Periodic Solutions for Coupled Systems of Differential-Difference and Difference Equations (*) (**).

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Summary. – *The objective of this paper is to give necessary and sufficient conditions for the existence of periodic solutions of coupled systems of differential-difference and difference equations. By differentiating the difference equation, we obtain a system of neutral differential-difference equations and we get the original problem by putting a side condition on the neutral equation; that is, by restricting the initial data to lie on certain manifold in the space of all initial data. This allows us to treat the problem using the methods of neutral functional differential equations. In* [8], *Hale and Martinez-Amore exploited a certain change of variables to obtain some results on the stability of this systems. In Section 2, we summarize those ideas. The effect of the side condition is reflected in the variation of constants formula in Section 3. In this section, the variation of constants formula is decomposed via eigenspaces. In Section 4, we give a theorem on the Fredholm alternative for periodic solutions which is basic to the application of the usual theory to perturbed linear problems. I want to express my most deep gratitude to Professor J. K. Hale for his advice and suggestions which led to considerable improvements of this paper.*

1. – Notations and background.

Let $R = (-\infty, \infty)$ and let R^n be an n-dimensional linear vector space with norm $|\cdot|$. For $r \geqslant 0$, let $C = C([-r, 0], R^n)$ be the space of continuous functions mapping $[-r, 0]$ into R^n with the topology of uniform convergence. The norm in C will also be designated by $|\varphi| = \sup_{-r\leqslant\theta\leqslant 0}|\varphi(\theta)|$, $\varphi \in C$. Suppose D, L are bounded linear operators from C to R^n,

$$\begin{aligned} D(\varphi) &= H\varphi(0) - \int_{-r}^{0}[d\mu(\theta)]\varphi(\theta) \\ L(\varphi) &= \int_{-r}^{0}[d\eta(\theta)]\varphi(\theta) \end{aligned} \tag{1.1}$$

where H is an $n\times n$ matrix, $\det H \neq 0$, μ, η are $n\times n$ matrix functions of bounded variation on $[-r, 0]$ with μ nonatomic at zero. This latter hypothesis is equivalent to the existence of a continuous nondecreasing function $\gamma\colon [0, r] \to R$ such that

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$\gamma(0)=0$ and

$$\Big|\int_{-\varepsilon}^{0}[d\mu(\theta)]\varphi(\theta)\Big|\leqslant\gamma(\varepsilon)|\varphi|$$

for $\varepsilon\in[0,r]$, $\varphi\in C$.

If x is a function from $[\sigma-r,\infty)$ to R^n, let x_t, $t\in[\sigma,\infty)$, be the function from $[-r,0]$ to R^n defined by $x_t(\theta)=x(t+\theta)$, $\theta\in[-r,0]$. An autonomous linear homogeneous neutral functional differential equation (NFDE) is defined to be

$$\frac{d}{dt}D(x_t)=L(x_t)\,. \tag{1.2}$$

A solution $x=x(\varphi)$ of (1.2) through $\varphi\in C$ at $t=0$ is a continuous function taking $[-r,A)$, $A>0$, into R^n such that $x_0=\varphi$, $D(x_t)$ [not $x(t)$] is continuously differentiable on $[0,A)$ and satisfies (1.2) on this interval. It is proved in [9] that there is a unique solution $x(\varphi)$ through $(0,\varphi)$ defined on $[-r,\infty)$ and depends continuously on φ. If the transformation $T(t)\colon C\to C$, $t\geqslant0$, is defined by $T(t)\varphi=x_t(\varphi)$, then it is also skown in [9] that $\{T(t),t\geqslant0\}$ is a strongly continuous semigroup of linear operators with infinitesimal generator $A\colon\mathfrak{D}(A)\to C$, $A\varphi(\theta)=\dot\varphi(\theta)$, $-r\leqslant\theta\leqslant0$, $\mathfrak{D}(A)=\{\varphi\in C\colon\varphi\in C, D(\dot\varphi)=L(\varphi)\}$ and the spectrum $\sigma(A)$ of A consists of all those λ which satisfy the characteristic equation

$$\det\Delta(\lambda)=0\,,\qquad \Delta(\lambda)=\lambda D(\exp[\lambda\cdot]I)-L(\exp[\lambda\cdot]I)= \\ =\lambda H-\lambda\int_{-r}^{0}\exp[\lambda\theta]\,d\mu(\theta)-\int_{-e}^{0}\exp[\lambda\theta]\,d\eta(\theta)\,. \tag{1.3}$$

DEFINITION 1.1. – The operator D is said to be stable if there is a $\nu>0$ such that all roots of the equation

$$\det D(\exp[\lambda\cdot]I)=0$$

satisfy $\operatorname{Re}\lambda\leqslant-\nu$.

From the results of CRUZ and HALE [1] and HENRY [10] an operator D is stable if and only if the zero solution of the functional equation

$$D(y_t)=0\,,\qquad t\geqslant0$$

is uniformly asymptotically stable; that is, there are constants $K,\alpha>0$ such that

$$|y_t(\varphi)|\leqslant K\exp[-\alpha t]|\varphi|\,,\qquad t\geqslant0,\ \varphi\in C,\ D(\varphi)=0\,.$$

If $D(\varphi)=H\varphi(0)-J\varphi(-r)$, then D is stable if the roots of the polynomial equation

$$\det[H-\varrho J]=0 \tag{1.4}$$

satisfy $|\varrho|<1$.

An important property of equation (1.2) when D is stable is the following (see [2]): If D is stable, then there is a constant $a_D < 0$ such that for any $a > a_D$, there are only a finite number of roots λ of (1.3) with $\operatorname{Re}\lambda > a$.

Let D be stable. If $\Lambda = \{\lambda : \det\Delta(\lambda) = 0, \operatorname{Re}\lambda \geqslant 0\}$, then Δ is a finite set and it follows from [9] that the space C can be decomposed as $C = P \oplus Q$ where P, Q are subspaces of C invariant under $T(t)$, the space P is finite dimensional and corresponds to the initial values of all those solutions of (1.2) which are of the form $p(t)\exp[\lambda t]$, where $p(t)$ is a polynomial in t and $\lambda \in \Delta$.

The fundamental matrix $X(t)$ of (1.2) is defined to be the $n \times n$ matrix solution of the equation

$$(1.5)\qquad \begin{aligned} D(X_t) &= I + \int_0^t L(X_s)\,ds\,, \qquad t \geqslant 0 \\ X_0(\theta) &= \begin{cases} 0\,, & -r \leqslant 0 < 0 \\ H^{-1}\,, & \theta = 0 \end{cases} \end{aligned}$$

If $F, G: R \to R^n$ are continuous, a nonhomogeneous linear NFDE is defined as

$$(1.6)\qquad \frac{p}{dt}\{D(x_t) - G(t)\} = L(x_t) + F(t)\,.$$

A solution through φ at $t = \sigma$ is defined as before and is known to exist on $[\sigma - r, \infty)$.

The variation of constants formula for (1.6) (see [7], [3]) states that the solution of (1.6) through (σ, φ) is given by

$$(1.7)\qquad x(t) = T(t-\sigma)\varphi(0) + \int_\sigma^t X(t-s)F(s)\,ds - \int_\sigma^{t^+} [d_s X(t-s)][G(s) - G(\sigma)]$$

for $t \geqslant \sigma$ where X is the fundamental matrix solution given by (1.5). Another convenient equivalent form for equation (1.7) is the following:

$$(1.8)\qquad x(t) - X(0)G(t) = T(t-\sigma)\varphi(0) - X(t-\sigma)G(\sigma) + \int_\sigma^t X(t-s)F(s)\,ds - \\ - \int_\sigma^t [d_s X(t-s)]G(s)\,, \qquad t \geqslant \sigma\,.$$

Let us make a few observations on the variation of constants formula which suggest changes of variables which will be useful in later sections. Let PC be the space of functions taking $[-r, 0]$ into R^n which are uniformly continuous on $[-r, 0)$

and may be discontinuous at zero. With the matrix X_0 as defined before, it is clear that

$$PC = C + \langle X_0 \rangle$$

where $\langle X_0 \rangle$ is the span of X_0; that is, any $\psi \in PC$ is given as $\psi = \varphi + X_0 b$ where $\varphi \in C$, $b \in R^n$. We make PC a normed vector space by defining the norm $|\psi| = \sup_{-r \leqslant \theta \leqslant 0} |\psi(\theta)|$.

Let us define $x_t(\psi) = T(t)\psi$ where $\psi \in PC$ and $x(\psi)$ is the solution of (1.2) through ψ. The operator $T(t)\colon PC \to$ (functions on $[-r, 0]$) is linear, but $T(t)$ does not take $PC \to PC$. The operator $T(t)$ is an extension of the original semigroup $T(t)$ on C. If we use this notation, then the variation of constants formula (1.8) can be written as

$$x_t - X_0 G(t) = T(t-\sigma)[\varphi - X_0 G(\sigma)] + \int_\sigma^t T(t-s) X_0 F(s)\, ds - \int_\sigma^t [d_s T(t-s) X_0] G(s) \tag{1.9}$$

for $t \geqslant \sigma$, $\varphi \in C$. As usual in the theory of functional differential equations, these integrals are evaluated at each θ in $[-r, 0]$ as ordinary integrals in R^n.

If C is decomposed by Λ as $C = P \oplus Q$, then equation (1.9) is equivalent to

$$\begin{aligned} &a)\ x_t^P - X_0^P G(t) = T(t-\sigma)[\varphi^P - X_0^P G(\sigma)] \\ &\qquad + \int_\sigma^t T(t-s) X_0^P F(s)\, ds - \int_\sigma^t [d_s T(t-s) X_0^P] G(s) \\ &b)\ x_t^Q - X_0^Q G(t) = T[t-\sigma)[\varphi^Q - X_0^Q G(\sigma)] \\ &\qquad + \int_\sigma^t T(t-s) X_0^Q F(s)\, ds - \int_\sigma^t [d_s T(t-s) X_0^Q] G(s) \end{aligned} \tag{1.10}$$

where the superscripts P and Q designate the projections of the corresponding functions onto the subspaces P and Q, respectively.

Projection operators taking C onto P and a are determined by means of the adjoint differential equation

$$\frac{d}{dt}\left[y(t) - \int_{-r}^{0} y(t-\theta) H^{-1} d\mu(\theta) \right] = -\int_{-r}^{0} y(t-\theta) H^{-1} d\eta(\theta) \tag{1.11}$$

and the bilinear form

$$(1.12)\qquad (\alpha, \varphi) = \alpha(0)D_1(\varphi) + \int_{-r}^{0}\int_{0}^{\theta} \dot{\alpha}(\xi - \theta)H^{-1}[d\mu(\theta)]\varphi(\xi)\, d\xi - \int_{-r}^{0}\int_{0}^{\theta} \alpha(\xi - \theta)H^{-1}[d\eta(\theta)]\varphi(\xi)\, d\xi$$

defined for all $\alpha \in C^* = C([0, r], R^{n*})$, $\dot{\alpha} \in C^*$, $\varphi \in C$,

$$D_1(\varphi) = \varphi(0) - H^{-1}\int_{-r}^{0}[d\eta(\theta)]\varphi(\theta)\,.$$

If $\Phi = (\varphi_1, \ldots, \varphi_p)$ is a basis for the initial values of those solutions of (1.2) of the form $p(t)\exp[\lambda t]$ and $\Psi = \operatorname{col}(\psi_1, \ldots, \psi_p)$ is a basis for the initial values of those solution of (1.11) of the form $p(t)\exp[-\lambda t]$, then it is shown in [9] that the $p \times p$ matrix $(\Psi, \Phi) = ((\psi_i, \varphi_j))$, $i, j = 1, \ldots, p$ is nonsingular and, therefore, can be assumed to be the identity. If Φ, Ψ are defined in this way, then for any $\varphi \in C$, we define φ^P, φ^Q by $\varphi = \varphi^P + \varphi^Q$, $(\Psi, X_0) = \Psi(0)H^{-1}$. Hence, if we put $X_0^P = \Phi\Psi(0)H^{-1}$, $X_0^Q = X_0 - X_0^P$ we get formulas (1.10).

The following result of HENRY [10] (see also [2]) will be fundamental to our investigation.

LEMMA 1.1. – If $\operatorname{Re}\lambda \leqslant \delta$ for all λ satisfying (1.3), then for any $\varepsilon > 0$, there is a $K = K(\varepsilon)$ such that

$$(1.13)\qquad \begin{aligned} &a)\quad |T(t)\varphi^P| \leqslant K\exp[(\delta + \varepsilon)t]|\varphi^P|\,, \\ &\qquad |T(t)X_0|\,,\quad \left|\frac{d}{dT}\,T(t)X_0^P\right| \leqslant K\exp[\delta + \varepsilon)t]\,, \qquad t \leqslant 0\,, \\ &b)\quad |T(t)\varphi^Q| \leqslant K\exp[(\delta - \varepsilon)t]\,|\varphi^Q|\,, \\ &\qquad |T(t)X_0|\,,\quad \left|\frac{d}{dt}\,T(t)X_0^Q\right| \leqslant K\exp[(\delta - \varepsilon)t]\,, \qquad t \geqslant 0\,. \end{aligned}$$

Formula (1.9) certainly suggest the change of variables

$$(1.14)\qquad x_t - X_0G(t) = z_t\,, \qquad \varphi - X_0G(\sigma) = \psi$$

from $C \to PC$. If this is done, equation (1.9) becomes

$$(1.15)\qquad z_t = T(t - \sigma)\psi + \int_{\sigma}^{t} T(t - s)X_0F(s)\, ds - \int_{\sigma}^{t}[d_sT(t - s)X_0]G(s)$$

a formula much simpler than (1.9).

Let Φ, Ψ be the matrices defined as before for the decomposition $C = P \oplus Q$, $(\Psi, \Phi) = I$, and let E be the $p \times p$ matrix such that $T(t)\Phi = \Phi \exp[Et]$, $t \in (-\infty, \infty)$. The spectrum of E is Λ. For any $\psi \in PC$ one can define (Ψ, ψ) and therefore, it is meaningful to put

$$\psi^P = \Phi(\Psi, \psi)\,, \qquad \psi^Q = \psi - \psi^P\,, \qquad \psi \in PC\,.$$

Then equation (1.15) can be split as (1.10). If $z_t^P = \Phi u(t)$ then it follows from (1.10) and the transformation (1.14) that equation (1.15) is equivalent to

$$\begin{aligned} &z_t = u(t) + z_t^Q\,,\\ &\dot{u}(t) = Eu(t) + \Psi(0)H^{-1}F(t) + E\Psi(0)H^{-1}G(t)\,,\\ &z_t^Q = T(t-\sigma)\psi^Q + \int_\sigma^t T(t-s)X_0^Q F(s)\,ds - \int_\sigma^t [d_s T(t-s)X_0^Q]G(s)\,. \end{aligned} \tag{1.16}$$

We are now in a position for the discussion of the mixed differential and difference equations of the next section.

2. – A special equation.

In this section, we consider the equation

$$\begin{aligned} &a)\ \dot{x}(t) = Ax(t) + By(t-r)\\ &b)\ y(t) - E'x(t) - Jy(t-r) = 0 \end{aligned} \tag{2.1}$$

where x, y are n-vectors, all matrices are constants and E' is the ranspose of E. For any $a \in R^n$, $\psi \in C$, one can define a solution of (2.1) with initial value $x(0) = a$, $y_0 = \psi$.

Define $C = C([-r, 0], R^n)$

$$\begin{aligned} &D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}: R^n \times C \to R^{n\times n}\\ &D_1(a, \psi) = a\\ &D_2(a, \psi) = \psi(0) - E'a - J\psi(-r) \end{aligned} \tag{2.2}$$

For $(a, \psi) \in R^n \times C$, let

$$L = \begin{pmatrix} L_1 \\ 0 \end{pmatrix}: R^n \times C \to R^{n\times n}\,, \qquad L_1(a, \psi) = Aa + B\psi(-r)\,. \tag{2.3}$$

Equation (2.1) is a special case of the N.F.D.E.

$$\frac{d}{dt}D(x(t), y_t) = L(x(t), y_t) \tag{2.4}$$

and one obtains the equation (2.1) by requiring that

$$D_2(a, \psi) = 0 \,. \tag{2.5}$$

Equation (2.4) defines a semigroup $T(t)$ on $R^n \times C$. If we define $(R^n \times C)_0 = \{(a, \psi) \in R^n \times C \colon D_2(a, \psi) = 0\}$ then $(R^n \times C)_0$ can be considered as a Banach space. Furthermore, for any $(a, \psi) \in (R^n \times C)_0$, the solution of (2.4) through (a, ψ) will be in $(R^n \times C)_0$ since it corresponds to the solution of (2.1) through (a, ψ). Consequently,

$$T_0(t) \stackrel{\text{def}}{=} T(t)|_{(R^n \times C)_0} \colon (R^n \times C)_0 \to (R^n \times C)_0$$

is a strongly continuous semigroup. The infinitesimal generator $\mathcal{A}_0$ of $T_0(t)$ is $\mathcal{A}_0 = \mathcal{A}|_{(R^n \times C)_0}$ where $\mathcal{A}$ is the infinitesimal generator of $T(t)$. One easily shows that

$$\sigma(\mathcal{A}_0) = \{\lambda \in C \colon \det \Delta(\lambda) = 0\}\,, \qquad \Delta(\lambda) = \begin{bmatrix} \lambda I - A & -B\exp[-\lambda r] \\ -E' & I - J\exp[-\lambda r] \end{bmatrix}. \tag{2.6}$$

Observe that

$$D(a, \psi) = \begin{bmatrix} a \\ \psi(0) - E'a - J\,\psi(-r) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -E' & I \end{bmatrix}\begin{bmatrix} a \\ \psi(0) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} a \\ \psi(-r) \end{bmatrix} \stackrel{\text{def}}{=} H\varphi(0) - M\varphi(-r) \tag{2.7}$$

$$L(a, \psi) = \begin{bmatrix} Aa + B\psi(-r) \\ 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ \psi(0) \end{bmatrix} + \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ \psi(-r) \end{bmatrix} \stackrel{\text{def}}{=} N\varphi(0) + P\varphi(-r)$$

and, therefore, if the eigenvalues of the matrix J have modulii less than 1, then D is stable.

Also, from (1.5) we have that the fundamental matrix solution $X(t)$ of (2.4) is

$$X(0) = \begin{bmatrix} I & 0 \\ E' & I \end{bmatrix} = H^{-1} \tag{2.8}$$

and the other initial data < 0 necessary to define $X(t)$ are zero.

If

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

then X must be a solution of (2.4) with the initial data specified above. Therefore, the matrices X_{ij} must automatically satisfy

$$D_2(X_{11}(t), X_{21,t}) = 0\,, \qquad t \geqslant 0\,, \tag{2.9}$$

$$D_2(X_{12}(t), X_{22,t}) = I\,, \qquad t \geqslant 0\,. \tag{2.10}$$

Notice that (2.9) implies X_{11}, X_{21} are solutions of (2.1). The functions X_{12}, X_{22} do not satisfy (2.1*b*), but a nonhomogeneous version of it. However, it is important to notice that if these functions were differentiable, the derivatives would satisfy (2.1*b*). This is an important remark since it essentially implies that the variation of $X(t)$ satisfies the equation (2.1).

If we let $w_t = \operatorname{col}(x(t), y_t)$, by (2.7), equation (2.4) may be written as

$$\frac{d}{dt}[Hw(t) - Mw(t-r)] = Nw(t) + Pw(t-r)\,, \qquad w_0 = \varphi \in R^n \times C \tag{2.11}$$

which is equivalent to

$$\frac{d}{dt}[w(t) - Mw(t-r)] = \bar{N}w(t) + \bar{P}w(t-r) \tag{2.12}$$

since $H^{-1}M = M$ and where $H^{-1}N = \bar{N}$, $H^{-1}P = \bar{P}$.

We define the adjoint equation to (2.12) as

$$\frac{d}{dt}[v(t) - v(t+r)\,M] = -\,v(t)\;\bar{N} - v(t+r)\,\bar{P} \tag{2.13}$$

with the initial data $v_0 = \alpha = (b, \beta) \in R^{n*} \times C^*$.

From HALE and MAYER [9] if follows that if α belongs to the corresponding domain of $\mathcal{A}$, then the solution $v(\alpha)$ of (2.13) on $(-\infty, r]$ is continuously differentiable and, therefore, we may write (2.13) in the form

$$\dot{v}(t) - \dot{v}(t+r)\,M = -\,v(t)\bar{N} - v(t+r)\bar{P}\,. \tag{2.14}$$

For any $\alpha \in R^{n*} \times C^*$, let $v(\sigma, \alpha)$ be the solution of (2.14) on $(-\infty, \sigma + r]$ with $v(\sigma, \alpha)(\sigma + \theta) = \alpha(\theta)$, $\theta \in [0, r]$. Also, let $v^t(\sigma, \alpha) \in R^{n*} \times C^*$, $t < \sigma$, be defined by

$$v^t(\sigma, \alpha)(\theta) = v(\sigma, \alpha)(t + \theta)\,, \qquad \theta \in [0, r]\,.$$

For any $\varphi \in R^n \times C$, we define the following bilinear form

$$(\alpha, \varphi) = \alpha(0)[\varphi(0) - M\varphi(-r)] - \int_{-r}^{0} \dot{\alpha}(\theta + r)\,M\varphi(\theta)\,d\theta + \int_{-r}^{0} \alpha(\theta + r)\bar{P}\varphi(\theta)\,d\theta \tag{2.15}$$

for all those $\alpha \in R^{n*} \times C^*$ for which the expression is meaningful.

LEMMA 2.1. – If $v(t)$ is a solution of (2.14) on $(-\infty, \sigma+r]$, $\sigma>0$, and $w(t)$ is a solution of (2.12) on $[-r, \infty)$ then

$$(v^t, w_t) = \text{constant}\,, \qquad \text{for all } t\in[0,\sigma]\,. \tag{2.16}$$

PROOF. – By (2.15), we have

$$(v^t, w_t) = v(t)[w(t)-Mw(t-r)] - \int_{t-r}^{t}\dot v(\theta+r)\,Mw(\theta)\,d\theta + \int_{t-r}^{t} v(\theta+r)\bar P w(\theta)\,d\theta\,.$$

Then

$$\begin{aligned}\frac{d}{dt}(v^t, w_t) &= \dot v(t)\,[w(t)-Mw(t-r)] + v(t)\,[\bar N w(t) + \bar P w(t-r)] -\\ &\quad - \dot v(t+r)\,Mw(t) - \dot v(t)\,Mw(t-r) + v(t+r)\,\bar P w(t) - v(t)\,\bar P w(t-r) =\\ &= \dot v(t)\,w(t) + [-\dot v(t+r)\,M + v(t)\,\bar N + v(t+r)\,\bar P]\,w(t) = 0\end{aligned}$$

by (2.12) and (2.14).

We consider now the nonhomogeneous equation

$$\begin{aligned}\dot x(t) &= Ax(t) + By(t-r) + f(t)\\ y(t) &- E'x(t) - Jy(t-r) - g(t) = 0\end{aligned} \tag{2.17}$$

where f, g are continuous functions from $[0,\infty)$ to R^n. With D, L defined as in (2.2), (2.3), respectively,

$$G = \begin{bmatrix}0\\ g\end{bmatrix}\in R^{n\times n}\,, \qquad F = \begin{bmatrix}f\\ 0\end{bmatrix}\in R^{n\times n}\,,$$

the equation (2.17) is a special case of the NFDE

$$\frac{d}{dt}[D(w_t) - G(t)] = L(w_t) + F(t)\,, \qquad w_t = \operatorname{col}(x(t), y_t)\,, \qquad w_0 = \varphi\in R^n\times C \tag{2.18}$$

and one obtains the equation (2.17) by requiring that

$$D_2(\varphi) = g(0)\,. \tag{2.19}$$

As before we may write (2.18) in the form

$$\frac{d}{dt}[w(t) - Mw(t-r) - H^{-1}G(t)] = \bar N w(t) + \bar P w(t-r) + H^{-1}F(t) \tag{2.20}$$

and then can prove the following:

LEMMA 2.2. – If $v(t)$ is a solution of the adjoint equation (2.14) on $(-\infty, \infty)$ and $w(t)$ is a solution of (2.20) on $[-r, \infty)$, then

$$(v^t, w_t) = (v^0, w_0) + \int_0^t v(s)H^{-1}F(s)\, ds - \int_0^t [dv(s)H^{-1}]G(s)\,, \qquad t \geq 0\,. \tag{2.21}$$

PROOF. – By (2.15), we have

$$(v^t, w_t) = v(t)[w(t) - Mw(t-r) - H^{-1}G(t)] - \int_{t-r}^{t} \dot{v}(\theta + r)Mw(\theta)\, d\theta + \int_{t-r}^{t} v(\theta + r)\bar{P}w(\theta)\, d\theta\,.$$

Then

$$\begin{aligned}\frac{d}{dt}(v^t, w_t) &= \dot{v}(t)[w(t) - Mw(t-r) - H^{-1}G(t)] + v(t)[\bar{N}w(t) + \bar{P}w(t-r) + H^{-1}F(t)] - \\ &\quad - \dot{v}(t+r)Mw(t) - \dot{v}(t)Mw(t-r) + v(t+r)\bar{P}w(t) - v(t)\bar{P}w(t-r) = \\ &= \dot{v}(t)w(t) + [-\dot{v}(t+r)M + v(t)\bar{N} + v(t+r)\bar{P}]w(t) + v(t)H^{-1}F(t) - \dot{v}(t)H^{-1}G(t) = \\ &= v(t)H^{-1}F(t) - \dot{v}(t)H^{-1}G(t)\,.\end{aligned}$$

Integrating this expression from 0 to t yields the formula (2.21) which proves Lemma 2.2.

3. – Decomposition of the variation of constants formula.

The general solution of (2.18) is given by the variation of constants formula

$$\begin{aligned} w_t - X_0G(t) = T(t)[\varphi - X_0G(0)] &+ \int_0^t T(t-s)X_0F(s)\, ds \\ &- \int_0^t [d_sT(t-s)X_0]G(s)\,, \qquad t \geq 0 \end{aligned} \tag{3.1}$$

where $T(t)$ is an extension to $R^n \times C + \langle X_0 \rangle \overset{\text{def}}{=} Y$ of the original semigroup $T(t)$ on $R^n \times C$ and $X(t) \overset{\text{def}}{=} (T(t)X_0)(0)$ is the corresponding fundamental matrix solution of (2.4) given by (2.8).

As in Section 1, we will simplify the variation of constants formula. Let us make a few observations which will be useful later. Let

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \overset{\text{def}}{=} [U, V]\,, \qquad U = \operatorname{col}(X_{11}, X_{21})\,, \qquad V = \operatorname{col}(X_{12}, X_{22})\,.$$

By (2.9), $D_2(U_t) = 0$, $t \geqslant 0$. Since $F = \operatorname{col}(f, 0)$, it is reasonable to let

$$T(t-s)X_0F(s) = T(t-s)\,U_0 f(s) = T_0(t-s)\,U_0 f(s)$$

where $T_0(t)$ is the restriction of the semigroup $T(t)$ to $(R^n \times C)_0 + \langle X_0 \rangle \stackrel{\text{def}}{=} Y_0$; that is, $T_0(t)$ is an extension to Y_0 of the original semigroup $T_0(t)$ on $(R^n \times C)_0$.

By (2.10), $D_2(V_t) = I$, $t \geqslant 0$. Assume that D is stable and define the transformation $V \to W$, $W = \operatorname{col}(X^*_{12}, X^*_{22})$, by

$$\begin{aligned} X_{12} &= X^*_{12}\,, \\ X_{22} &= X^*_{22} + (I-J)^{-1}I\,. \end{aligned} \tag{3.2}$$

We can prove that $D_2(W_t) = 0$. In fact

$$D_2(W_t) = D_2(V_t) - (I-J)^{-1}I + J(I-J)^{-1} = I - (I-J)(I-J)^{-1} = 0\,.$$

Hence, we can write

$$[d_s T(t-s)X_0]G(s) = [d_s T(t-s)\,V_0]g(s) = [d_s T_0(t-s)\,W_0]g(s)$$

since $G = \operatorname{col}(0, g)$ and we are only interested in the variation of $T(t-s)X_0$.

We do now the change of variables

$$w_t - X_0 G(t) = z_t\,, \qquad \varphi - X_0 G(0) = \xi$$

from $R^n \times C \to R^n \times C + \langle X_0 \rangle$. If $\xi = \operatorname{col}(a, \xi_1)$, then

$$\begin{bmatrix} a \\ \xi_1 \end{bmatrix} = \begin{bmatrix} a \\ \psi \end{bmatrix} - \begin{bmatrix} 0 \\ X_{22,0}\cdot g(0) \end{bmatrix} = \begin{bmatrix} a \\ \psi - X_{22,0}\, g(0) \end{bmatrix} = \begin{bmatrix} a \\ \psi - \left(X^*_{22,0} + (I-J)^{-1}I\right) g(0) \end{bmatrix}.$$

If we define

$$\xi_1(\theta) = \begin{cases} \psi(\theta)\,, & -r \leqslant \theta < 0 \\ \psi(0) - \left(X^*_{22,0}(0) + (I-J)^{-1}I\right) g(0)\,, & \theta = 0 \end{cases} \tag{3.3}$$

then $D_2(\xi) = 0$. In fact, by (3.2) and (3.3), we have

$$\begin{aligned} D_2(\xi) &= \xi_1(0) - E'a - J\xi_1(-r) \\ &= \psi(0) - X^*_{22,0}(0)g(0) - (I-J)^{-1}g(0) - E'a - J\psi(-r) \\ &= \psi(0) - X^*_{22,0}(0)g(0) + (I-J)^{-1}g(0) - (I-J)^{-1}g(0) \\ &\qquad - E'a - J\psi(-r) = 0 \end{aligned}$$

after taking formula (2.19) into account.

If these transformations are done, equation (3.1) becomes

$$z_t = T_0(t)\xi + \int_0^t T_0(t-s)\,U_0 f(s)\,ds - \int_0^t [d_s T_0(t-s)\,W_0]g(s)\,, \qquad t \geq 0\,. \tag{3.4}$$

Suppose that the eigenvalues of J have modulii less than one. If $\Lambda = \{\lambda \in \sigma(\mathcal{A}_0):$ $\operatorname{Re}\lambda \geq 0\}$, then Λ is a finite set and the space $(R^n \times C)_0$ can be decomposed as $P \oplus Q$ where P, Q are subspaces of $(R^n \times C)_0$ invariant under $T_0(t)$ and the space P is finite dimensional. If Φ is a basis for the initial values of those solutions of (2.1) of the form $p(t)\exp[\lambda t]$ and Ψ is a basis for the initial values of those solutions of (2.14) of the form $p(t)\exp[-\lambda t]$, p a polynomial, $\lambda \in \Lambda$, then the matrix (Ψ, Φ) is nonsingular, by (2.16), and, therefore, can be assumed to be the identity. Let E be the matrix defined by the relation $\mathcal{A}_0\Phi = \Phi E$. The spectrum of E is Λ. For any $\xi \in Y_0$ we can define (Ψ, ξ) and put $\xi^P = \Phi(\Psi, \xi)$, $\xi^Q = \xi - \xi^P$, $\xi \in Y_0$. Each row of $\exp[-Et]\Psi$, $\Psi(\theta) = \exp[-E\theta]\Psi(0)$, $0 \leq \theta \leq r$, is a solution of the adjoint equation on $(-\infty, \infty)$ and, therefore, by (2.21),

$$(\exp[-Et]\Psi, z_t) = (\Psi, \xi) + \int_0^t \exp[-Es]\Psi(0)H^{-1}F(s)\,ds - \int_0^t [d_s \exp[-Es]\Psi(0)H^{-1}]G(s)$$

and

$$(\Psi, z_t) = \exp[Et](\Psi, \xi) + \int_0^t \exp[E(t-s)]\Psi(0)H^{-1}F(s)\,ds - \\ - \int_0^t [d_s \exp[E(t-s)]\Psi(0)H^{-1}]G(s)\,.$$

If $z_t = z_t^P + z_t^Q$, $z_t^P = \Phi(\Psi, z_t)$, then

$$\begin{aligned} z_t^P &= \Phi\exp[Et](\Phi, \xi) + \int_0^t \Phi\exp[E(t-s)]\Psi(0)H^{-1}F(s)\,ds - \\ &\qquad - \int_0^t [d_s\Phi\exp[E(t-s)]\Psi(0)H^{-1}[G(s) \\ &= T_0(t)\Phi(\Psi, \xi) + \int_0^t T_0(t-s)\Phi\Psi(0)H^{-1}F(s)\,ds - \int_0^t [d_s T_0(t-s)\Phi\Psi(0)H^{-1}]G(s) \\ &= T_0(t)\xi^P + \int_0^t T_0(t-s)\,U_0^P f(s)\,ds - \int_0^t [d_s T_0(t-s)\,W_0^P]g(s) \end{aligned} \tag{3.5}$$

where $U_0^P f(s) = \Phi\Psi(0)H^{-1}F(s)$ and $[d_s T_0(t-s)\Phi\Psi(0)H^{-1}]G(s) = [d_s T_0(t-s)\,W_0^P]g(s)$, after taking the above observations into account. If we let $z_t^P = \Phi u(t)$, then

$$\dot{u}(t) = Eu(t) + \Psi(0)H^{-1}F(t) + E\Psi(0)H^{-1}G(t) \tag{3.6}$$

or

$$\dot{u}(t) = Eu(t) + \Psi(0)\,U_0(0)f(t) + E\Psi(0)\,V_0(0)g(t)$$

since $H^{-1} = [U_0(0), V_0(0)]$, $F = \operatorname{col}(f, 0)$, $G = \operatorname{col}(0, g)$.

With z_t^P given above, define $U_0^Q = U_0 - U_0^P$, $W_0^Q = W_0 - W_0^P$. Then

$$z_t^Q = T_0(t)\xi^Q + \int_0^t T_0(t-s)\,U_0^Q f(s)\,ds - \int_0^t [d_s T_0(t-s)\,W_0^Q]g(s)\,. \tag{3.7}$$

We have proved

THEOREM 3.1. – If the eigenvalues of J have modulii less than one and $(R^n \times C)_0$ is decomposed by $\Lambda = \{\lambda \in \sigma(\mathcal{A}_0)\colon \operatorname{Re}\lambda \geqslant 0\}$ as $P \oplus Q$, then the solution of (2.17) satisfies (3.5) and (3.7). Furthermore, if $z_t^P = \Phi u(t)$, then $u(t)$ satisfies (3.6).

Since the semigroup $T_0(t)$ is defined on $Y_0 = \{\varphi \in Y\colon D_2(\varphi) = 0\}$ we can state an analogous lemma to Lemma 1.1 which is obtained by applying the same arguments in HENRY [10].

LEMMA 3.1. – If the eigenvalues of J have modulii less than one and all roots of (2.6) satisfy $\operatorname{Re}\lambda \leqslant -\delta < 0$, then are positive constants K, α such that

$$(3.8)\quad \begin{aligned} &a)\quad |T_0(t)\xi^P| \leqslant K\exp[\alpha t]\,|\xi^P|\,,\quad |T_0(t)\,U^P|\,,\quad \left|\frac{d}{dt}T_0(t)\,W_0^P\right| \leqslant K\exp[\alpha t]\,,\quad t \leqslant 0\\ &b)\quad |T_0(t)\xi^Q| \leqslant K\exp[\alpha t]\,|\xi^Q|\,,\quad |T_0(t)\,U_0^Q|\,,\quad \left|\frac{d}{dt}T_0(t)\,W_0^Q\right| \leqslant K\exp[-\alpha t]\,,\quad t \geqslant 0\,.\end{aligned}$$

4. – Fredholm alternative for periodic solutions.

In this section, we shall study the necessary and sufficient conditions that (2.17) has periodic solutions (see Hale [4], [5]). We assume that $f, g \in \mathcal{B}$, the set of bounded continuous functions mapping $(-\infty, \infty)$ into R^n with the topology of uniform convergence. Let $\mathcal{P}_T$ be the subset of $\mathcal{B}$ of periodic functions of period T.

For any $\sigma \in (-\infty, \infty)$ we know from the variation of constants formula (3.4) that the solution z of (2.17) with initial value z_σ at σ must satisfy

$$z_t = T_0(t-\sigma)\,z_\sigma + \int_\sigma^t T_0(t-s)\,U_0 f(s)\,ds - \int_\sigma^t [d_s T_0(t-s)\,W_0]g(s)\,,\qquad t \geqslant \sigma\,. \tag{4.1}$$

We shall be interested in solutions of (4.1) which are bounded on $(-\infty, \infty)$. If D is stable, then it follows from [6] that the solution is continuous and continuously differentiable. Suppose

$$\Lambda = \Lambda_0 \cup \Lambda_1\,,\qquad \Lambda_0 = \{\lambda \in \sigma(\mathcal{A}_0)\colon \operatorname{Re}\lambda = 0\},\ \Lambda_1 = \{\lambda \in \sigma(\mathcal{A}_0)\colon \operatorname{Re}\lambda > 0\}\,,$$

P_0, P_1 are the generalized eigenspaces of (2.1) associated with Λ_0, Λ_1, respectively, and that $(R_n \times C)_0$ is decomposed by Λ as $P_0 \oplus P_1 \oplus Q$. If $z_t = z_t^{P_0} + z_t^{P_1} + z_t^Q$, then, equation (4.1) is equivalent to

$$\text{(4.2)}\qquad \begin{aligned} &a)\ z_t^{P_0} = T_0(t-\sigma) z_\sigma^{P_0} + \int_\sigma^t T_0(t-s) U_0^{P_0} f(s)\, ds - \int_\sigma^t [d_s T_0(t-s) W_0^{P_0}] g(s)\,, \\ &b)\ z_t^{P_1} = T_0(t-\sigma) z_0^{P_1} + \int_\sigma^t T_0(t-s) U_0^{P_1} f(s)\, ds - \int_\sigma^t [d_s T_0(t-s) W_0^{P_1}] g(s)\,, \\ &c)\ z_t^Q = T_0(t-\sigma) z_\sigma^Q + \int_\sigma^t T_0(t-s) U_0^Q f(s)\, ds - \int_\sigma^t [d_s T_0(t-s) W_0^Q] g(s)\,, \qquad t \geqslant \sigma. \end{aligned}$$

LEMMA 4.1. – Equations (4.2*b*) and (4.2*c*) have unique solutions, $z_t^{P_1}, z_t^Q$, which are bounded for $t \in (-\infty, \infty)$ and these functions are given by

$$\text{(4.3)}\qquad \begin{aligned} &a)\ z_t^{P_1} = \int_\infty^t T_0(t-s) U f_0^{P_1}(s)\, ds - \int_\infty^t [d_s T_0(t-s) W_0^{P_0}] g(s)\,, \\ &b)\ z_t^Q = \int_{-\infty}^t T_0(t-s) U_0^Q f(s)\, ds - \int_{-\infty}^t [d_s T_0(t-s) W_0^Q] g(s)\,, \qquad t \in (-\infty, \infty)\,. \end{aligned}$$

Furthermore, $z_t^{P^1}, z_t^Q$ are continuous linear functions on $\mathcal{B}$ in the sense that there is a constant $L > 0$ such that

$$|z_t^{P_1}|\,, \qquad |z_t^Q| \leqslant L(|f| + |g|)\,.$$

Also, if $f, g \in \mathcal{P}_T$, then $z_t^{P_1}, z_t^Q \in \mathcal{P}_T$.

PROOF. – The same as in HALE [4] by using Lemma 3.1.

THEOREM 4.1 (Fredholm alternative [5]). – If the eigenvalues of J have modulii less than one, $f, g \in \mathcal{P}_T$, then the equation (2.17) has a solution in $\mathcal{P}_T$ if and only if

$$\text{(4.4)}\qquad \int_0^T v(t) H^{-1} F(s)\, ds - \int_0^T [dv(t) H^{-1}] G(s) = 0$$

for all T-periodic solutions v of the adjoint equation (2.14).

PROOF. – From Lemma 4.1, it is clear that we only need to consider equation (4.2*a*). Furthermore, if $z_t^{P_0} = \Phi u(t)$, then (4.2*a*) is equivalent, by (3.6), to

$$\text{(4.5)}\qquad \dot{u}(t) = E u(t) + \Psi(0) H^{-1} F(t) + E \Psi(0) H^{-1} G(t)$$

where Φ is a basis for P_0, Ψ is a basis for the generalized eigenspace of the adjoint equation associated with Λ_0 and E is defined by $\mathcal{A}_0\Phi = \Phi E$. The eigenvalues of E coincide with Λ_0 and thus have real parts equal to zero. Equation (4.5) is equivalent to

$$u(t) = \exp[Et]u(0) + \int_0^t \exp[E(t-s)]\Psi(0)H^{-1}F(s)\,ds - \int_0^t [d_s \exp[E(t-s)]\Psi(0)H^{-1}]G(s)\,.$$

As $\exp[Et]u(0)$ is T-periodic, in order to have $u(t)$ T-periodic it is necessary and sufficient that

$$\int_0^t \exp[E(t-s)]\Psi(0)H^{-1}F(s)\,ds - \int_0^t [d_s \exp[E(t-s)]\Psi(0)H^{-1}]G(s)$$

be T-periodic; that is, we require that

$$\int_0^T \exp[-Es]\Psi(0)H^{-1}F(s)\,ds - \int_0^T [d_s \exp[-Es]\Psi(0)H^{-1}]G(s) = 0\,.$$

But $\exp[-Es]\Psi(0) = \Psi(s)$ and Ψ is a basis for the T-periodic solutions of the adjoint equation. This completes the proof of the theorem.

It follows from [5] that Theorem 4.1 implies there is a continuous projection operator $\mathcal{J}\colon \mathcal{P}_T \to \mathcal{P}_T$ such that the set of all F, G satisfying (4.4) is the null space of $\mathcal{J}$, that is, $\mathcal{R}(I-\mathcal{J}) = (I-\mathcal{J})\mathcal{P}_T$.

Equation (2.17) is equivalent to

$$D(w_t) - D(\varphi) - \int_0^t L(w_s)\,ds = G(t) - G(0) + \int_0^t F(s)\,ds\,.$$

Let $\mathcal{H}\colon \mathcal{P}_T \to \mathcal{P}_T$ be defined by

$$\mathcal{H}(t) = D(w_t) - D(\varphi) - \int_0^t L(w_s)\,ds\,.$$

$\mathcal{H}$ is continuous and linear and the null space $\mathcal{N}(\mathcal{H})$ is the range of some continuous projection $\mathcal{S}\colon \mathcal{P}_T \to \mathcal{P}_T$. Also, there is a continuous linear operator $\mathcal{K}\colon (I-\mathcal{J})\mathcal{P}_T \to \mathcal{P}_T$ such that $\mathcal{K}(F, G)$ is a solution of (2.17) for each $F, G \in (I-\mathcal{J})\mathcal{P}_T$ and the solution $\mathcal{K}(F, G)$ will be unique if we require that $\mathcal{S}\mathcal{K} = 0$. Furthermore, the range $\mathcal{R}(\mathcal{H}) = \mathcal{R}(I-\mathcal{J})$ and $\mathcal{H}\mathcal{K} = I$, that is, $\mathcal{H}$ has a bounded right inverse $\mathcal{K}$ on the range of $\mathcal{H}$.

This remark is important because it would allow one to study perturbed linear problems.

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