Computation of the Homology of $\Omega(X \vee Y)$ (*).

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Summary. – *We compute the homology of $\Omega(X \vee Y)$ (the loop space of the wedge of the spaces X and Y), in terms of the homogies of ΩX and ΩY. To do this we use the fact that our problem is equivalent to the computation of the homology of the free product of two topological groups in terms of the homologies of the topological groups. We establish a multiple Kunneth formula with coefficients over a Dedekind domain, which is used to prove a Kunneth like formula involves homologies over a Dedekind domain and generalizes similar results with integral or field coefficients. Over a principal ideal domain the formula for a free product is made more specific.*

1. – Introduction.

The classical Kunneth formula for two complexes was extended by MACLANE [9] for three complexes. BOCKSTEIN [3] and HUNGERFORD [5] proved a multiple Kunneth formula for chain complexes of abelian groups. In § 2, these results are generalized to a multiple Kunneth formula for chain complexes of modules over a Dedekind domain (D.D.).

A Kunneth like formula for the coproduct of two simplicial groups with homology taken over a D.D. (theorem 3) is stated in § 3. This is a generalization of a similar result with field coefficients [2], [4], as well as for integral coefficients [8]. Up to homotopy type, ΩX can be replaced by a topological group [12], and $\Omega(X \vee Y)$ can be replaced by the free product of topological groups of the homotopy type of $\Omega(X)$ and $\Omega(Y)$ [7]. Thus, by theorem 3 the homology of $\Omega(X \vee Y)$ is expressed in terms of the homologies of ΩX and ΩY. In § 4 we make the formula of theorem 3 applicable to concrete computations when the homologies are given over a principal ideal domain (P.I.D.). § 5 is devoted to a demonstration of the use of the formulas.

All chain complexes are non negative. The ring R is a D.D. throughout the paper, except for § 5 in which it is assumed to be a P.I.D.

2. – A multiple Kunneth formula.

Let L^j be free resolutions of the modules A^j, $j = 1, 2, \ldots, n$ [13, p. 219]. According to the definitions in [6] we have:

$$\mathrm{Mult}_i^n(A^1, \ldots, A^n) = H_i(L^1 \otimes \ldots L^n)$$

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($H_i(\ )$ denotes the i dimensional homology functor). We also use the following notation:

$$A^1 * A^2 = \mathrm{Tor}_1(A^1, A^2) = H_1(L^1 \otimes A^2) = H_1(L^1 \otimes L^2) = \mathrm{mult}_1^2(A^1, A^2)$$

[13, p. 219]. The main result of this section is the following:

THEOREM 1. – Let $K^1, \ldots, K^n$ be chain complexes of modules over a D.D. such that for each $1 \leqslant \lambda \leqslant n-1$

$$H_k\Big(\bigotimes_{j=1}^{\lambda+1} K^j\Big) \cong \Big(H\Big(\bigotimes_{j=1}^{\lambda} K^j\Big) \otimes H(K^{\lambda+1})\Big)_k \oplus \Big(H\Big(\bigotimes_{j=1}^{\lambda} K^j\Big) * H(K^{\lambda+1})\Big)_{k-1}.$$

Then:

$$H_k\Big(\bigotimes_{j=1}^{n} K^j\Big) = \Big(\bigotimes_{j=1}^{n} H(K_j)\Big)_k \oplus \sum_{\sum_{j=1}^{n} r_j + i = k,\, i>0} \mathrm{mult}_i^n\big(H_{r_1}(K^1), \ldots, H_{r_n}(K^n)\big).$$

Note that the isomorphism conditions in the theorem are just the requirements that the Kunneth formula holds for the pairs of chain complexes $\bigotimes_{j=1}^{\lambda} K^j$ and $K^{\lambda+1}$, $1 \leqslant \lambda \leqslant n-1$. These conditions are obviously satisfied if each K^j is a free complex, which will be the situation in our applications.

PROOF. – First we note that for given moduls $A^1, \ldots, A^n$ we have:

$$\mathrm{mult}_i^n(A^1, \ldots, A^n) \cong [\mathrm{mult}_i^{n-1}(A^1, \ldots, A^{n-1}) \otimes A^n] \oplus [\mathrm{mult}_{i-1}^{n-1}(A^1, \ldots, A^{n-1}) * A^n].$$

Immediate consequences of this are the following:

$$\mathrm{mult}_i^n(A^1, \ldots, A^n) = 0 \quad \text{for } i \geqslant n,$$
$$\mathrm{mult}_0^n(A^1, \ldots, A^n) \cong A^1 \otimes \ldots \otimes A^n.$$

The proof of the theorem now follows by induction.

3. – The homology of $\Omega(X \vee Y)$.

Let us consider connected simplicial groups G_1, G_2 [10]. We define the following graded module associated with G_1 and G_2:

$$\mathrm{mult}(G_1, G_2) = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \sum_{\substack{(r_1, \ldots, r_n)\\(j_1, \ldots, j_n)}} \mathrm{mult}_i^n\big(\tilde{H}_{r_1}(G_{j_1}), \ldots, \tilde{H}_{r_n}(G_{j_n})\big),$$

where $(r_1, \ldots, r_n)$ is a sequence of non-negative integers, $(j_1, \ldots, j_n)$ is a sequence alternating on the numbers 1, 2, and $\tilde{H}_{r_t}(G_{j_t})$ is the augmented r_t dimensional homology of G_{j_t}, $t = 1, 2, \ldots, n$. The k-dimensional elements of $\operatorname{mult}(G_1, G_2)$ are:

$$(\operatorname{mult}(G_1, G_2))_k = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \sum_{\substack{\Sigma r_t + i = k \\ (j_1, \ldots, j_n)}} \operatorname{mult}_i^n (\tilde{H}_{r_1}(G_{j_1}), \ldots, \tilde{H}_{r_n}(G_{j_n})) .$$

Let $R(G)$ be the associated element of the simplicial group G in the category of augmented differential graded algebras, and let $B_1 \coprod B_2$ denote the coproduct in this category of the objects B_1 and B_2 [2]. We thus obtain a generalization of theorem 2 of [8].

THEOREM 2. – Let R be a D.D. Then we have: $H(R(G_1) \coprod R(G_2)) \cong R \oplus \oplus \operatorname{mult}(G_1, G_2)$. The proof is by a multiple use of theorem 1, and the fact that homology commutes with direct sums.

Combining the latter theorem with theorem 1 of [8] i.e. that $R(G_1 * G_2)$ and $R(G_1) \coprod R(G_2)$ are chain homotopy equivalent, we immediately obtain the following:

THEOREM 3. – Let G_1, G_2 be two connected simplicial groups. Then over a D.D. R we have:

$$H(G_1 * G_2) \cong R \oplus \operatorname{mult}(G_1, G_2) .$$

The simplicial group $G_1 * G_2$ is the coproduct of G_1 and G_2 in the category of simplicial groups [8]. We note that $*$ denotes both the torsion product of two modules and the free product of two groups. However, it will always be clear from the context which of them we mean.

Now let X and Y be well pointed spaces such that ΩX and ΩY are connected. Since ΩX and ΩY are of the homotopy type of topological groups, say G_1 and G_2 respectively, and $\Omega(X \vee Y)$ is of the homotopy type of $G_1 * G_2$ we conclude:

THEOREM 3′. – Let ΩX and ΩY be connected. Then over a Dedekind domain R we have:

$$H(\Omega(X \vee Y)) = R \oplus \operatorname{mult}(\Omega X, \Omega Y) .$$

4. – Computations of $\operatorname{mult}(\Omega X, \Omega Y)$.

In this section G_1, G_2 stand for two groups or for ΩX, ΩY respectively and $G_1 * G_2$ stands for the co-product of the groups or for $\Omega(X \vee Y)$. In order to apply the formulas of theorems 3, 3′ to concrete examples we should be able to compute $\operatorname{mult}(G_1, G_2)$. We restrict ourselves to connected spaces whose homology is of finite type, i.e. $H_r(G_i)$ is a finitely generated R module for $r = 1, 2, \ldots$ and $i = 1, 2$. In this case it is quite obvious that $(\mathrm{mult}(G_1, G_2))_k$ is a direct sum of a finite number of terms of the form $\mathrm{mult}_i^n(\tilde{H}_{r_1}(G_{j_1}), \dots, \tilde{H}_{r_n}(G_{j_n}))$, and thus the problem is reduced to compute those terms. Furthermore we require that the ring R is a P.I.D. This last restriction is imposed to assure the following known fact, which makes future computations manageable.

PROPOSITION 1. – $R_p \otimes R_q \cong R_p * R_q = R_{(p,q)}$, where R_p, R_q are the quotients of R by the ideals generated by p, q respectively, and (p, q) is the greatest common diviser of p and q.

We also use the following fundamental structure theorem [1, p. 370]:

PROPOSITION 2. – Let A be a finitely generated module over R. Then A is a finite direct sum of the form: $A \cong \sum_{i=1}^{n} R_{p_i}^{h_i} \oplus \sum_{j=1}^{m} R$, where p_i are primes in R and h_i are positive integers. This representation is unique up to the order of the summation.

As a direct consequence of the definition of $\mathrm{mult}_i^n(\ )$, for finitely generated R modules $A^1, \dots, A^n$ we have:

(i) $\mathrm{mult}_i^n(A^1, \dots, A^n)$ is a finite direct sum of modules of the form $\mathrm{mult}_i^n(R_{q_1}, \dots, R_{q_n})$, where q_j are either positive powers of primes or o $(R_0 = R)$.

(ii) $\mathrm{mult}_i^n(A^1, \dots, A^n) \cong \mathrm{mult}_i^n(A^{\nu(1)}, \dots, A^{\nu(n)})$, where ν is a permutation of the ordered set $(1, \dots, n)$.

(iii) If $R_{q_{i_1}}, \dots, R_{q_{i_s}}$ are the torsion modules out of the modules $R_{q_1}, \dots, R_{q_n}$, then $\mathrm{mult}_i^n(R_{q_1}, \dots, R_{q_n}) \cong \mathrm{mult}_i^s(R_{q_{i_1}}, \dots, R_{q_{i_s}})$.

(iv) Let $R_{q_1}, \dots, R_{q_n}$ be torsion modules. Then $\mathrm{mult}_i^n(R_{q_1}, \dots, R_{q_n}) \cong \bigoplus_{\binom{n}{i-1}} R_{(q_1,\dots,q_n)}$, where $(q_1, \dots, q_n)$ stands for the greatest common diviser of $q_1, \dots, q_n$ and $\binom{j}{i}$ is the binomial coefficient. The proof of (iv) is by induction.

At this point we can draw some conclusions.

COROLLARY 1. – The module R_{p^h} is a direct summand of $\mathrm{mult}_i^n(\tilde{H}_{r_1}(G_{j_1}), \dots, \tilde{H}_{r_n}(G_{j_n}))$, only if R_{p^h} is a direct summand of at least one of the modules $\tilde{H}_{r_1}(G_{j_1}), \dots, \tilde{H}_{r_n}(G_{j_n})$.

COROLLARY 2. – The module R_{p^h} is a direct summand of $\tilde{H}_k(G_1 * G_2)$ if and only if R_{p^h} is a direct summand of one of the modules $\tilde{H}_r(G_1), \tilde{H}_r(G_2)$, $0 \leqslant r \leqslant k$.

We introduce the following notation associated with the modules

$$\{\tilde{H}_{n_t}(G_j)\}, \qquad t = 1, 2, \dots, n:$$

$f_t \equiv$ the number of R direct summands in $\tilde{H}_{r_t}(G_{j_t})$,

$g_t \equiv$ the number of $R_{p^{h'}}$, $h' \geqslant h$ direct summands in $\tilde{H}_{r_t}(G_{j_t})$,

$d_t \equiv$ the number of $R_{p^{h'}}$, $h' > h$ direct summands in $\tilde{H}_{r_t}(G_{j_t})$.

COROLLARY 3. – The number of R^{p^h} direct summands in $\mathrm{mult}_i^n\,(H_{r_1}(G_{j_n}), \ldots, H_{r_1}(G_{j_n}))$ is given by the expression:

$$\sum_{\alpha \in I} f_1^{(1-\alpha(1))} \cdots f_n^{(1-\alpha(n))} \cdot (g_1^{\alpha(1)} \cdots g_n^{\alpha(n)} - d_1^{\alpha(1)} \cdots d_n^{\alpha(n)}) \cdot \binom{\sum_{t=1}^{n} \alpha(t) - 1}{i},$$

where I is the collection of the functions $\{0, 1\}^{\{1, 2, \ldots, n\}}$.

We are now ready to point out a procedure of computing $H_k(G_1 * G_2)$. This is done in several steps.

I) We find modules R_{p^h} that can appear as direct summands in $H_k(G_1 * G_2)$ (Corollary 2).

II) We consider all $\mathrm{mult}_i^n\,(H_{r_1}(G_{j_n}), \ldots, H_{r_1}(G_{j_n}))$ which appear in $(\mathrm{mult}\,(G_1, G_2))_k$, that R_{p^h} may be a direct summand of (Corollary 1).

III) We use Corollary 3 to obtain the number of R_{p^h} direct summands in $H_k(G_1 * G_2)$.

IV) The number of R direct summands in $H_k(G_1 * G_2)$ is obtained by adding up the number of R direct sums in each of the modules $H_{r_1}(G_{j_1}) \otimes \cdots \otimes H_{r_n}(G_{j_n})$, which make up $(\mathrm{mult}\,(G_1, G_2))_k$. The number of the R direct summands in such a module equals $f_1 \cdot f_2 \cdots f_n$.

5. – An example.

We demonstrate our method of computation, on the free product of the special ortogonal group SO_3 with itself. The homology of SO_3 as computed in [11] is:

$$H_j(SO_3) = \begin{cases} Z & j = 0, 3\,, \\ Z_2 & j = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $G_1 = G_2 = SO_3$. To compute $H_3(G_1 * G_2)$ we have to consider the number of its Z and Z_2, direct summands. The number of Z and Z_2 direct summands is obtained by considering the following terms only:

$$\begin{aligned}
&\mathrm{mult}_0^1\,(\tilde{H}_3(G_1)) = Z\,,\\
&\mathrm{mult}_0^1\,(\tilde{H}_3(G_2)) = Z\,,\\
&\mathrm{mult}_0^3\,(\tilde{H}_1(G_1), \tilde{H}_1(G_2), \tilde{H}_1(G_1)) = Z_2\,,\\
&\mathrm{mult}_0^3\,(\tilde{H}_1(G_2), \tilde{H}_1(G_1), \tilde{H}_1(G_2)) = Z_2\,,\\
&\mathrm{mult}_1^2\,(\tilde{H}_1(G_1), \tilde{H}(G_2)) = Z_2\,,\\
&\mathrm{mult}_1^2\,(\tilde{H}_1(G_2), \tilde{H}_1(G_1)) = Z_2\,.
\end{aligned}$$

We conclude that:

$$H_3(SO_3 * SO_3) = \bigoplus_2 Z \bigoplus_4 Z_2 .$$

The reader should have no difficulty in finding that $H_k(SO_3 * SO_3)$ has $Z \oplus Z$ as summand only in dimensions which are divisible by 3, and the number of the Z_2 summands in each dimension forms the following sequence:

$$(2, 2, 4, 10, 16, 30, 58, 104, 192, 356, 652, 1200, 2210, 4062, \ldots) .$$

We note that though the homology of SO_3 is quite simple the homology of $SO_3 * SO_3$ is not.

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