Asymptotic Nature of Nonoscillatory Solutions of Nonlinear Deviating Differential Equations (*) (**).

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Dedicated to Professor Giovanni Sansone on his 90th birthday

Summary. – *The higher order nonlinear deviating equations*

$$x^{(n)}(t) + \delta \sum_{i=1}^{m} f_i(t, x[g_{i1}(t)], \ldots, x[g_{ik}(t)]) = h(t)$$

are considered, where $\delta = \pm 1$. *Our main purpose is to characterize the asymptotic behavior of nonoscillatory solutions of above equations.*

1. – Introduction.

The purpose of this paper is to characterize the asymptotic behavior of nonoscillatory solutions of nonlinear deviating equations

$$E(\delta): x^{(n)}(t) + \delta \sum_{i=1}^{m} f_i\big(t, x[g_{i1}(t)], \ldots, x[g_{ik}(t)]\big) = h(t)\,, \qquad \delta = \pm 1\,.$$

In what follows, we are only going to consider continuous solutions of $E(\delta)$ which are extendable on some positive half-line $I \equiv [t_0, \infty)$, $t_0 > 0$. We call a function on I *oscillatory* if it has arbitrarily large zeros, otherwise it is called *nonoscillatory*.

Throughout this paper, we assume the following conditions always hold:

(i) $g_{ij}, h \in C[I, R \equiv (-\infty, \infty)], \operatorname{ilm}_{t\to\infty} g_{ij}(t) = \infty$ *for* $i = 1, 2, \ldots, m; j = 1, 2, \ldots, k$.

(ii) $h(t) = 0$ *or there exists an oscillatory function* $r(t)$ *such that*

$$r^{(n)}(t) = h(t)\,, \lim_{t\to\infty} r^{(\varkappa)}(t) = 0 \qquad \textit{for } \varkappa = 0, 1, \ldots, n-1\,.$$

(iii) $f_i \in C[I \times R^k, R]$, *for* $x_j > 0$, $j = 1, 2, \ldots, k$ *and all* $t \geq t_0$ *imply*

$$0 < f_i(t, x_1, \ldots, x_k) \leq - f_i(t, -x_1, \ldots, -x_k)$$

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for $i = 1, 2, \ldots, m$. *Moreover there is an index* p, $1 \leqslant p \leqslant n$ *such that* $f_p(t, x_1, \ldots, x_k)$ *is nondecreasing with respect to* $x_1, x_2, \ldots, x_k$ *and for all* $t \geqslant t_0$.

Using condition (ii), $E(\delta)$ may be written as

$$L(\delta):\ y^{(n)}(t) + \delta \sum_{i=1}^{m} f_i(t, y[g_{i1}(t)] + r[g_{i1}(t)], \ldots, y[g_{ik}(t)] + r[g_{ik}(t)]) = 0$$

where $y(t) = x(t) - r(t)$.

In order to obtain our main results, we need the following two lemmas. The first lemma is an analog of a result due to KIGURADZE [10], the other is due to STAIKOS and SFICAS [19].

LEMMA 1. – *If* $x(t)$ *is a positive (negative) solution of* $E(\delta)$ *for* $t \geqslant t_0$, *then there is a* $T \geqslant t_0$ *for which* $y(t) = x(t) - r(t)$ *is a positive (negative) solution of* $L(\delta)$ *for* $t \geqslant T$. *Also there is an integer* l, $0 \leqslant l \leqslant n$ *with* $n + l$ *odd if* $y^{(n)}(t) \leqslant 0$, $n + l$ *even if* $y^{(n)}(t) \geqslant 0$ *and such that for* $t \geqslant T$

$$\text{(1)}\qquad \begin{cases} l > 0 \ \textit{imply} \ y^{(\varkappa)}(t) > 0\,, & \varkappa = 0, 1, \ldots, l-1 \\ l \leqslant n-1 \ \textit{imply} \ (-1)^{l+\varkappa} y^{(\varkappa)}(t) > 0\,, & \varkappa = l, l+1, \ldots, n \end{cases}$$

$$\text{(2)}\qquad x^{(\varkappa)}(t) y^{(\varkappa)}(t) > 0 \quad \textit{for} \quad \varkappa = 0, 1, \ldots, n\,.$$

PROOF. – We only consider $E(+1)$. We assume that $x(t)$ and $x[g_{ij}(t)]$ are positive for $t \geqslant t_0$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$, since $x(t) < 0$ can be discussed similarly. Then $y(t) = x(t) - r(t)$ is a solution of $L(+1)$. Condition (iii) implies $y^{(n)}(t) < 0$. If $y(t) \leqslant 0$ for t large enough, then $r(t) > -y(t) \geqslant 0$, a contradiction to the oscillatory character of $r(t)$. Hence $y(t) > 0$. By KIGURADZE's lemma [10], we have (1). Let $y^{(\varkappa)}(t) > 0$ (< 0). If $x^{(\varkappa)}(t) < 0$ (> 0), then $r^{(\varkappa)}(t) = x^{(\varkappa)}(t) - y^{(\varkappa)}(t) < 0$ (> 0), a contradiction. This contradiction proves (2).

LEMMA 2. – *If* $y(t)$ *is as in Lemma* 1 *and for some* $0 \leqslant \tilde{\varkappa} \leqslant n-2$, $\lim_{t\to\infty} y^{(\tilde{\varkappa})}(t) = c$, $c \in R$, *then*

$$\lim_{t\to\infty} y^{(\varkappa)}(t) = 0\,, \qquad \varkappa = \tilde{\varkappa} + 1, \ldots, n-1\,.$$

2. – Main results.

THEOREM 1. – *Let* n *be even. Assume that*

$$(\mathrm{C}_1)\qquad \int^{\infty} f_p(t, cg_{p1}(t), \ldots, cg_{pk}(t))\, dt = \pm\infty$$

for any constant $c \neq 0$, *and*

$$(\mathrm{C}_2)\qquad \begin{aligned} &f_p(t, cg_{p1}(s), \ldots, cg_{pk}(s)) \leqslant s f_p(t, c, \ldots, c) \quad \textit{for} \quad c > 0 \\ &f_p(t, cg_{p1}(s), \ldots, cg_{pk}(s)) \geqslant s f_p(t, c, \ldots, c) \quad \textit{for} \quad c < 0 \end{aligned}$$

for all large t and $s > 0$. Then each nonoscillatory solution of $E(-1)$ has either $x^{(\varkappa)}(t) \to 0$ or $|x^{(\varkappa)}(t)| \to \infty$ as $t \to \infty$ for $\varkappa = 0, 1, \ldots, n-1$.

PROOF. – Without any loss of generality, we can assume that $x(t)$ and $x[g_{ij}(t)]$ are positive for $t \geqslant t_0$ and $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$. Let $y(t) = x(t) - r(t)$. Then we have $L(-1)$. Condition (iii) implies $y^{(n)}(t) > 0$. It follows from Lemma 1 that there exist a $t_1 \geqslant t_0$ and an integer l (even) such that (1) and (2) hold for $t \geqslant t_1$. If $x'(t) > 0$, then by Lemma 1, $x''(t) > 0$. Therefore $x(t) \to \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{x(t)}{t} = \lim_{t\to\infty} \frac{x(t) - x(t_1)}{t - t_1} \geqslant x'(t_1) > 0 .$$

Let $x'(t_1) = 2c$. Then there is a $t_2 \geqslant t_1$ such that $x(t)/t > c$ for $t \geqslant t_2$. By (i) there is a $T \geqslant t_2$ such that $g_{ij}(t) \geqslant t_2$ for $t \geqslant T$. Thus for $t \geqslant T$

$$x[g_{ij}(t)] > c g_{ij}(t) , \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, k . \tag{3}$$

Integrating $L(-1)$ from T to t and using (3), (iii), we have

$$y^{(n-1)}(t) \geqslant y^{(n-1)}(T) + \int_T^t f_p(s, c g_{p1}(s), \ldots, c g_{pk}(s))\, ds \to \infty$$

as $t \to \infty$. Thus $y^{(\varkappa)}(t) \to \infty$ as $t \to \infty$ for $\varkappa = 0, 1, \ldots, n-1$. Condition (ii) implies $\lim_{t\to\infty} x^{(\varkappa)}(t) = \infty$ for $\varkappa = 0, 1, \ldots, n-1$.

If $x'(t) < 0$, then $\lim_{t\to\infty} x(t)$ exists and is nonnegative. Hence by Lemma 2, $\lim_{t\to\infty} x^{(\varkappa)}(t) = 0$ for $\varkappa = 1, 2, \ldots, n-1$. Since $x'(t) < 0$, by Lemma 1, $(-1)^{\varkappa} x^{(\varkappa)}(t) > 0$, for $t \geqslant t_0$, $\varkappa = 1, 2, \ldots, n$. If $\lim_{t\to\infty} x(t) = 2c > 0$, then there exists a $T \geqslant t_0$ such that for $t \geqslant T$

$$x[g_{ij}(t)] \geqslant c , \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, k . \tag{4}$$

Integrating $L(-1)$ from T to t

$$y^{(n-1)}(t) = y^{(n-1)}(T) + \int_T^t \sum_{i=1}^m f_i(s, x[g_{i1}(s)], \ldots, x[g_{ik}(s)])\, ds . \tag{5}$$

Hence

$$y^{(n-1)}(T) = -\int_T^\infty \sum_{i=1}^m f_i(s, x[g_{i1}(s)], \ldots, x[g_{ik}(s)])\, ds . \tag{6}$$

Integrating (5) from T to t and using (i), (iii), (5) and (6), we have

$$y^{(n-2)}(t) = y^{(n-2)}(T) + (t-T)y^{(n-1)}(T) + \int_T^t (t-s)\sum_{i=1}^m f_i(s, x[g_{i1}(s)], \ldots, x[g_{ik}(s)])\,ds =$$

$$= y^{(n-2)}(T) + \int_T^t (T-s)\sum_{i=1}^m f_i(s, x[g_{i1}(s)], \ldots, x[g_{ik}(s)])\,ds -$$

$$-(t-T)\int_t^\infty \sum_{i=1}^m f_i(s, x[g_{i1}(s)], \ldots, x[g_{ik}(s)])\,ds \leqslant$$

$$\leqslant y^{(n-2)}(T) - Ty^{(n-1)}(T) - \int_T^t f_p(s, cg_{p1}(s), \ldots, cg_{pk}(s))\,ds \to -\infty$$

as $t \to \infty$, a contradiction. Hence $c = 0$.

EXAMPLE 1. – Consider the equation for $t \geqslant \pi$

$$x''(t) - e^{\pi}x(t-\pi) = e^{-t}(e^{2\pi}\sin t - 2\cos t)$$

which has $x(t) = e^t + e^{-t}\sin t$ as a nonoscillatory solution.

EXAMPLE 2. – Equation

$$x''(t) - e^{-\pi}x(t-\pi) = e^{-t}(2\cos t - \sin t)$$

has $x(t) = e^{-t}(2 - \sin t)$ as a nonoscillatory solution.

THEOREM 2. – *Let $n \geqslant 3$ be odd and conditions* (C_1), (C_2) *hold. If $x(t)$ is a nonoscillatory solution of $E(+1)$, then $x^{(\varkappa)}(t) \to 0$ as $t \to \infty$ for $\varkappa = 0, 1, \ldots, n-1$.*

PROOF. – As in the proof of Theorem 1, we only discuss the case where $x(t)$ and $x[g_{ij}(t)]$ are positive for $t \geqslant t_0$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$. Let $y(t) = x(t) - r(t)$. Then $E(+1)$ can be written as $L(+1)$. Condition (iii) implies $y^{(n)}(t) < 0$ for $t \geqslant t_0$. It follows from Lemma 1 that there is a $t_1 \geqslant t_0$ and an integer l (even) such that (1) and (2) hold for $t \geqslant t_1$. If $x'(t) > 0$ for $t \geqslant t_1$, then $l \geqslant 2$. Thus as in the proof of Theorem 1, there is a $c > 0$ and $T \geqslant t_1$ such that (3) holds for $t \geqslant T$. Integrating $L(+1)$ from T to t and using (iii), (4), we have

$$y^{(n-1)}(t) \leqslant y^{(n-1)}(T) - \int_T^t f_p(s, cg_{p1}(s), \ldots, cg_{pk}(s))\,ds \to -\infty$$

as $t\to\infty$, a contradiction. Hence $x'(t)<0$ for $t\geqslant t_1$, then $\lim_{t\to\infty} x(t)$ exists and is nonnegative. By Lemma 2, $\lim_{t\to\infty} x^{(\varkappa)}(t)=0$ for $\varkappa=1,2,\dots,n-1$. If $\lim_{t\to\infty} x(t)=2c>0$, then there exists a $T\geqslant t_1$ such that (4) holds for $t\geqslant T$. Since $x'(t)<0$ for $t\geqslant T$ by Lemma 1, $(-1)^{\varkappa}x^{(\varkappa)}(t)>0$ for $t\geqslant T$, $\varkappa=0,1,\dots,n$. Integrating $L(+1)$ from T to t, we have

$$y^{(n-1)}(t)=y^{(n-1)}(T)-\int_T^t\sum_{i=1}^m f_i\big(s,x[g_{i1}(s)],\dots,x[g_{ik}(s)]\big)\,ds\,. \tag{7}$$

Hence

$$y^{(n-1)}(T)=\int_T^\infty\sum_{i=1}^m f_i\big(s,x[g_{i1}(s)],\dots,x[g_{ik}(s)]\big)\,ds\,. \tag{8}$$

Integrating (7) and using (4), (8), (iii) we have

$$y^{(n-2)}(t)\geqslant y^{(n-2)}(T)-Ty^{(n-1)}(T)+\int_T^t f_p\big(s,cg_{p1}(s),\dots,cg_{pk}(s)\big)\,ds\to\infty$$

as $t\to\infty$, a contradiction. Hence $c=0$.

EXAMPLE 3. – Equation

$$x'''(t)+e^{-\pi}x(t-\pi)=-e^{-t}(2\cos t+\sin t)$$

has $x(t)=e^{-t}(2-\sin t)$ as a nonoscillatory solution.

THEOREM 3. – *Let $n\geqslant 2$ be even. Assume that*

$$\int^\infty t^{n-1}f_p(t,c,\dots,c)\,dt=\pm\infty \quad \textit{for any constant} \quad c\neq 0\,.$$

If $x(t)$ is a nonoscillatory solution of $E(+1)$, then $x(t)\to\pm\infty$ as $t\to\infty$.

PROOF. – Without loss of generality, we may assume that $x(t)$ and $x[g_{ij}(t)]$ are positive for $t\geqslant t_0$, $i=1,2,\dots,m$, $j=1,2,\dots,k$. Let $y(t)=x(t)-r(t)$. Thus as in the proof of Theorem 2, we have $y^{(n)}(t)<0$ for $t\geqslant t_0$. By Lemma 1 there exist a $T\geqslant t_0$ and an integer l (odd) such that (1) and (2) hold for $t\geqslant T$. If $l=3$, then we see easily that $y(t)\to\infty$. Hence $x(t)\to\infty$ as $t\to\infty$. If $l=1$, then

$$(-1)^{\varkappa+1}y^{(\varkappa)}(t)>0 \quad \text{for} \quad \varkappa=0,1,\dots,n\,. \tag{9}$$

If $y(t)$ is unbounded, then we have our theorem. Now consider the case $y(t)$ is bounded. Multipling $L(+1)$ by t^{n-1} and integrating it from T to t

$$(10)\qquad Q(t)-Q(T)+(-1)^{(n-1)}(n-1)!\big(y(t)-y(T)\big)+\int_T^t s^{(n-1)}\sum_{i=1}^{m} f_i\big(s, x[g_{i1}(s)], \dots, x[g_{ik}(s)]\big)\,ds=0$$

where $Q(t)=\sum_{\varkappa=0}^{n-2}(-1)^{\varkappa}[t^{n-1}]^{(\varkappa)}y^{(n-\varkappa-1)}(t)$.

By (9), $Q(t)>0$. Hence

$$y(t)>Q(T)+\int_T^t s^{n-1}f_p(s, c, \dots, c)\,ds\to\infty \quad \text{as} \quad t\to\infty$$

where $c=x(T)$, a contradiction. Hence our proof is complete.

Example 4. – Equation $x''(t)+(1/4t^2)x(t)=0$ has a nonoscillatory solution $x(t)=t^{\frac{1}{2}}$.

Theorem 4. – *Let $n\geqslant 3$ be odd and condition* (C_1) *hold. Assume that*

$$\int^{\infty} f_p(t, c, \dots, c)\,dt=\pm\infty \quad \textit{for any constant} \quad c\neq 0\,.$$

If $x(t)$ is a nonoscillatory solution of $E(-1)$, then $|x^{(\varkappa)}(t)|\to\infty$ as $t\to\infty$ for $\varkappa=0, 1, \dots, n-1$.

Proof. – Without any loss of generality, we assume that $x(t)$ and $x[g_{ij}(t)]$ are positive for $t\geqslant t_0$ and $i=1, 2, \dots, m$, $j=1, 2, \dots, k$. Let $y(t)=x(t)-r(t)$. As in the proof of Theorem 1, we have $y^{(n)}(t)>0$ and (1), (2) hold for $t\geqslant t_0$, where t_0 is large enough. If $y^{(n-1)}(t)>0$, then as in the proof of Theorem 1, $y^{(\varkappa)}(t)\to\infty$ thus $x^{(\varkappa)}(t)\to\infty$ as $t\to\infty$ for $\varkappa=1, 2, \dots, n-1$. If $y^{(n-1)}(t)<0$, then $y'(t)>0$. There exist a $T\geqslant t_0$ and a constant $c>0$ such that $x[g_{ij}(t)]>c$ for $t\geqslant T$. Hence

$$y^{(n-1)}(t)=y^{(n-1)}(T)+\int_T^t\sum_{i=1}^{m} f_i\big(s, x[g_{i1}(s)], \dots, x[g_{ik}(s)]\big)\,ds> \\ > y^{(n-1)}(T)+\int_T^t f_p(s, c, \dots, c)\,ds\to\infty\,.$$

as $t\to\infty$, a contradiction.

Example 5. – Equation $x'''(t)-e^{\pi}x(t-\pi)=e^{-t}(2\cos t+2\sin t+e^{2\pi}\sin t)$ has a nonoscillatory solution $x(t)=e^{t}+e^{-t}\sin t$.

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