''(u)$ does not change its sign for $u \geqslant 0$, and

$$|\varphi''(s)| \leqslant C\,\frac{\varphi'(s)}{s} , \qquad s > 0 \tag{1.2}$$

for some positive constant C.

If at the time $t = 0$ the fluid occupies a bounded region of the space, and if the motion is laminar-unidimensional, we are led to the Cauchy problem

$$u_t = \frac{\partial^2}{\partial x^2}\varphi(u) , \qquad (x, t) \in S_T \equiv \mathbf{R}\times(0, T] , \qquad 0 < T < \infty \tag{1.3}$$

$$u(0) = u_0(x) \tag{1.4}$$

$$u_0(x) > 0 \qquad \text{for } x \in (a, b)$$

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and

$$u_0(x) = 0 \qquad \text{for } x \in \boldsymbol{R} \sim (a, b)\,,$$

where

$$(a, b) \text{ is a finite interval}\,.$$

A consequence of the degeneracy of $\varphi'(\cdot)$ when $u = 0$, is that if the quantity

$$\psi = \int_0^u \frac{\varphi'(s)}{s}\, ds \tag{1.5}$$

is finite, then the fluis diffuses with finite speed, and the solution $u(x, t)$ of (1.2)-(1.3) is concentrated in the region

$$\mathfrak{D} \equiv \{[\zeta_1(t) < x < \zeta_2(t)] \times (0, T]\}$$

where $\zeta_i(t)$, $i = 1, 2$, $t \in [0, T]$ are two Lipschitz-continuous curves, non-increasing and non-decreasing respectively with $\zeta_1(0) = a$, $\zeta_2(0) = b$. In other words $u(x, t) > 0$ for $(x, t) \in \mathfrak{D}$ and $u(x, t) = 0$ for $(x, t) \in S_T \sim \mathfrak{D}$.

It is known that problem (1.3)-(1.4), in general does not admit a classical solution but is solvable in a weak sense. This paper is concerned with some regularity properties of the solution $u(x, t)$ with respect to the time-variable. In particular, in the polytropic case an Hölder estimate of the form

$$|u(x, t + \Delta t) - u(x, t)| \leqslant \boldsymbol{C}(\Delta t)^{1/(m-1)}$$

is obtained under certain monotonicity assumptions on $|\psi_{xx}|$ in the vicinity of the interface $x = \zeta_i(t)$, $i = 1, 2$. A related result is given by GILDING [2].

The plan of the paper is as follows. In Section 2 we recall the construction of the weak solution and certain relevant information. In Section 3 we prove Lemma 1 which is itself of interest, and will be exploited in Section 4, where it is shown that

$$\sqrt{\varphi'(u)}\, \psi_{xx} \in L^2_{\text{loc}}(S_T)\,.$$

The point here is to consider the situation where $|\psi_{xx}|$ is permitted to grow to infinity at the interface. In Section 5 finally we discuss time-regularity and the Hölder continuity expressed above. An example is given that shows that the Hölder coefficient $1/(m-1)$, in the time-variable, is the best possible. I wish to thank Professor R. E. SHOWALTER for having read the manuscript and for many valuable suggestions.

2. – The weak solution.

We say that a continuous non-negative function $u(x, t)$ is a weak solution of (1.3)-(1.4) in S_T, if

(i) $(\partial/\partial x)\varphi(u)$ exists in the weak sense and is summable in S_T;

(ii) $u(x, t)$ satisfies the integral equation

$$\iint_{S_T}\left\{uf_t - f_x\frac{\partial}{\partial x}\varphi(u)\right\}dx\,dt + \int_{\boldsymbol{R}} f(x, 0)\,u_0(x) = 0$$

for all $f(x, t)$ with compact support in the space-variable, such that f_x, f_t exist in the weak sense and are summable in S_T, and such that $f(x, T) = 0$.

Existence, uniqueness and continuous dependence upon the data have been established in [4]. The Lipschitz-continuity of $\varphi(u_0(x))$ is required. The solution $u(x, t)$ satisfies

$$0 \leqslant u(x, t) \leqslant M\,, \qquad (x, t) \in S: \tag{2.1}$$

$$M = \sup_{x\in\boldsymbol{R}} u_0(x),$$

while $(\partial/\partial t)u$, $(\partial^2/\partial x^2)\varphi(u)$, exist in the classical sense for $(x, t) \in \mathfrak{D}$, and (1.3) is satisfied in $\mathfrak{D}$.

The monotonicity of $\psi(\cdot)$ in u suggests we write (1.3)-(1.4) as

$$\psi_t = \varphi'[H(\psi)]\psi_{xx} + \psi_x^2\,, \qquad (x, t) \in S_T \tag{2.2}$$

$$\psi(0) = \psi(u_0) = \psi_0(x) \tag{2.3}$$

$$\psi_0(x) > 0 \quad x \in (a, b)\,, \qquad \psi_0(x) = 0 \quad x \in \boldsymbol{R} \sim (a, b)\,,$$

where $H[\psi(u)] = u$, $u > 0$. Aronson in [1] shows that the equivalence between (1.3)-(1.4) and (2.2)-(2.3) is not only formal; i.e., the weak solution of one of these allows the recovery of the other.

Consider the sequence of problems

$$\begin{cases} \dfrac{\partial \psi_n}{\partial t} = \varphi'(u_n)\dfrac{\partial^2}{\partial x^2}\psi_n + \psi_{nx}^2 \\ (x, t) \in (-n, n)\times(0, T] \\ \psi_n(x, 0) = \psi_{0n}(x) \qquad x \in [-n, n] \\ \psi_n(\pm n, t) = \sup \psi_{0n} = N \qquad t \in [0, T] \end{cases} \tag{2.4}$$

where $u_n = H(\psi_n)$ and $\{\psi_{0n}\} \searrow \psi_0(x)$, $0 < 1/n \leqslant \psi_{0n}(x) \leqslant N$, $\psi_{0n}(x) = N$ for

$$x \in [-n, -(n-1)] \cup [(n-1), n]$$

and it is smoothly connected to $\psi_n(\pm n, t)$ in

$$[-(n-1), -(n-2)] \cup [(n-2), (n-1)] .$$

For all $n > 2$, $n \in \boldsymbol{N}$, $\psi_n(x, t)$ the solution of (2.4) is $C^\infty[(-n, n) \times (0, T]]$ and

$$\{\psi_n(x, t)\} \searrow \psi(x, t)$$

where $u(x, t) = H(\psi(x, t))$ is the weak solution of (1.3)-(1.4). The convergence is uniform on compacts $K \subset S_T$.

By the maximum-principle

$$\varphi'(M) \geqslant \varphi'(H(\psi_n)) = \varphi'(u_n) \geqslant 0 , \qquad n \in \boldsymbol{N} , \tag{2.5}$$

and the inequality is preserved at the limit. For $(x, t) \in \mathfrak{D}$ we have

$$\varphi'(u) > 0$$

so that from classical Schauder-type estimates it follows that for all $h, k \in \boldsymbol{N}$

$$\frac{\partial t^h}{\partial^h} \frac{\partial^k}{\partial x^k} \psi_n \to \frac{\partial^h}{\partial t^h} \frac{\partial^k}{\partial x^k} \psi , \qquad (x, t) \in \mathfrak{D}$$

and the convergence is uniform on compact sets contained in $\mathfrak{D}$. In particular we notice that

$$\frac{\partial}{\partial t} \psi \in C^\infty(\mathfrak{D}) , \qquad \frac{\partial^2}{\partial x^2} \psi \in C^\infty(\mathfrak{D}) . \tag{2.6}$$

Finally we observe the following sharp result due to Aronson [1] and Kalashnikov [3]. Set

$$R = \{[x_1, x_2] \times [\tau, T]\} , \qquad \tau > 0;$$

then for every $n \in \boldsymbol{N}$

$$\left| \frac{\partial}{\partial x} \psi_n(x, t) \right| \leqslant C_0 , \qquad (x, t) \in R \tag{2.7}$$

where C_0 is a constant depending upon M, φ, τ. The estimate (2.7) is preserved at the limit, and C_0 does not depend upon τ if

$$\left| \frac{\partial}{\partial x} \psi_0(x) \right| \leqslant \text{const} .$$

3. – Fundamental lemma.

From now on C will indicate a generic non-negative constant, depending upon quantities that will be specified later.

Lemma 1. – Let $0 \le f \in C[(0,1]] \cap L^1[0,1]$ and $\varphi \geq 0$, Lipschitz-continuous in $[0,1]$ with $\varphi(0) = 0$. Assume moreover that there is a $\delta > 0$ such that f is non-increasing in the interval $(0, \delta]$. Then

$$\varphi \cdot f \in L^\infty[0,1] .$$

Proof. – The Hardy-Littlewood maximal function defined by

$$M(f)(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} f(\xi)\, d\xi$$

is an operator of weak-type (1, 1), [5], i.e. there is a constant $C > 0$ such that for all $g \in L^1(\boldsymbol{R})$ and all $\lambda > 0$

$$m\{x\colon M(g)(x) > \lambda\} \le \frac{C}{\lambda} \|g\|_1 \tag{3.1}$$

where $m(\Omega)$ indicates the Lebesgue measure of the set Ω, and

$$\|g\|_1 = \int_{\boldsymbol{R}} |g|\, dx .$$

By extending f to be zero outside $[0, 1]$ we can apply (3.1) to the f. Notice that for $x \in (0, 1)$ in view of the continuity of f in x, by the Lebesgue theorem we have

$$M(f)(x) \geq \lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} f(\xi)\, d\xi = f(x)$$

so that if λ is any positive number and $x \in [0,1]$

$$\{x\colon f(x) > \lambda\} \subseteq \{x\colon M(f)(x) > \lambda\} .$$

To prove the lemma we will show that there is a number $\bar{\bar{M}} > 0$ such that

$$m\{x\colon \varphi(x) f(x) > \bar{\bar{M}}\} = 0 .$$

Let k be the Lipschitz-constant of φ and let λ be so large that

$$\{x\colon \varphi(x) f(x) > \lambda\} \cup \left\{x\colon f(x) > \frac{\lambda}{k\delta}\right\} \subseteq [0, \delta] .$$

From the above remarks it follows that

$$\{x\colon \varphi(x)f(x) > \lambda\} \subseteq \left\{x\colon f(x) > \frac{\lambda}{k\delta}\right\} \subseteq \left\{x\colon M(f)(x) > \frac{\lambda}{k\delta}\right\}.$$

Hence

$$m\left\{x\colon f(x) > \frac{\lambda}{k\delta}\right\} \leqslant \frac{Ck\delta}{\lambda}\|f\|_1 .$$

Since f is monotone in $(0, \delta]$

$$\left\{x\colon f(x) > \frac{\lambda}{k\delta}\right\} \subseteq \left[0, \frac{Ck\delta}{\lambda}\|f\|_1\right].$$

Therefore

$$\{x\colon \varphi(x)f(x) > \lambda\} \subseteq \left[0, \frac{Ck\delta}{\lambda}\|f\|_1\right].$$

Using again the Lipschitz-continuity of φ, we have

$$\{x\colon \varphi(x)f(x) > \lambda\} \subseteq \left\{x\colon f(x) > \frac{\lambda^2}{Ck^2\delta\|f\|_1}\right\}$$

and

$$m\{x\colon \varphi(x)f(x) > \lambda\} \leqslant \frac{C^2k^2\|f\|_1^2}{\lambda^2}\delta .$$

Iterating, for all $n \in \boldsymbol{N}$

$$m\{x\colon \varphi(x)f(x) > \lambda\} \leqslant \left[\frac{Ck\|f\|_1}{\lambda}\right]^n \delta .$$

Hence, if $\lambda > Ck\|f\|_1$

$$m\{x\colon \varphi(x)f(x) > \lambda\} = 0 .$$

REMARK. – If we had $0 \leqslant \varphi$, Hölder continuous with exponent $\alpha \in (0, 1)$, the same argument would have given for all $n \in \boldsymbol{N}$

$$m\{x\colon \varphi(x)f(x) > \lambda\} \leqslant \left[\frac{Ck\|f\|_1}{\lambda}\right]^{1+\alpha+\alpha^2+\dots+\alpha^n} \delta$$

and the series $\sum_{n\geqslant 0}\alpha^n$ is convergent for $\alpha \in (0, 1)$.

4. – The estimates.

Lemma 2. – Let $R \equiv \{[x_1, x_2]\times[\tau_1, \tau_2]\}$, $0<\tau_1<\tau_2$ and $\gamma \in C^1(R)$. Moreover let n be so large that $R \subset (-n, n)\times(0, T]$, and let $\psi_n(x, t)$ be the classical solution of (2.4). Then there is a constant C, depending upon φ, τ_1, τ_2, γ, but not upon n, such that

$$\left|\iint_R \gamma\psi_{nt}\cdot\psi_{nx}\,dx\,dt\right| \leqslant C\,.$$

Proof. – Multiply the first of (2.4) by $\gamma\psi_{nx}$, and integrate over R, to get

$$\iint_R \gamma\psi_{nt}\psi_{nx}\,dx\,dt = \iint_R \gamma\varphi'[H(\psi_n)]\psi_{nxx}\psi_{nx}\,dx\,dt + \iint_R \gamma\psi_{nx}^3\,dx\,dt\,.$$

By virtue of (2.7), the last integral is uniformly bounded. For the first we have

$$2\iint_R \gamma\varphi'[H(\psi_n)]\psi_{nxx}\psi_{nx}\,dx\,dt = \int_{\tau_1}^{\tau_2} \gamma\psi_{nx}^2\varphi'[H(\psi_n)]\Big|_{x_1}^{x_2} dt - \iint_R \gamma_x\varphi'[H(\psi_n)]\psi_{nx}^2\,dx\,dt - \iint_R \gamma\psi_{nx}^2\frac{\partial}{\partial x}\varphi'[H(\psi_n)]\,dx\,dt\,.$$

In this last expression, the first two terms are uniformly bounded because of (2.5) and (2.7). To derive uniform bound for the last term, observe that by (1.2), (2.5), (2.7) we have

$$\left|\frac{\partial}{\partial x}\varphi'(H(\psi_n))\right| \leqslant C\left|\varphi'(u_n)\,\frac{u_{nx}}{u_n}\right| = C\left|\frac{\partial}{\partial x}\psi_n(x, t)\right| \leqslant \tilde{C}_0\,.$$

Lemma 3. – Let $Q \equiv \{[\xi_1, \xi_2]\times[\tau_1, \tau_2]\}$, and let $Q \subset R \equiv \{[x_1, x_2]\times[\tau_1, \tau_2]\}$, where $x_1<\xi_1<\xi_2<x_2$, $0<\tau_1<t_1<t_2<\tau_2$. Let $u(x, t)$, $\psi(x, t)$ be respectively weak solutions of (1.3)-(1.4) and (2.2)-(2.3). Then

$$\sqrt{\varphi'(u)}\,\psi_{xx} \in L^2(\mathfrak{D}\cap Q)\,. \tag{4.1}$$

Proof. – Let $\alpha \in C_0^\infty(R)$ such that $\alpha = 1$ for $(x, t) \in Q$. Multiplying the first of (2.4) by $\alpha(x, t)\psi_{nxx}(x, t)$ and integrating over R, we get

$$\iint_R \alpha\varphi'(H(\psi_n))\,\psi_{nxx}^2\,dx\,dt \leqslant \left|\iint_R \alpha\psi_{nt}\psi_{nxx}\,dx\,dt\right| + \left|\iint_R \alpha\psi_{nxx}\psi_{nx}^2\,dx\,dt\right| = |I_1| + |I_2|\,,$$

$$I_2 = \int_{\tau_1}^{\tau_2}\int_{x_1}^{x_2} \alpha\psi_{nxx}\psi_{nx}^2\,dx\,dt = -\frac{1}{3}\int_{\tau_1}^{\tau_2}\int_{x_1}^{x_2} \alpha_x\,\psi_{nx}^3\,dx\,dt$$

so (2.7) implies that $|I_2|$ is uniformly bounded. For I_1 we have

$$I_1=\int_{\tau_1}^{\tau_2}\int_{x_1}^{x_2}\alpha\psi_{nt}\psi_{nxx}\,dx\,dt=-\int_{x_1}^{x_2}\int_{\tau_1}^{\tau_2}\alpha\psi_{nx}\psi_{nxt}\,dx\,dt-\int_{x_1}^{x_2}\int_{\tau_1}^{\tau_2}\alpha_x\psi_{nx}\psi_{nt}\,.$$

The second integral is uniformly bounded by virtue of Lemma 2, and the first equals

$$-\frac{1}{2}\int_{x_1}^{x_2}\int_{\tau_1}^{\tau_2}\alpha\frac{\partial}{\partial t}(\psi_{nx})^2\,dx\,dt=\frac{1}{2}\int_{x_1}^{x_2}\int_{\tau_1}^{\tau_2}\psi_{nx}^2\alpha_t\,dx\,dt$$

which can be uniformly bounded applying (2.7). We conclude that there is a constant C depending upon α, R, φ, but not upon n, such that

$$\iint_{Q\cap\mathfrak{D}}\varphi'[H(\psi_n)]\psi_{nxx}^2\,dx\,dt\leqslant\iint_R\alpha\varphi'[H(\psi_n)]\psi_{nxx}^2\,dx\,dt\leqslant C\,.$$

Taking now the lim inf as $n\to\infty$, yields

$$\iint_{Q\cap\mathfrak{D}}\varphi'(u)\psi_{xx}^2\,dx\,dt\leqslant C$$

by Fatou's lemma because

$$H(\psi_n)\to u\,,\qquad \frac{\partial^2\psi_n}{\partial x^2}\to\frac{\partial^2\psi}{\partial x^2}$$

pointwise in $\mathfrak{D}$.

REMARK. – The argument being independent of Q, Lemma 3 says that

$$\sqrt{\varphi'(u)}\,\psi_{xx}\in L^2_{\mathrm{loc}}(S_T)\,,$$

where ψ_{xx} is defined to be zero outside $\mathfrak{D}$.

Lemma 3 and Fubini's theorem imply that

$$I(t)=\int_{\zeta_1(t)}^{\zeta_2(t)}\varphi'(u)\psi_{xx}^2(x,t)\,dx$$

is finite for almost all $t\in[t_1,t_2]$.

Set $f=\varphi'(u)\psi_{xx}^2$, $(x,t)\in Q\cap\mathfrak{D}$; then $f\in L^1(Q\cap\mathfrak{D})$. Since $\varphi'(u)$ is Lipschitz-continuous in the x-variable (uniformly on $t\in[t_1,t_2]$), and $\varphi'[u(\zeta_i(t),t)]=0$, if f is monotone in the vicinity of the interface by Lemma 1 we have

$$\varphi'(u)f\in L^\infty[\zeta_1(t),\zeta_2(t)] \tag{4.2}$$

for almost all $t \in [t_1, t_2]$. Moreover if (x, t) is close enough to $(\zeta_i(t), t)$, then from Lemma 1 and (2.7) it follows that there is a constant C depending upon φ, t_1, t_2 but not upon $t \in [t_1, t_2]$, such that

$$(\varphi' f)(x) \leqslant C\|f\|_1(t) \qquad \text{a.e.} \qquad t \in [t_1, t_2]\,. \tag{4.3}$$

The above can be rewritten as

$$|\varphi'(u)\,\psi_{xx}|(x) \leqslant C^{\frac{1}{2}}\|\varphi'(u)\,\psi_{xx}^2\|_1^{\frac{1}{2}}(t) \tag{4.4}$$

for almost all $t \in [t_1, t_2]$.

LEMMA 4. – Let $\psi(x, t)$ a weak solution of (2.2)-(2.3). Suppose that for any rectangle $Q \equiv [\xi_1, \xi_2] \times [t_1, t_2]$, $0 < t_1 < t_2 \leqslant T$, $\xi_1 < \zeta_1(t_2)$, $\xi_2 > \zeta_2(t_2)$, there is an $\varepsilon > 0$ such that the quantity $\varphi'(u)\,\psi_{xx}^2$ is respectively non-increasing in $[\zeta_1(t), \zeta_1(t) + \varepsilon]$ and non-decreasing in $[\zeta_2(t) - \varepsilon, \zeta_2(t)]$. Then there is a constant C depending upon φ, ε, t_1, t_2, but not upon $t \in [t_1, t_2]$, such that

$$I(t) = \int_{\zeta_1(t)}^{\zeta_2(t)} \varphi'(u)\,\psi_{xx}^2\,dx \leqslant C \tag{4.5}$$

for almost all $t \in [t_1, t_2]$.

PROOF. – Without loss of generality, we can assume that $\zeta_1(0) < 0 < \zeta_2(0)$, i.e. $a < 0 < b$. We will show that

$$I_\alpha(t) = \int_0^{\zeta_2(t)} \varphi'(u)\,\psi_{xx}^2(x, t)\,dx \leqslant C\,, \qquad \text{a.e.} \qquad t \in [t_1, t_2]\,.$$

The proof for the analogous integral over $\zeta_1(t) \leqslant x \leqslant 0$ will be similar.

By Fubini's theorem $I_\alpha(t)$ is finite for a.e. $t \in [t_1, t_2]$; let t_0 be such a t and let $\{x_n\}$ be an increasing sequence such that

$$(x_n, t_0) \to \big(\zeta_2(t_0), t_0\big)\,.$$

Then

$$\begin{aligned}\int_0^{\zeta_2(t_0)} \varphi'(u)\,\psi_{xx}^2\,dx &= \lim_{n\to\infty} \int_0^{x_n} \varphi'(u)\,\psi_{xx}^2\,dx \\ &= \int_0^{\zeta_2(t_0)-\varepsilon} \varphi'(u)\,\psi_{xx}^2\,dx + \lim_{n\to\infty} \int_{\zeta_2(t_0)-\varepsilon}^{x_n} \varphi'(u)\,\psi_{xx}^2 = I_1 + \lim_{n\to\infty} I_n\,.\end{aligned} \tag{4.6}$$

The first integral is bounded uniformly in $t \in [t_1, t_2]$ since the compact $[0, \zeta_2(t) - \varepsilon] \times$ $\times [t_1, t_2]$ is contained in $\mathfrak{D}$.

For I_n we have

$$I_n = \int\limits_{\zeta_2(t_0)-\varepsilon}^{x_n} \varphi'(u)\,\psi_{xx}\psi_{xx}\,dx = \varphi'(u)\,\psi_x\psi_{xx}\Big|_{\zeta_2(t_0)-\varepsilon}^{x_n} - \int\limits_{\zeta_2(t_0)-\varepsilon}^{x_n} \psi_x \frac{\partial}{\partial x}\varphi'(u)\,\psi_{xx}\,dx - \int\limits_{\zeta_2(t_0)-\varepsilon}^{x_n} \psi_x\varphi'(u)\,\psi_{xxx}\,dx\,.$$

So

$$|I_n| \leqslant |\varphi'(u)\,\psi_x\psi_{xx}(x_n, t_0)| + |\varphi'(u)\,\psi_{xx}\psi_{xx}(\zeta_2(t_0-\varepsilon), t_0)| + |I_n^1| + |I_n^2|\,.$$

Recalling (1.2), (1.5) we have

$$|I_n^1| = \left|\int\limits_{\zeta_2(t_0)-\varepsilon}^{x_n} \psi_x \frac{\partial}{\partial x}\varphi'(u)\,\psi_{xx}\,dx\right| \leqslant C\left|\int\limits_{\zeta_2(t_0)-\varepsilon}^{x_n} \frac{\partial}{\partial x}\,\psi_x^2(x, t_0)\,dx\right|$$

for some constant C independent of t. Hence in view of (2.7), we conclude that there is a constant C independent of t and n such that

$$|I_n^1| \leqslant C\,.$$

We estimate now $|I_n^2|$. From the monotonicity assumptions on $\varphi'(u)\,\psi_{xx}^2$ it follows that for $x \in (\zeta_2(t_0)-\varepsilon, \zeta_2(t_0))$

(i) $\dfrac{\partial}{\partial x}\varphi'(u)\,\psi_{xx}^2 + 2\varphi'(u)\,\psi_{xx}\psi_{xxx} \geqslant 0$

(ii) $\psi_{xx}(x, t_0)$ has a fixed sign

(iii) $\psi_{xx}(x, t_0)$ and $[\psi_x(x, t_0) - \psi_x(\zeta_2(t_0)-\varepsilon, t_0)]$ have the same sign.

Hence dividing the above inequality by $\psi_{xx}(x, t_0)$ and multiplying it by

$$[\psi_x(x, t_0) - \psi_x(\zeta_2(t_0)-\varepsilon, t_0)]$$

yields

$$(4.7)\qquad -2[\psi_x(x, t_0) - \psi_x(\zeta_2(t_0)-\varepsilon, t_0)]\,\varphi'(u)\,\psi_{xxx} \leqslant [\psi_x(x, t_0) - \psi_x(\zeta_2(t_0)-\varepsilon, t_0)]\,\frac{\partial}{\partial x}\varphi'(u)\cdot\psi_{xx}\,.$$

We have

$$I_n^2 = -\int\limits_{\zeta_2(t_0)-\varepsilon}^{x_n} [\psi_x(x, t_0) - \psi_x(\zeta_2(t_0)-\varepsilon, t_0)]\varphi'(u)\,\psi_{xxx}\,dx - \int\limits_{\zeta_2(t_0)-\varepsilon}^{x_n} \psi_x(\zeta_2(t_0)-\varepsilon, t_0)\,\varphi'(u)\,\psi_{xxx}\,dx = J_1^n + J_2^n\,.$$

Using (4.7), for J_1^n we have

$$2J_1^n \leqslant \int_{\zeta_2(t_0)-\varepsilon}^{x_n} [\psi_x(x, t_0) - \psi_x(\zeta_2(t_0) - \varepsilon, t_0)] \frac{\partial}{\partial x} \varphi'(u)\psi_{xx} dx$$

and uniform upper bound on $|J_1^n|$ can be obtained employing the same procedure used to estimate $|I_n'|$.

For J_2^n, integrating by parts and applying (2.7) we obtain

$$|J_2^n| \leqslant C|\varphi'(u)\psi_{xx}(x_n, t_0)| + C|\varphi'(u)\psi_{xx}(\zeta_2(t_0) - \varepsilon, t_0)| + C\left| \int_{\zeta_2(t_0)-\varepsilon}^{x_n} \frac{\partial}{\partial x}\varphi'(u)\psi_{xx} dx \right|$$

where the last integral can be uniformly bounded by the method used for $|I_1^n|$.

Now we observe that the quantity

$$|\varphi'(u)\psi_{xx}(\zeta_2(t_0) - \varepsilon, t_0)|$$

is uniformly bounded for $\varepsilon > 0$ fixed by interior Schauder-type estimates, whereas if x_n is close enough to $\zeta_2(t_0)$, by (4.4) we have

$$|\varphi'(u)\psi_{xx}(x_n, t_0)| \leqslant CI_\alpha^{\frac{1}{2}}(t_0) .$$

Carrying the above estimates in (4.6) gives

$$I_\alpha(t_0) \leqslant C_1 + C_2 I_\alpha^{\frac{1}{2}}(t_0)$$

where C_i, $i = 1, 2$ depend upon φ, ε, t_1, t_2, but not upon $t \in [t_1, t_2]$. This proves the lemma.

5. – Regularity results.

Next we will look for an essential bound on $\psi_t(x, t)$. It is clear from (2.2) that if $\varphi'(u)\psi_{xx} \in L^\infty\{\boldsymbol{R}\times[\tau, T]\}$, $\tau > 0$, then so also does $\psi_t(x, t)$. The interesting case occurs when $|\psi_{xx}(x, t)|$ grows to infinity as (x, t) approaches the interface.

Our main result deals with this case. However, we were unable to separate completely the behavior of $\psi_{xx}(x, t)$ and $\psi_t(x, t)$ in a neighborhood of the interface.

THEOREM 1. – Suppose that $\psi(x, t)$ is a weak solution of (2.2)-(2.3) and assume that for any rectangle

$$Q \equiv \{[\xi_1, \xi_2]\times[t_1, t_2]\}, \quad 0 < t_1 < t_2 \leqslant T ,$$

$\xi_1 \leq \zeta_1(t_2) < \zeta_2(t_2) \leq \xi_2$, there is an $\varepsilon > 0$ such that $\varphi'(u)\psi_{xx}^2(x,t)$, $(x,t) \in \mathfrak{D}$ is non-increasing in $[\zeta_1(t), \zeta_1(t)+\varepsilon]$ and non-decreasing in $[\zeta_2(t)-\varepsilon, \zeta_2(t)]$. Then $\psi_t(x,t)$ exists almost everywhere in S_T and

$$\psi_t \in L^\infty\{\mathbf{R}\times[\tau, T]\}, \qquad \forall \tau > 0 .$$

Proof. – It will be enough to show that for almost all $t \in [t_1, t_2]$

$$\limsup_{(x,t)\to(\zeta_i(t),t)} |\psi_t(x,t)| \leq C, \qquad (x,t) \in \mathfrak{D}$$

and the bound does not depend upon t. Let's show this for $\zeta_2(t)$; the proof for $\zeta_1(t)$ is similar.

For $(x,t) \in \mathfrak{D}$, (2.2) is satisfied in the classical sense, hence

$$\limsup_{(x,t)\to(\zeta_2(t),t)} |\psi_t(x,t)| \leq \limsup_{(x,t)\to(\zeta_2(t),t)} |\varphi'(u)\psi_{xx}(x,t)| + \limsup_{(x,t)\to(\zeta_2(t),t)} \psi_x^2(x,t) .$$

The last term is bounded because of (2.7) and a control on the first is supplied by (4.4) and Lemma 4.

We will now exploit the results of Theorem 1 to deduce some regularity properties for the solution of (1.3). Let's put ourselves in the assumptions of Theorem 1. Since

$$|\psi(x,t_2) - \psi(x,t_1)| \leq \int_{t_1}^{t_2} |\psi_t|\, dt, \qquad t_1 > 0 ,$$

Theorem 1 and (2.7) imply that $\psi(x,t)$ is Lipschitz-continuous in $\mathbf{R}\times[\tau, T]$, $\tau > 0$, i.e. there is a constant C such that

$$|\psi(x_1,t_1) - \psi(x_2,t_2)| \leq C\{|x_1 - x_2| + |t_1 - t_2|\} .$$

Theorem 2. – Consider the $\psi(\cdot)$ defined by (1.5) as a function of u.

(i) If $\psi''(u) > 0$ for $u \in [0, M]$, then for all $h > 0$ and $(x,t) \in \{\mathbf{R}\times[\tau, T]\}$, $\tau > 0$ there is a constant C such that

$$|u(x, t+h) - u(x,t)| \leq H(Ch) \tag{5.2}$$

where $H(\cdot)$ is such that $H[\psi(u)] = u$.

(ii) If $\psi''(u) \leq 0$, $u \in [0, M]$, then for all $h > 0$ and $(x,t) \in \{\mathbf{R}\times[\tau, T]\}$, $t > 0$, there is a constant C such that

$$|u(x, t+h) - u(x,t)| \leq Ch . \tag{5.3}$$

PROOF. – (i) $\psi''(u) > 0$ is a convexity condition which can be expressed in the form

$$\psi(|u(x, t+h) - u(x, t)|) \leqslant |\psi(u(x, t+h)) - \psi(u(x, t))| .$$

Applying (5.1) and observing that $H(\cdot)$ is monotone increasing, yields (5.2).

(ii) If $\psi''(u) \leqslant 0$, $u \in [0, M]$, we have

$$\psi'(M) = \frac{\varphi'(M)}{M} \geqslant a > 0$$

and since $\psi'(\cdot)$ is decreasing

$$\psi'(u) \geqslant \psi'(M) \geqslant a > 0 . \tag{5.4}$$

Notice that (5.3) holds for u replaced by u_n, for n large enough, so integrating between $u_n(x, t)$ and $u_n(x, t+h)$ and applying (5.1) gives

$$a|u_n(x, t+h) - u_n(x, t)| \leqslant |\psi(u_n(x, t+h)) - \psi(u_n(x, t))| \leqslant Ch$$

letting $n \to \infty$ proves (5.3).

It is interesting to observe the results of Theorem 2 in the situation of polytropic flow. In this case we have

$$\psi(u) = \frac{1}{m-1} u^{m-1}(x, t)$$
$$\psi''(u) = (m-2) u^{m-3}(x, t)$$

therefore if $m > 2$, $\psi''(u) > 0$ and

$$|u(x, t+h) - u(x, t)| \leqslant Ch^{1/(m-1)} . \tag{5.5}$$

If $m \leqslant 2$, $\psi''(u) \leqslant 0$, hence by Theorem 2

$$|u(x, t+h) - u(x, t)| \leqslant Ch . \tag{5.6}$$

The Hölder exponent $1/(m-1)$ in (5.5) is the best possible. This is shown by the following example

$$u(x,t) = \begin{cases} \dfrac{1}{\lambda(t)}\left\{1 - \left[\dfrac{x}{\lambda(t)}\right]^2\right\}^{m-1/1} & |x| \leqslant \lambda(t) \\ 0 & |x| > \lambda(t) \end{cases}$$

$$\lambda(t) = \left\{\frac{2m(m+1)}{m-1}(t+1)\right\}^{1/m-1} \qquad \begin{matrix} m > 1 \\ t \geqslant 0 . \end{matrix}$$

Such a $u(x, t)$ is the unique solution of (1.3), with initial value

$$u_0(x) = \frac{1}{\lambda(0)}\left\{1 - \left[\frac{x}{\lambda(0)}\right]^2\right\} \qquad |x| < \lambda(0) .$$

It is not difficult to see that if $x = \lambda(t)$, then

$$|u(\lambda(t), t + \Delta t) - u(\lambda(t), t)| = F(m, t)(\Delta t)^{1/(m-1)}$$

where

$$0 < \alpha \leqslant F(m, t) \leqslant \beta$$

for some α, β positive constants. The above explicit solution of (1.3)-(1.4) is due to PATTLE [6].

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