Relative Continuum Mechanics in General Relativity (*) (**).
II. - The Lagrangian Viewpoint.

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Summary. – *An axiomatic approach to the study of relative continuum mechanics in curved space-time is proposed. The explicit assumptions are: a) existence of the energy-momentum tensor* T^{ij}, *satisfying the equations of motion* $T^{ij}{}_{//j} = 0$, *and b) existence of the congruence of stream-lines of the given continuum. The argument relies on a relativistic extension of the Lagrangian viewpoint, and involves the analysis of the relative dynamical behaviour of an arbitrary infinitesimal globule* Δ *of continuum in a given frame of reference* $[\Gamma]$. *The plan is fulfilled in two steps:* 1) *geometrical theory of the Lagrangian viewpoint, valid for any type of continua satisfying the stated requirements;* 2) *physical applications, illustrating the general theory in the case of an energy-momentum tensor of the form* $T^{ij} = \mu_0 V^i V^j - S^{ij}$.

Introduction.

In a previous paper [1] (henceforth denoted by I) we have discussed the kinematical foundations of relative continuum mechanics in General Relativity. We shall now examine the *dynamical* aspects of the theory.

In the space-time manifold $\mathcal{V}_4$, we consider an ideal physical system, completely characterized by a symmetric energy-momentum tensor T^{ij}, and by a corresponding congruence Ξ of stream-lines. Our plan is to develop a mathematical scheme for the study of the dynamical implications of the equations of motion

$$T^{ij}{}_{//j} = 0 \tag{0.1}$$

in any given frame of reference $[\Gamma]$.

The line of approach is based on a relativistic extension of the so called *Lagrangian viewpoint*, commonly used in Classical Continuum Mechanics [2, 3].

In other words, we shall consider an arbitrary infinitesimal globule Δ of continuum, and shall analyse its evolution relative to $[\Gamma]$, within the framework of point particle dynamics [5 ÷ 8].

The mathematical preliminaries are dealt with in Section 1. These include a short review of the concept of *spatial volume* (see e.g. I), and a subsequent application to the study of an arbitrary equation of the form $W^{ik\ldots l}{}_{//j} = 0$.

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The relativistic formulation of the Lagrangian viewpoint is outlined in Section 2. We prove that, as a consequence of eq. (0.1), the dynamical behaviour of every infinitesimal globule Δ of continuum in the frame of reference $[\Gamma]$ is determined by a set of equations which are formally identical to the standard relative equations of motion for a point particle in General Relativity, i.e.

$$\frac{\delta^* \boldsymbol{p}_\Delta}{\delta T} = \boldsymbol{F}_\Delta\,, \qquad \frac{\delta^* E_\Delta}{\delta T} = W_\Delta \tag{0.2}$$

the quantities $\boldsymbol{p}_\Delta$ (standard momentum), E_Δ (standard energy), $\boldsymbol{F}_\Delta$ (standard force), W_Δ (standard power) being determined by a corresponding set of *densities* (momentum density $\tilde{\boldsymbol{\pi}}$, energy density ε, force density $\boldsymbol{f}$ and power density w), each multiplied by the spatial volume of Δ relative to $[\Gamma]$.

The theory is completed by a suitable set of *identifications*, providing definite expressions for the densities $\tilde{\boldsymbol{\pi}}$, $\varepsilon, \boldsymbol{f}$ and w in terms of the energy-momentum tensor T^{ij}, of the standard velocity $\boldsymbol{v}$, and of the geometry of $\mathcal{V}_4$. In particular, the description of $\boldsymbol{f}$ and w is seen to involve a further dynamical object, called the *effective stress* of the continuum relative to $[\Gamma]$, and described by a generally non-symmetric space-tensor $\tilde{t}^{\alpha\beta}$.

We next come to the physical implications of the theory. As pointed out in I, these depend on the choice of an explicit relation between the four-velocity field V^i and T^{ij}. The case discussed here is based on the *ansatz*

$$(T_{ij} + \mu^0 g_{ij})\, V^j = 0 \tag{0.3}$$

μ^0 denoting the invariant mass density of the system [9].

As a consequence of eq. (0.3) we have now a linear relation between momentum density $\tilde{\boldsymbol{\pi}}$ and standard velocity $\boldsymbol{v}$. This allows a precise description of the *inertial* properties of the continuum in the frame of reference $[\Gamma]$, and leads quite naturally to the introduction of a *relativistic mass density tensor* $\tilde{\varrho}^{\alpha\beta}$ (1). A remarkable consequence of this fact is that, in the Lagrangian scheme, an arbitrary infinitesimal globule Δ of continuum is not regarded as a point particle in the usual sense, but as an « anomalous » particle, whose inertial properties are described by a relativistic *mass tensor* $\tilde{m}_\Delta^{\alpha\beta} = \tilde{\varrho}^{\alpha\beta} \cdot \mathrm{vol}\, \Delta$.

A detailed discussion of this point shows that the previous conclusion is unavoidable, and that it is in complete agreement with Einstein's equivalence principle.

A conclusive evidence in this sense comes from the physical interpretation of the equations of motion (0.2). The relevant results are listed below:

a) splitting of $\boldsymbol{F}_\Delta$ into a part entirely due to the internal stresses, and a grav-

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(1) The need for a mass density *tensor* $\tilde{\varrho}^{\alpha\beta}$ is due to the lack of parallelism between $\tilde{\boldsymbol{\pi}}$ and $\boldsymbol{v}$. This property is not peculiar of the Lagrangian scheme developed here, but is already present in Special Relativity (see e.g. [11]).

itational term. The analysis clarifies the physical meaning of the effective stress tensor $\tilde{t}^{\alpha\beta}$, and points out the role of the mass tensor $\tilde{m}_{\Delta}^{\alpha\beta}$ in the determination of the gravitational effects acting upon Δ in the given frame of reference (equivalence principle);

b) splitting of the power W_{Δ} into a mechanical part W_{mech}, and a dissipative one. A difficulty here arises from the fact that W_{mech} is not simply given by $\boldsymbol{F}_{\Delta}\cdot\boldsymbol{v}$, but includes an extra term, which depends on the *internal structure* of the globule Δ. Again, the use of the mass tensor $\tilde{m}_{\Delta}^{\alpha\beta}$ plays a fundamental role in the interpretation of the results.

1. – Mathematical preliminaries.

In this Section we set up the necessary mathematical tools for the relativistic extension of the Lagrangian viewpoint. For the notation, terminology, etc. the reader is referred to I.

(i) In the space-time manifold $\mathcal{V}_4$, let Ξ denote a sufficiently smooth congruence of world-lines, defined on a open world-tube $\mathcal{C}$. As in I, we regard Ξ as a *kinematical* object, rather than as a purely geometrical one. Mathematically, this is obtained by representing Ξ as a differentiable map θ of the domain $\mathcal{C}$ onto an abstract three-space $\mathcal{B}$, subject to the condition

$$\theta(x)=\theta(y) \qquad \text{if and only if both events } x, y \text{ belong to the } \textit{same} \text{ curve} \qquad a\in\Xi\,.$$

This allows to interpret $\mathcal{B}$ as the set of points of an ideal physical system, and Ξ as the congruence of stream-lines describing the *evolution* of $\mathcal{B}$ in $\mathcal{V}_4$.

The introduction of a physical frame of reference $[\Gamma]$ specializes the situation as follows:

1) The congruence Ξ admits a distinguished tangent vector field $\boldsymbol{\partial}_T$, related to the temporal 1-form $\boldsymbol{\omega}^0$ of $[\Gamma]$ by the duality relation

$$\langle\boldsymbol{\partial}_T,\,\boldsymbol{\omega}^0\rangle=1\,. \tag{1.1a}$$

Equation (1.1a) is mathematically equivalent to

$$\boldsymbol{\partial}_T=\boldsymbol{v}+\tilde{\boldsymbol{\partial}}_0 \tag{1.1b}$$

$\boldsymbol{v}=\mathcal{P}_\Sigma(\boldsymbol{\partial}_T)\in\tilde{D}^1$ denoting the *standard velocity* of Ξ relative to $[\Gamma]$.

2) The standard affine connection ∇^* associated with $[\Gamma]$ determines a differential operator $\nabla^*_{\boldsymbol{\partial}_T}$, called the *standard time derivative* along the curves of Ξ. The restriction of $\nabla^*_{\boldsymbol{\partial}_T}$ to an arbitrary curve $a\in\Xi$ is denoted by $\delta^*/\delta T$, and is called

the standard time derivative along a (I, 2.1). If we indicate by D/DT the absolute derivative on a determined by the Riemannian connection ∇ of $\mathcal{V}_4$ we have the explicit relations [5]

$$\frac{D}{DT}\tilde{\partial}_0 = \left[\frac{1}{2}(\tilde{K}^\alpha{}_\lambda - \tilde{\Omega}^\alpha{}_\lambda)\tilde{v}^\lambda + \tilde{C}^\alpha\right]\tilde{\partial}_\alpha \overset{\text{def}}{=} \tilde{b}^\alpha\tilde{\partial}_\alpha \tag{1.2a}$$

and

$$\frac{DX}{DT} = \frac{\delta^* X}{\delta T} + \tilde{X}^\beta\left(\tilde{b}_\beta\tilde{\partial}_0 - \frac{1}{2}\tilde{\Omega}^\alpha{}_\beta\tilde{\partial}_\alpha\right) \tag{1.2b}$$

for all spatial vector fields $X = \tilde{X}^\alpha\tilde{\partial}_\alpha$ defined on a (2).

3) The local kinematical behaviour of Ξ in the frame of reference $[\Gamma]$ is described by the spatial tensor field

$$\tilde{s} = \tilde{s}_{\alpha\beta}\omega^\alpha\otimes\omega^\beta = (\tilde{\nabla}^*_\alpha\tilde{v}_\beta + \tilde{C}_\alpha\tilde{v}_\beta + \tilde{\Omega}_{\lambda\alpha}\tilde{v}^\lambda\tilde{v}_\beta + \tfrac{1}{2}\tilde{K}_{\alpha\beta})\omega^\alpha\otimes\omega^\beta \tag{1.3}$$

called the *velocity gradient* of Ξ relative to $[\Gamma]$. The latter is further decomposed into *relative angular velocity tensor* $\tilde{s}_{\alpha\beta}\omega^\alpha\wedge\omega^\beta$, *relative dilatation* $\tilde{s}^\lambda{}_\lambda$ and *relative shear* $(\tilde{s}_{\alpha\beta} - \frac{1}{3}\tilde{s}^\lambda{}_\lambda\tilde{\gamma}_{\alpha\beta})\omega^\alpha\circledcirc\omega^\beta$ (I, 2.3).

4) The frame of reference $[\Gamma]$ induces a time-dependent *Riemannian structure* over the manifold $\mathcal{B}$, completely described by the time-dependent fundamental form

$$\hat{\varphi} = \varphi - 2g(\partial_T)\circledcirc\omega^0 + \langle\partial_T, g(\partial_T)\rangle\omega^0\otimes\omega^0 \tag{1.4a}$$

and by the associated Ricci tensor

$$\hat{\eta} = \partial_T \,\lrcorner\, \eta \tag{1.4b}$$

Φ and η denoting respectively the fundamental form and the Ricci tensor of $\mathcal{V}_4$ (I 3.2).

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(2) Throughout the subsequent discussion, a tilde placed over the components of an arbitrary tensor field will indicate that these components are referred to a *natural basis* $\{\tilde{\partial}_i, \tilde{\omega}^i\}$ of the tensor algebra over $(\mathcal{V}_4, \Gamma)$ [4]. The same convention applies to the symbols $\tilde{\nabla}_i$ and $\tilde{\nabla}^*_i$ (covariant derivatives induced respectively by the Riemannian connection ∇ and by the standard affine connection ∇^* associated with $[\Gamma]$). For later use, we recall the following relation between the connection coefficients $\tilde{\Gamma}_{ij}{}^k$ of ∇ and the spatial connection coefficients $\tilde{\Gamma}^*_{\alpha\beta}{}^\lambda$ of ∇^* in natural bases:

$$\tilde{\Gamma}_{\alpha\beta}{}^\lambda = \tilde{\Gamma}^*_{\alpha\beta}{}^\lambda\,;\quad \tilde{\Gamma}_{i0}{}^0 = 0\,;\quad \tilde{\Gamma}_{00}{}^\alpha = \tilde{\gamma}^{\alpha\lambda}\tilde{\Gamma}_{0\lambda}{}^0 = \tilde{C}^\alpha$$

$$\tilde{\Gamma}_{0\beta}{}^\lambda = \tilde{\Gamma}_{\beta 0}{}^\lambda = \tfrac{1}{2}(\tilde{K}^\lambda{}_\beta - \tilde{\Omega}^\lambda{}_\beta)\,;\quad \Gamma_{\beta\lambda}{}^0 = \tfrac{1}{2}(\tilde{K}_{\beta\lambda} + \tilde{\Omega}_{\beta\lambda})$$

the meaning of the symbols being as in [4, 5].

The differential forms (1.4a, b) determine a T-dependent *euclidean structure* on each tangent space $T_\xi(\mathcal{B})$, $\xi \in \mathcal{B}$, i.e. a T-dependent fundamental form $\hat{\boldsymbol{\varphi}}_\xi(T)$ and a T-dependent Ricci tensor $\hat{\boldsymbol{\eta}}_\xi(T)$ on $T_\xi(\mathcal{B})$, the time T being evaluated on the world-line $\theta^{-1}(\xi) \in \Xi$.

Neglecting higher order infinitesimals, this gives rise to a *local* T-dependent measure for distances, volumes, etc. in $\mathcal{B}$, based on the fact that, within first order approximation, every sufficiently small neighbourhood Δ of a point $\xi \in \mathcal{B}$ may be identified with a corresponding neighbourhood $\tilde{\Delta}$ of the origin in $T_\xi(\mathcal{B})$.

In particular, under the stated assumption, the *volume* of Δ at the instant T in the frame of reference $[\Gamma]$ is given explicitly by

$$(\mathrm{vol}\, \Delta)_T = \int_{\tilde{\Delta}} \hat{\boldsymbol{\eta}}_\xi(T) \tag{1.5}$$

the right-hand side being the ordinary integral of the 3-form $\hat{\boldsymbol{\eta}}_\xi(T)$ over the domain $\tilde{\Delta} \subset T_\xi(\mathcal{B})$.

(ii) As pointed out in I, the Lie derivative $\mathfrak{L}_{\boldsymbol{\partial}_T}\hat{\boldsymbol{\eta}}$ of the field (1.4b) satisfies the identity

$$\mathfrak{L}_{\boldsymbol{\partial}_T}\hat{\boldsymbol{\eta}} = (\partial_T)^j{}_{//j}\hat{\boldsymbol{\eta}} = [\tfrac{1}{2}\tilde{K}^\lambda{}_\lambda + (\tilde{\nabla}^*_\lambda + \tilde{C}_\lambda)\tilde{v}^\lambda]\hat{\boldsymbol{\eta}} = \tilde{s}^\lambda{}_\lambda\hat{\boldsymbol{\eta}} \tag{1.6}$$

$\tilde{s}^\lambda{}_\lambda$ being the relative dilatation of Ξ in the frame of reference $[\Gamma]$.

Now, let $\chi \in \mathcal{F}$ denote an arbitrary solution of the equation (3)

$$\boldsymbol{\partial}_T(\chi) = (\partial_T)^j{}_{//j} = \tilde{\nabla}_j(\tilde{\partial}_T{}^j) = \tilde{s}^\lambda{}_\lambda\,. \tag{1.7a}$$

Then, by eq. (1.6), we have the identity

$$\mathfrak{L}_{\boldsymbol{\partial}_T}(\exp[-\chi]\hat{\boldsymbol{\eta}}) = -\exp[-\chi]\tilde{s}^\lambda{}_\lambda\hat{\boldsymbol{\eta}} + \exp[-\chi]\tilde{s}^\lambda{}_\lambda\hat{\boldsymbol{\eta}} = 0 \tag{1.7b}$$

showing that the field $\exp[-\chi]\hat{\boldsymbol{\eta}}$ is an *ordinary* (time independent) geometrical object over $\mathcal{B}$.

More generally, given any point $\xi \in \mathcal{B}$, let us indicate by $\chi(T)$ the restriction of an arbitrary solution of eq. (1.7a) to the curve $\theta^{-1}(\xi) \in \Xi$, parametrized by means of standard time T. Then, recalling the definition of the Ricci tensor $\hat{\boldsymbol{\eta}}_\xi(T)$ induced by $\hat{\boldsymbol{\eta}}$ on the tangent space $T_\xi(\mathcal{B})$ (I, 3.2), the previous arguments imply that the product $\exp[-\chi(T)]\hat{\boldsymbol{\eta}}_\xi(T)$ is an *ordinary* (T-independent) field over $T_\xi(\mathcal{B})$. From this, keeping the same notation as above, we conclude that, if $\Delta \subset \mathcal{B}$ is any suffi-

(3) Eq. (1.7*a*) determines χ up to an arbitrary solution of the homogeneous equation $\boldsymbol{\partial}_T(f) = 0$. In particular, the restriction of χ to an arbitrary curve $a \in \Xi$ is determined up to an additive constant.

ciently small neighbourhood of the point ξ, the volume (1.5) factorizes into

$$(1.8)\qquad (\operatorname{vol}\Delta)_T = \exp[\chi(T)]\int_{\tilde{\Delta}} \exp[-\chi(T)]\,\hat{\eta}_\xi \overset{\text{def}}{=} \exp[\chi(T)]\,\delta\chi(\Delta)$$

with $\delta\chi(\Delta) \overset{\text{def}}{=} \int_{\tilde{\Delta}} \exp[-\chi(T)]\hat{\eta}_\xi =$ constant along the curve $\theta^{-1}(\xi)$.

REMARK 1.1. – From a geometrical viewpoint, the quantity $\delta\chi(\Delta)$ may be interpreted as a statical measure of Δ, *induced* by the given solution χ of eq. (1.7a). The freedom in the choice of χ (see footnote (3)) is then reflected in the fact that the function $\exp[\chi(T)]$ and the measure $\delta\chi(\Delta)$ in the factorization (1.8) are defined up to arbitrary transformations of the form $\exp[\chi(T)] \to \alpha \exp[\chi(T)]$, $\delta\chi(\Delta) \to \alpha^{-1}\delta\chi(\Delta)$, $\quad \alpha \in R$.

(iii) To complete our mathematical scheme, we shall now set up a general technique for handling any equation of the form $\tilde{\nabla}_j \tilde{W}^{ji_1\dots i_r} = 0$ in the world-tube $\mathcal{T}$. This will provide the formal basis for the relativistic formulation of the Lagrangian viewpoint, to be discussed in Section 2.

The method relies on the use of the projection operator $\hat{\mathfrak{I}}: \mathfrak{D}^1 \to \tilde{\mathfrak{D}}^1$ introduced in I. In natural bases, recalling eq. (1.1*b*), we have the explicit relation

$$\hat{\mathfrak{I}}(\boldsymbol{X}) \overset{\text{def}}{=} \boldsymbol{X} - \langle \boldsymbol{X}, \boldsymbol{\omega}^0\rangle\, \boldsymbol{\partial}_T = (\tilde{X}^\alpha - \tilde{X}^0 \tilde{v}^\alpha)\, \tilde{\boldsymbol{\partial}}_\alpha$$

for all $\boldsymbol{X} = \tilde{X}^i \tilde{\boldsymbol{\partial}}_i \in \mathfrak{D}^1$.

By means of $\hat{\mathfrak{I}}$, every tensor field $\boldsymbol{W} \in \mathfrak{D}^{r+1}$ $(r = 0, 1, \dots)$ defined on $\mathcal{T}$ may be factorized into

$$(1.9a)\qquad \boldsymbol{W} = -\boldsymbol{t} + \boldsymbol{\partial}_T \otimes \boldsymbol{\pi}$$

with $\boldsymbol{t} \in \mathfrak{D}^{r+1}$ and $\boldsymbol{\pi} \in \mathfrak{D}^r$ given respectively by the equations

$$(1.9b)\qquad \boldsymbol{t} \overset{\text{def}}{=} -\hat{\mathfrak{I}} \otimes \text{id} \otimes \dots \otimes \text{id}\, \boldsymbol{W} = (-\tilde{W}^{\alpha i_1\dots i_r} + \tilde{v}^\alpha \tilde{W}^{0 i_1\dots i_r})\, \tilde{\boldsymbol{\partial}}_\alpha \otimes \tilde{\boldsymbol{\partial}}_{i_1} \otimes \dots \otimes \tilde{\boldsymbol{\partial}}_{i_r}$$

$$(1.9c)\qquad \boldsymbol{\pi} \overset{\text{def}}{=} \tilde{W}^{ji_1\dots i_r}\langle \tilde{\boldsymbol{\partial}}_j, \boldsymbol{\omega}^0\rangle\, \tilde{\boldsymbol{\partial}}_{i_1} \otimes \dots \otimes \boldsymbol{\partial}_{i_r} = \tilde{W}^{0 i_1\dots i_r}\, \tilde{\boldsymbol{\partial}}_{i_1} \otimes \dots \otimes \tilde{\boldsymbol{\partial}}_{i_r}$$

Let us now define the field $\boldsymbol{z} \in \mathfrak{D}^r$ by

$$(1.10)\qquad \boldsymbol{z} \overset{\text{def}}{=} \tilde{\nabla}_j \tilde{t}^{ji_1,\dots i_r}\, \tilde{\boldsymbol{\partial}}_{i_1} \otimes \dots \otimes \tilde{\boldsymbol{\partial}}_{i_r}\,.$$

Then, setting for simplicity $\operatorname{Div} \boldsymbol{W} \overset{\text{def}}{=} \tilde{\nabla}_j \tilde{W}^{ji_1\dots i_r}\, \tilde{\boldsymbol{\partial}}_{i_1} \otimes \dots \otimes \boldsymbol{\partial}_{i_r}$ we have the following

LEMMA 1.1. – If $\chi \in \mathcal{F}$ is any solution of eq. (1.7a), the equation $\operatorname{Div} W = 0$ is mathematically equivalent to the transport law

$$\frac{D}{DT}(\exp[\chi(T)]\boldsymbol{\pi}) = \boldsymbol{z}\cdot\exp[\chi(T)] \tag{1.11}$$

along each curve $a \in \Xi$ (where, by abuse of language, we are using the same symbols $\boldsymbol{\pi}$, $\boldsymbol{z}$ to indicate the restrictions of the fields (1.9c), (1.10) to the curve a).

The proof is almost obvious: in view of eqs. (1.7a), (1.9a), (1.10), the equation $\operatorname{Div} \boldsymbol{W} = 0$ may be written in the form

$$0 = -\boldsymbol{z} + \tilde{\nabla}_j(\tilde{\partial}_T{}^j \tilde{\pi}^{i_1 \dots i_r})\, \tilde{\partial}_{i_1} \otimes \dots \otimes \tilde{\partial}_{i_r} = -\boldsymbol{z} + \partial_T(\chi)\boldsymbol{\pi} + \nabla_{\partial_T}\boldsymbol{\pi} =$$
$$= -\boldsymbol{z} + \exp[-\chi]\nabla_{\partial_T}(\exp[\chi]\boldsymbol{\pi})$$

which is equivalent to the transport law (1.11) along each curve $a \in \Xi$(D/DT being, by definition, the absolute derivative along a determined by the Riemannian connection of $\mathcal{V}_4$, see footnote (2)). Q.E.D.

The importance of Lemma 1.1 lies in the following observation: suppose we concentrate our attention on a single world-line $a \in \Xi$. Then, if Δ is any sufficiently small neighbourhood of the point $\theta(a) \in \mathcal{B}$, eq. (1.8) implies $\exp[\chi(T)] = \operatorname{vol}\Delta/\delta\chi(\Delta)$, with $\delta\chi(\Delta) = \text{const.}$ on a ([4]). We may therefore express eq. (1.11) in the equivalent form

$$\frac{D}{DT}(\boldsymbol{\pi}\operatorname{vol}\Delta) = \boldsymbol{z}\operatorname{vol}\Delta\,. \tag{1.12}$$

This formulation of the original equation $\operatorname{Div}\boldsymbol{W} = 0$ is especially relevant whenever the quantities $\boldsymbol{\pi}$ and $\boldsymbol{z}$ correspond to fields of *densities* associated with the physical system in study (momentum density, energy density, force density, power density, charge density, current density, etc.). In this case, eq. (1.12) determines the behaviour of the corresponding « integrated » quantities $\boldsymbol{\pi}_\Delta \overset{\text{def}}{=} \boldsymbol{\pi}\cdot\operatorname{vol}\Delta$, etc. associated with any infinitesimal globule Δ of the system, thus leading to a precise description of what is usually called the *Lagrangian viewpoint.*

II. - The Lagrangian Viewpoint.

2.1. *General theory.*

(i) Let us now assign to the physical system in study a symmetric energy-momentum tensor $\tilde{T}^{ij}\tilde{\partial}_i \otimes \tilde{\partial}_j$, satisfying the equations of motion

$$\tilde{\nabla}_i \tilde{T}^{ij} = 0\,. \tag{2.1}$$

([4]) From here on, for simplicity, we shall drop the subscript T in $(\operatorname{vol}\Delta)_T$.

Our plan is then to apply Lemma 1.1 in order to derive the dynamical implications of eqs. (2.1) in the frame of reference $[\Gamma]$. At the present stage, we shall not impose any a-priori assumption concerning the relation between $\tilde{T}^{ij}$ and the congruence Ξ of stream-lines of the given physical system.

As pointed out in I, this will not give rise to a *complete* dynamical scheme, but rather to a *geometrical theory* of the Lagrangian viewpoint, valid for any type of continua. Dynamical completeness is then restored in any specific physical situation, by adding to the present scheme the correct relationship between Ξ and $\tilde{T}^{ij}$. An example of this procedure, covering a large class of material continua, will be illustrated in Subsection 2.2.

To start our analysis, we observe that the factorization (1.9*a*, *b*, *c*) for the energy-momentum tensor $\tilde{T}^{ij}$ reads

$$\tilde{T}^{ij}\,\tilde{\partial}_i\otimes\tilde{\partial}_j = -\,\boldsymbol{t} + \partial_T\otimes\boldsymbol{\pi} = -\,\boldsymbol{t} + (\tilde{\partial}_0 + \boldsymbol{v})\otimes\boldsymbol{\pi} \tag{2.2a}$$

with

$$\boldsymbol{\pi} = \tilde{T}^{0i}\,\tilde{\partial}_i = \tilde{T}^{i0}\,\tilde{\partial}_i \overset{\text{def}}{=} \tilde{\pi}^i\,\tilde{\partial}_i \tag{2.2b}$$

$$\boldsymbol{t} = -\,\hat{\mathcal{T}}\otimes\mathrm{id}\,(\tilde{T}^{ij}\,\tilde{\partial}_i\otimes\tilde{\partial}_j) = (-\,\tilde{T}^{\alpha i} + \tilde{v}^\alpha\tilde{\pi}^i)\tilde{\partial}_\alpha\otimes\tilde{\partial}_i\,. \tag{2.2c}$$

The previous relations may be written more synthetically by introducing the *spatial* fields

$$\tilde{\boldsymbol{\pi}} \overset{\text{def}}{=} \mathcal{T}_\Sigma\boldsymbol{\pi} = \tilde{\pi}^\alpha\,\tilde{\partial}_\alpha = \tilde{T}^{0\alpha}\,\tilde{\partial}_\alpha = \tilde{T}^{\alpha 0}\,\tilde{\partial}_\alpha \tag{2.3a}$$

$$\varepsilon \overset{\text{def}}{=} \langle\boldsymbol{\pi}, \boldsymbol{\omega}^0\rangle = \tilde{\pi}^0 = \tilde{T}^{00} \tag{2.3b}$$

$$\tilde{\boldsymbol{t}} \overset{\text{def}}{=} \mathcal{T}_\Sigma\otimes\mathcal{T}_\Sigma\boldsymbol{t} = \tilde{t}^{\alpha\beta}\,\tilde{\partial}_\alpha\otimes\tilde{\partial}_\beta = (-\,\tilde{T}^{\alpha\beta} + \tilde{v}^\alpha\tilde{\pi}^\beta)\,\tilde{\partial}_\alpha\otimes\tilde{\partial}_\beta\,. \tag{2.3c}$$

We have then easily

$$\boldsymbol{\pi} = \tilde{\boldsymbol{\pi}} + \varepsilon\cdot\tilde{\partial}_0 \tag{2.4a}$$

$$\boldsymbol{t} = \tilde{\boldsymbol{t}} - (\tilde{T}^{\alpha 0} - \tilde{v}^\alpha\cdot\tilde{\pi}^0)\,\tilde{\partial}_\alpha\otimes\tilde{\partial}_0 = \tilde{\boldsymbol{t}} - (\tilde{\boldsymbol{\pi}} - \varepsilon\cdot\boldsymbol{v})\otimes\tilde{\partial}_0 \tag{2.4b}$$

Moreover, eq. (2.3*c*) implies the identity

$$(\tilde{t}^{\alpha\beta} - \tilde{v}^\alpha\tilde{\pi}^\beta)\tilde{\partial}_\alpha\wedge\tilde{\partial}_\beta = (\tilde{t}^{\alpha\beta} + \tilde{\pi}^\alpha\tilde{v}^\beta)\tilde{\partial}_\alpha\wedge\tilde{\partial}_\beta = 0 \tag{2.5}$$

due to the assumed symmetry of the components $\tilde{T}^{\alpha\beta}$.

In view of eq. (2.2*b*), the field $\boldsymbol{\pi}$ coincides with the density of four-momentum of the given continuum in the frame of reference $[\Gamma]$. This property is implicit in the definition of the energy-momentum tensor, and may be proved directly on the basis of the statistical model proposed by SYNGE [9, 10]. In particular, the fields

(2.3a, b) describe respectively the momentum and energy density of the continuum relative to $[\Gamma]$.

An equally simple interpretation for the field (2.3c) is not possible at this stage, and must be postponed until we have discussed the dynamical equations (2.1). For reasons that will be clear soon, we call $\tilde{\boldsymbol{t}}$ the *effective stress* of the continuum relative to $[\Gamma]$. Notice that, in general, the components $\tilde{t}^{\alpha\beta}$ are not symmetric, the only exception arising in the case $\boldsymbol{\pi}\|\boldsymbol{v}$ (see eq. (2.5)).

To sum up, taking eqs. (2.2a), (2.4a, b) into account, we conclude that momentum density $\tilde{\boldsymbol{\pi}}$, energy density ε and effective stress $\tilde{\boldsymbol{t}}$, together with standard velocity $\boldsymbol{v}$, are the natural *relative* quantities involved in the description of the evolution of the given continuum in the frame of reference $[\Gamma]$. These quantities, however, are not independent, but satisfy the inner identity (2.5).

(ii) The next step in our analysis involves the evaluation of the field (1.10), which, in the present case, reads

$$\boldsymbol{z} = \tilde{\nabla}_j(\tilde{t}^{ji})\,\tilde{\boldsymbol{\partial}}_i = [\tilde{\boldsymbol{\partial}}_j(\tilde{t}^{ji}) + \tilde{\Gamma}_{jp}{}^j\tilde{t}^{ip} + \tilde{\Gamma}_{jp}{}^i\tilde{t}^{jp}]\,\tilde{\boldsymbol{\partial}}_i\,.$$

Setting for simplicity $\tilde{\boldsymbol{z}} = \mathcal{T}_\Sigma \boldsymbol{z} = \tilde{z}^\alpha\tilde{\boldsymbol{\partial}}_\alpha$, $\tilde{z}^0 = \langle \boldsymbol{z}, \boldsymbol{\omega}^0\rangle$, and recalling eq. (2.4$b$) and footnote (2), a straightforward calculation yields the explicit expressions

$$\tilde{\boldsymbol{z}} = [(\tilde{\nabla}^*_\lambda + \tilde{C}_\lambda)\,\tilde{t}^{\lambda\alpha} - \tfrac{1}{2}(\tilde{K}^\alpha{}_\lambda - \tilde{\Omega}^\alpha{}_\lambda)(\tilde{\pi}^\lambda - \varepsilon\tilde{v}^\lambda)]\tilde{\boldsymbol{\partial}}_\alpha \tag{2.6a}$$

$$\tilde{z}^0 = -(\tilde{\nabla}^*_\lambda + \tilde{C}_\lambda)(\tilde{\pi}^\lambda - \varepsilon\tilde{v}^\lambda) + \tfrac{1}{2}(\tilde{K}_{\lambda\mu} + \tilde{\Omega}_{\lambda\mu})\,\tilde{t}^{\lambda\mu}\,. \tag{2.6b}$$

In view of Lemma 1.1, the dynamical equations (2.1) may now be written in the equivalent form

$$\frac{D}{DT}\{\exp[\chi(T)](\tilde{\boldsymbol{\pi}} + \varepsilon\,\tilde{\boldsymbol{\partial}}_0)\} = \exp[\chi(T)](\tilde{\boldsymbol{z}} + \tilde{z}^0\tilde{\boldsymbol{\partial}}_0)$$

with the usual meaning of the symbols. From this, taking eqs. (1.2a, b) into account, we get the final result

$$\frac{\delta^*}{\delta T}(\exp[\chi(T)]\tilde{\boldsymbol{\pi}}) = \exp[\chi(T)]\left(\tilde{z}^\alpha - \varepsilon\tilde{b}^\alpha + \frac{1}{2}\,\tilde{\Omega}^\alpha{}_\beta\tilde{\pi}^\beta\right)\tilde{\boldsymbol{\partial}}_\alpha \overset{\text{def}}{=} \exp[\chi(T)]\boldsymbol{f} \tag{2.7a}$$

$$\frac{d}{dT}(\exp[\chi(T)]\varepsilon) = \exp[\chi(T)]\,(\tilde{z}^0 - \tilde{\pi}^\beta\tilde{b}_\beta) \overset{\text{def}}{=} \exp[\chi(T)]\,w \tag{2.7b}$$

providing the transport law for the fields $\exp[\chi(T)]\tilde{\pi}$ and $\exp[\chi(T)]\varepsilon$ along each world-line $a\in\Xi$. In particular, by eqs. (1.2a), (2.6a, b), (2.7a, b), one can easily derive the following expressions for the fields $\boldsymbol{f}$ and w:

$$\boldsymbol{f} = \tilde{f}^\alpha\,\tilde{\boldsymbol{\partial}}_\alpha = [\tilde{\nabla}^*_\lambda\tilde{t}^{\lambda\alpha} + (\tilde{\Omega}^\alpha{}_\lambda - \tfrac{1}{2}\tilde{K}^\alpha{}_\lambda)\,\tilde{\pi}^\lambda - (\varepsilon\tilde{\gamma}^{\lambda\alpha} - \tilde{t}^{\lambda\alpha})\,\tilde{C}_\lambda]\,\tilde{\boldsymbol{\partial}}_\alpha \tag{2.8a}$$

$$w = (\tilde{\nabla}^*_\lambda + \tilde{C}_\lambda)(\varepsilon\tilde{v}^\lambda - \tilde{\pi}^\lambda) + \tfrac{1}{2}\tilde{K}_{\lambda\beta}\tilde{t}^{\lambda\beta} - \tilde{\pi}^\lambda(\tfrac{1}{2}\tilde{v}^\beta\tilde{K}_{\beta\lambda} + \tilde{C}_\lambda)\,. \tag{2.8b}$$

Given any curve $a \in \Xi$, let us now indicate by $\Delta \subset \mathcal{B}$ an arbitrary infinitesimal globule of continuum surrounding the point $\theta(a) \in \mathcal{B}$. Then, proceeding as in Section 1, we may replace the factor $\exp[\chi(T)]$ on both sides of eqs. (2.7*a*, *b*) by the ratio $\operatorname{vol}\Delta/\delta\chi(\Delta)$. Again, for fixed Δ, all terms $\delta\chi(\Delta)$ cancel out, and we are left with the equivalent system

$$\frac{\delta^*}{\delta T}(\tilde{\boldsymbol{\pi}} \operatorname{vol}\Delta) = \boldsymbol{f}\cdot\operatorname{vol}\Delta \tag{2.9a}$$

$$\frac{d}{dT}(\varepsilon \operatorname{vol}\Delta) = w\cdot\operatorname{vol}\Delta \tag{2.9b}$$

(compare with eq. (1.12)). In view of the stated interpretation for the fields $\tilde{\boldsymbol{\pi}}$ and ε, the quantities $\tilde{\boldsymbol{\pi}}\cdot\operatorname{vol}\Delta$ and $\varepsilon\cdot\operatorname{vol}\Delta$ coincide respectively with the *total momentum* $\boldsymbol{p}_\Delta$ and the *total energy* E_Δ of the globule Δ in the frame of reference $[\Gamma]$. This suggests a natural interpretation of the terms $\boldsymbol{f}\cdot\operatorname{vol}\Delta$ and $w\cdot\operatorname{vol}\Delta$ respectively as the *total force* $\boldsymbol{F}_\Delta$ and *total power* W_Δ acting on Δ in the given frame of reference (i.e. of $\boldsymbol{f}$ and w respectively as the *relative force density* and *relative power density* associated with the given continuum). In this sense, eqs. (2.9*a*, *b*) may be identified with the *standard relative equations of motion* for an arbitrary infinitesimal globule Δ of continuum in the frame of reference $[\Gamma]$, thus fulfilling the requirements of the Lagrangian viewpoint.

2.2. *Physical applications.*

(i) In order to complete our dynamical scheme, we have now to assume the existence of a functional relationship between the energy momentum tensor and the congruence of stream-lines of the physical system in study. As an example of this procedure, we shall consider an arbitrary *material* continuum, whose energy-momentum tensor $\tilde{T}^{ij}\tilde{\boldsymbol{\partial}}_i \otimes \tilde{\boldsymbol{\partial}}_j$ admits a canonical decomposition of the form [9]

$$\tilde{T}^{ij} = \mu^0 \tilde{V}^i \tilde{V}^j - \tilde{S}^{ij} \tag{2.10}$$

with $\tilde{V}^i \tilde{V}_i = -1$, and $\tilde{S}^{ij}\tilde{V}_j = 0$ (5). The fields μ^0, $\boldsymbol{V} = \tilde{V}^i\tilde{\boldsymbol{\partial}}_i$ and $\boldsymbol{S} = \tilde{S}^{ij}\tilde{\boldsymbol{\partial}}_i \otimes \tilde{\boldsymbol{\partial}}_j$ are called respectively the *invariant mass density*, the *four velocity* and the *stress tensor* of the continuum.

Recalling eq. (1.1*b*) and the identity

$$\boldsymbol{\partial}_T = (1 - v^2)^{\frac{1}{2}} \boldsymbol{V} \tag{2.11}$$

(5) The use of natural units ($c = 1$) is implicitly understood. Notice also that the ansatz (2.10) excludes explicitly the presence of electromagnetic interactions, at least on a macroscopic scale.

(I, 2.1), we have the explicit relations

$$\tilde V^\alpha = (1-v^2)^{-\frac12}\tilde v^\alpha\,,\qquad \tilde V^0 = -\tilde V_0 = (1-v^2)^{-\frac12}$$
$$0 = \tilde S^{ij}\tilde V_j = -\tilde S^{i0} + \tilde S^{i\beta}\tilde v_\beta$$

whence also

$$\tilde S^{\alpha 0} = \tilde S^{\alpha\beta}\tilde v_\beta\,;\qquad \tilde S^{00} = \tilde S^{0\beta}\tilde v_\beta = \tilde S^{\alpha\beta}\tilde v_\alpha\tilde v_\beta\,. \tag{2.12}$$

From these, setting for simplicity

$$\mu_0 = (1-v^2)^{-\frac12}\mu^0\,;\qquad \mu = (1-v^2)^{-1}\mu^0 \tag{2.13}$$

and making use of eqs. (2.3*a*, *b*, *c*), (2.10), we derive the following expressions for the fields $\tilde{\boldsymbol\pi}$, ε and $\boldsymbol t$:

$$\tilde\pi^\alpha = (\mu^0\tilde V^\alpha\tilde V^0 - \tilde S^{\alpha 0}) = (\mu\tilde v^\alpha - \tilde S^{\alpha\lambda}\tilde v_\lambda) \tag{2.14a}$$

$$\varepsilon = \mu^0\tilde V^0\tilde V^0 - \tilde S^{00} = \mu - \tilde S^{\alpha\beta}\tilde v_\alpha\tilde v_\beta \tag{2.14b}$$

$$\boldsymbol t = (-\mu^0\tilde V^\alpha\tilde V^\beta + \tilde S^{\alpha\beta} + \tilde v^\alpha\tilde\pi^\beta)\tilde{\boldsymbol\partial}_\alpha\otimes\tilde{\boldsymbol\partial}_\beta = (\tilde S^{\alpha\beta} - \tilde v^\alpha\tilde v_\lambda\tilde S^{\lambda\beta})\tilde{\boldsymbol\partial}_\alpha\otimes\tilde{\boldsymbol\partial}_\beta\,. \tag{2.14c}$$

Eqs. (2.14*a*, *b*, *c*) are mathematically equivalent to the system

$$\mu^0 = (1-v^2)\mu = \varepsilon - \tilde\pi^\alpha\tilde v_\alpha \tag{2.15a}$$

$$\tilde S^{\alpha\beta} = \tilde t^{\alpha\beta} + (1-v^2)^{-1}\tilde t^{\lambda\alpha}\tilde v_\lambda\tilde v^\beta \tag{2.15b}$$

$$\tilde\pi^\alpha = \varepsilon\tilde v^\alpha - \tilde t^{\alpha\lambda}\tilde v_\lambda \overset{\text{def}}{=} \tilde\varrho^{\alpha\lambda}\tilde v_\lambda \tag{2.15c}$$

with

$$\tilde\varrho^{\alpha\lambda} \overset{\text{def}}{=} \varepsilon\tilde\gamma^{\alpha\lambda} - \tilde t^{\alpha\lambda}\,. \tag{2.16}$$

Thus we see that, under the assumption (2.10), the basic dynamical objects associated with the continuum in study are reduced to two: effective stress $\tilde{\boldsymbol t}$ and energy density ε. These, together with the standard velocity $\boldsymbol v$, provide a complete spatial resolution of the energy-momentum tensor $\tilde T^{ij}$ in the frame of reference $[\Gamma]$, and determine all the relevant quantities involved in the equations of motion (2.9*a*, *b*) [6].

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[6] Notice that, in view of eq. (2.15*b*), the components $\tilde t^{\alpha\beta}$ and $\tilde v^\alpha$ are not independent, but satisfy the inner identity $(1-v^2)(\tilde t^{\alpha\beta} - \tilde t^{\beta\alpha}) = \tilde v_\lambda(\tilde t^{\lambda\beta}\tilde v^\alpha - \tilde t^{\lambda\alpha}\tilde v^\beta)$. From this (or also directly from eq. (2.14*c*)), it follows easily that, for $\|v\| \ll 1$, the antisymmetric part of $\tilde{\boldsymbol t}$ is negligible compared with the symmetric part.

In this sense, the present scheme is the natural generalization of the one adopted in Classical Continuum Mechanics, the only (substantial!) differences being *a*) the presence of a non-zero antisymmetric part in the effective stress $\tilde{\boldsymbol{t}}$, and *b*) the anisotropy of inertial mass, i.e. the lack of parallelism between momentum density $\tilde{\boldsymbol{\pi}}$ and standard velocity $\boldsymbol{v}$. The last property is of fundamental importance. As we shall see, it will lead to a natural interpretation of the field (2.16) as the *relativistic mass density tensor* of the continuum in the frame of reference $[\Gamma]$.

(ii) By eqs. (2.8*a*), (2.16), we obtain the following expression for the force density $\boldsymbol{f}$:

$$\boldsymbol{f} = [\tilde{\nabla}^*_\lambda \tilde{t}^{\lambda\alpha} + (\tilde{\Omega}^\alpha{}_\lambda - \tfrac{1}{2}\tilde{K}^\alpha{}_\lambda)\tilde{\pi}^\lambda - \tilde{\varrho}^{\lambda\alpha}\tilde{C}_\lambda]\tilde{\partial}_\alpha \,. \tag{2.17}$$

Let us now concentrate our attention on an arbitrary infinitesimal globule Δ of continuum. Then, setting for simplicity $\tilde{\boldsymbol{\pi}}\cdot\mathrm{vol}\,\Delta = \boldsymbol{p}$, $\varepsilon\cdot\mathrm{vol}\,\Delta = E$, $\tilde{\varrho}^{\alpha\beta}\cdot\mathrm{vol}\,\Delta = \tilde{m}^{\alpha\beta}$, $\tilde{\nabla}^*_\lambda \tilde{t}^{\lambda\alpha}\cdot\mathrm{vol}\,\Delta = \tilde{\sigma}^\alpha$, eqs. (2.9*a*), (2.15*c*), (2.17) yield

$$\boldsymbol{p} = \tilde{m}^{\alpha\lambda}\tilde{v}_\lambda\tilde{\partial}_\alpha \tag{2.18a}$$

$$\frac{\delta^* \boldsymbol{p}}{\delta T} = \left[\tilde{\sigma}^\alpha + \left(\tilde{\Omega}^\alpha{}_\lambda - \frac{1}{2}\tilde{K}^\alpha{}_\lambda\right)\tilde{p}^\lambda - \tilde{m}^{\lambda\alpha}\tilde{C}_\lambda\right]\tilde{\partial}_\alpha \,. \tag{2.18b}$$

In particular, when $\tilde{\boldsymbol{t}} = 0$ (i.e. $\boldsymbol{S} = 0$, corresponding to the case of an incoherent dust), eq. (2.16) implies

$$\tilde{m}^{\alpha\beta} = \varepsilon\,\tilde{\gamma}^{\alpha\beta}\cdot\mathrm{vol}\,\Delta = E\,\tilde{\gamma}^{\alpha\beta} \overset{\mathrm{def}}{=} \tilde{m}\,\tilde{\gamma}^{\alpha\beta} \,,$$

and eqs. (2.18*a*, *b*) take the more familiar form

$$\boldsymbol{p} = m\boldsymbol{v} \tag{2.19a}$$

$$\frac{\delta^* \boldsymbol{p}}{\delta T} = m\left[\left(\tilde{\Omega}^\alpha{}_\lambda - \frac{1}{2}\tilde{K}^\alpha{}_\lambda\right)\tilde{v}^\lambda - \tilde{C}^\alpha\right]\tilde{\partial}_\alpha \,. \tag{2.19b}$$

Eq. (2.19*b*) is identical to the equation of motion—already discussed in [5]—for a freely-falling test particle of relativistic mass m, subject only to the action of the external gravitational fields in the frame of reference $[\Gamma]$. Returning to the general case, it seems therefore quite natural to identify the quantity

$$[(\tilde{\Omega}^\alpha{}_\lambda - \tfrac{1}{2}\tilde{K}^\alpha{}_\lambda)\tilde{p}^\lambda - \tilde{m}^{\lambda\alpha}\tilde{C}_\lambda]\tilde{\partial}_\alpha = \tilde{m}^{\lambda\beta}[(\tilde{\Omega}^\alpha{}_\lambda - \tfrac{1}{2}\tilde{K}^\alpha{}_\lambda)\tilde{v}_\beta - \delta^\alpha_\beta\tilde{C}_\lambda]\tilde{\partial}_\alpha \tag{2.20}$$

with the *total gravitational force* acting upon the globule Δ of continuum in the given frame of reference. The remaining term $\tilde{\sigma}^\alpha\tilde{\partial}_\alpha = (\tilde{\nabla}^*_\lambda \tilde{t}^{\lambda\alpha}\cdot\mathrm{vol}\,\Delta)\tilde{\partial}_\alpha$ in eq. (2.18*b*) is then a description of the mechanical effects on Δ, due to the internal structure of the continuum.

The previous identification has a twofold advantage: on one side, it points out the physical relevance of the tensor $\tilde{\boldsymbol{t}}$, thus providing a complete justification for the term « effective stress » introduced before. On the other side, it clarifies the role of the tensor $\tilde{\varrho}^{\alpha\lambda}$ defined in eq. (2.16) (or, equivalently, of the tensor $\tilde{m}^{\alpha\lambda}=\tilde{\varrho}^{\alpha\lambda}\cdot\mathrm{vol}\,\Delta$). The latter, in fact, is not merely a formal object, relating momentum density $\tilde{\boldsymbol{\pi}}$ (or momentum $\boldsymbol{p}$) to the standard velocity $\boldsymbol{v}$ (as implicit in eqs. (2.15*c*), (2.18*a*)), but is indeed a *physical* quantity, which determines the coupling of every infinitesimal globule Δ of continuum with the external gravitational fields, in complete agreement with Einstein's equivalence principle.

Accordingly, we call $\tilde{\varrho}^{\alpha\lambda}$ the *relativistic mass density tensor* (and $\tilde{m}^{\alpha\lambda}$ the mass tensor of the globule Δ) in the frame of reference $[\Gamma]$.

By eq. (2.20), one can easily derive the expression for the density of gravitational force. Exactly as in [5], the latter is seen to consist of three different contributions, namely

1) a static effect $-\tilde{\varrho}^{\lambda\alpha}\tilde{C}_\lambda\tilde{\boldsymbol{\partial}}_\alpha$, depending only on the *acceleration* of $[\Gamma]$;

2) a Coriolis-type field $\tilde{\Omega}^{\alpha}{}_{\lambda}\tilde{\pi}^\lambda\tilde{\boldsymbol{\partial}}_\alpha$, depending on the *angular velocity* of $[\Gamma]$;

3) a deformation field (or Born field) $-\frac{1}{2}\tilde{K}^{\alpha}{}_{\lambda}\tilde{\pi}^\lambda\tilde{\boldsymbol{\partial}}_\alpha$, depending on the *non-rigidity* of the spatial geometry.

The discussion of these effects is analogous to the one presented in [5], and will not be repeatad here.

(ii) A similar analysis may be performed for the energy equation (2.9*b*). To this end, we observe that eq. (2.15*c*) implies the obvious identity

$$\tilde{\Omega}^{\beta}{}_{\lambda}(\tilde{\pi}^\lambda+\tilde{t}^{\lambda\mu}\tilde{v}_\mu)\tilde{v}_\beta=\tilde{\Omega}^{\beta}{}_{\lambda}(\varepsilon\cdot\tilde{v}^\lambda)\tilde{v}_\beta=0\,. \tag{2.21}$$

Taking eqs. (1.3), (2.8*b*), (2.15*c*), (2.17), (2.21) into account, we may now express the power density w in the form

$$w=(\tilde{\nabla}^*_\lambda+\tilde{C}_\lambda)(\tilde{t}^{\lambda\beta}\tilde{v}_\beta)+\tfrac{1}{2}\tilde{t}^{\lambda\beta}\tilde{K}_{\lambda\beta}-\tilde{v}_\beta(\tfrac{1}{2}\tilde{K}^{\beta}{}_{\lambda}\tilde{\pi}^\lambda+\tilde{\varrho}^{\lambda\beta}\tilde{C}_\lambda)=\tilde{v}_\beta\tilde{f}^\beta+\tilde{t}^{\lambda\beta}\tilde{s}_{\lambda\beta} \tag{2.22}$$

$\tilde{\boldsymbol{s}}=\tilde{s}_{\alpha\beta}\boldsymbol{\omega}^\alpha\otimes\boldsymbol{\omega}^\beta$ denoting the *velocity gradient* associated with the congruence of stream lines of the given continuum in the frame of reference $[\Gamma]$ (I, 2.3). The resulting expression for the energy equation (2.9*b*) is therefore

$$\frac{dE_\Delta}{dT}=W_\Delta=\boldsymbol{F}_\Delta\cdot\boldsymbol{v}+W'_\Delta \tag{2.23}$$

with the usual meaning of the symbols E_Δ, $\boldsymbol{F}_\Delta$, W_Δ, and with $W'_\Delta\overset{\text{def}}{=}\tilde{t}^{\lambda\beta}\tilde{s}_{\lambda\beta}\cdot\mathrm{vol}\,\Delta$ (7).

(7) For simplicity, we have indicated by $\boldsymbol{F}_\Delta\cdot\boldsymbol{v}$ the scalar product $\tilde{F}^\alpha_\Delta\tilde{v}_\alpha$.

From a physical viewpoint eq. (2.23) provides a natural splitting of the total power W_Δ into an *external part* $\boldsymbol{F}_\Delta \cdot \boldsymbol{v}$, due to the force $\boldsymbol{F}_\Delta$ acting on Δ, and an *internal contribution* W'_Δ, due to the *internal structure* of the globule Δ, i.e. to the presence of a non-zero effective stress *inside* Δ.

All this—and, in particular, the expression (2.22) for the power density w—is in complete agreement with the results of Classical Continuum Mechanics, and points out once again the validity of the identifications adopted so far.

2.3. *Further discussion of the energy equations.*

(i) For completeness, we shall now examine an alternative splitting of the power density w, into a *mechanical* part, and a *non-mechanical* (or dissipative) one. Among other advantages, the argument will throw new light on the physical meaning of the relativistic mass density tensor $\tilde{\varrho}^{\alpha\beta}$.

In carrying on the present analysis, we cannot rely on the usual four-dimensional formalism (spatial resolution of Minkowski's four force in the co-moving frame of reference), since the quantities $\boldsymbol{f}$ and w do not form a four-vector under transformations of the frame of reference $[\Gamma]$. We have therefore to resort to strictly three-dimensional methods.

Given an arbitrary infinitesimal globule Δ of continuum, we indicate by $(\text{vol}\,\Delta)_0$ and E^0 respectively the *proper volume* and the *proper energy* of Δ, i.e. the volume and energy of Δ, evaluated in the co-moving frame of reference (I, 2.1). Then, recalling eqs. (2.13), (2.15*a*) and the Lorentz contraction formula (I, 3.3), we have easily

$$E^0 = \mu^0 (\text{vol}\,\Delta)_0 = \mu^0 (1 - v^2)^{-\frac{1}{2}} \cdot \text{vol}\,\Delta = \mu_0 \cdot \text{vol}\,\Delta\,. \tag{2.24a}$$

Incidentally, this shows that the quantity μ_0 defined by eq. (2.13) coincides with the density of proper energy of the continuum relative to $[\Gamma]$. Moreover, by eqs. (2.15*a*), (2.24*a*), writing for simplicity $\boldsymbol{p}\cdot\boldsymbol{v}$ for $\tilde{\pi}^\alpha \tilde{v}_\alpha \cdot \text{vol}\,\Delta$, and E for $\varepsilon \cdot \text{vol}\,\Delta$, we obtain

$$E^0 = (1 - v^2)^{-\frac{1}{2}} (E - \boldsymbol{p}\cdot\boldsymbol{v}) \tag{2.24b}$$

in complete agreement with the corresponding formula, valid for an arbitrary point particle in Special Relativity. From this, dropping all subscripts Δ in the dynamical equations (2.9*a*), (2.23), we derive the explicit relation

$$W = \frac{dE}{dT} = \frac{d}{dT}[(1 - v^2)^{\frac{1}{2}} E^0 + \boldsymbol{p}\cdot\boldsymbol{v}] = (1 - v^2)^{\frac{1}{2}} \frac{dE^0}{dT} + \boldsymbol{F}\cdot\boldsymbol{v} + \\ + [\boldsymbol{p} - (1 - v^2)^{-\frac{1}{2}} E^0 \boldsymbol{v}] \cdot \frac{\delta^* \boldsymbol{v}}{\delta T} \tag{2.25}$$

the dot denoting scalar product.

From a physical viewpoint, it is now quite natural to identify the term $(1-v^2)^{\frac{1}{2}}(dE^0/dT)$ in eq. (2.25) with the dissipative power $W_{\text{diss}} \overset{\text{def}}{=} w_{\text{diss}} \cdot \text{vol}\,\Delta$ acting on Δ in the frame of reference $[\Gamma]$. In fact, by eqs. (2.23), (2.24*a*), recalling the relation between standard time T and proper time τ [5, 6], and denoting by w^0 the power density in the co-moving frame of reference, we have easily

$$(2.26a)\qquad W_{\text{diss}} = (1-v^2)^{\frac{1}{2}}\frac{dE^0}{dT} = (1-v^2)\frac{dE^0}{d\tau} = (1-v^2)\,w^0(\text{vol}\,\Delta)_0 = = (1-v^2)^{\frac{1}{2}}\,w^0\,\text{vol}\,\Delta$$

whence

$$(2.26b)\qquad w_{\text{diss}} = (1-v^2)^{\frac{1}{2}}\,w^0\,,$$

again in complete agreement with Special Relativity [11] (8).

If we now define the mechanical power W_{mech} acting on Δ as the difference $W - W_{\text{diss}}$, eq. (2.25) yields the explicit expression

$$(2.27a)\qquad W_{\text{mech}} = \boldsymbol{F}\cdot\boldsymbol{v} + [\boldsymbol{p} - (1-v^2)^{-\frac{1}{2}}E^0\boldsymbol{v}]\frac{\delta^*\boldsymbol{v}}{\delta T}\,.$$

From this, setting as usual $W_{\text{mech}} = w_{\text{mech}}\cdot\text{vol}\,\Delta$, and recalling eqs. (2.13), (2.24*a*) and the definition of $\boldsymbol{F}, \boldsymbol{p}$ and E^0, we derive the relation

$$(2.27b)\qquad w_{\text{mech}} = \tilde{f}^{\beta}\tilde{v}_{\beta} + [\tilde{\pi}_{\alpha} - \mu\tilde{v}_{\alpha}]\frac{\delta^*\tilde{v}^{\alpha}}{\delta T}\,,$$

thus completing the required splitting of the power density in the frame of reference $[\Gamma]$.

(ii) The nature of eq. (2.27*a*) is completely clarified, making use once again of the point particle model for the globule Δ developed in Subsection 2.2. More precisely, we shall now regard Δ as a ficticious « particle », completely characterized by a *proper energy* E^0, and by a *relativistic mass tensor* $\tilde{m}^{\alpha\beta}$, and moving under the action of an external force $\boldsymbol{F}$ and an external power W (the whole analysis being performed in the frame of reference $[\Gamma]$). The problem is then to compare the definition of W_{mech} given by eq. (2.27*a*) with the *mechanical work* done by the force $\boldsymbol{F}$ on Δ.

The main difficulty lies, of course, in the construction of a mathematical model for the study of point particles whose inertial properties have a tensorial character, rather than a scalar one. A suggestion in this sense comes from Classical Mechanics

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(8) Incidentally, eqs. (2.24*a*), (2.26*a*) allow a straightforward evaluation of w^0, based on the fact that the equation $w^0\,\text{vol}\,\Delta = dE^0/dT = (d/dT)(\mu_0\,\text{vol}\,\Delta)$ is mathematically equivalent to $w^0 = \tilde{\nabla}_i(\mu_0\tilde{\partial}^i_T)$ (compare with the remarks following Lemma 1.1). From this, recalling eqs. (2.11), (2.13), we get the familiar expression $w^0 = \tilde{\nabla}_i(\mu^0\tilde{V}^i)$ [9].

(more precisely from time-independent Lagrangian Dynamics), where the generalized momenta p_α are related to the « velocities » $\dot{q}^\alpha$ by an equation of the form [12]

$$p_\alpha = \frac{\partial \mathfrak{L}}{\partial \dot{q}^\alpha} = \frac{\partial \mathfrak{T}}{\partial \dot{q}^\alpha} = m_{\alpha\beta}\dot{q}^\beta \tag{2.28a}$$

the « mass tensor » $m_{\alpha\beta}$ being now directly involved in the definition of the kinetic energy

$$\mathfrak{T} = \tfrac{1}{2} m_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = \tfrac{1}{2} p_\alpha \dot{q}^\alpha . \tag{2.28b}$$

The mechanical work dL is then given by

$$dL \overset{\text{def}}{=} d\mathfrak{T} = \tfrac{1}{2}(dp_\alpha \dot{q}^\alpha + p_\alpha d\dot{q}^\alpha) = \tfrac{1}{2}(\dot{p}_\alpha dq^\alpha + p_\alpha d\dot{q}^\alpha) , \tag{2.28c}$$

the dot denoting time-derivative.

Setting for simplicity $\dot{p}_\alpha \overset{\text{def}}{=} F_\alpha$, eq. (2.28c) reads

$$dL = \tfrac{1}{2}(F_\alpha dq^\alpha + p_\alpha d\dot{q}^\alpha) . \tag{2.29}$$

Of course, all this is merely a consequence of the chosen definitions for p_α and F_α, and has nothing to do with effective mass anisotropies. However, it provides a useful mathematical model for the problem in study. In fact, suppose for a moment that we ascribe a real physical meaning to the « velocities » $\dot{q}^\alpha$ and momenta p_α —and therefore also to the « forces » F_α—by identifying them respectively with the components of the velocity $\boldsymbol{v}$, momentum $\boldsymbol{p}$ and force $\boldsymbol{F}$ acting upon a point particle Δ in a given frame of reference. Then, the suggestion of eq. (2.29) is that *the presence of an effectively anisotropic inertial mass $m_{\alpha\beta}$ modifies the structure of the elementary work dL, making it into a linear differential form*

$$dL = A_\alpha dx^\alpha + B_\alpha dv^\alpha \tag{2.30}$$

involving explicitly all differentials $dx^\alpha = dq^\alpha$, and $dv^\alpha = d\dot{q}^\alpha$ (i.e. into a linear differential form over the tangent space $T(M)$ associated with the configuration manifold M).

In general, of course we cannot expect eq. (2.30) to be *identical* to eq. (2.29), since the presence of effective mass anisotropies may in principle require a modification of the expression (2.28b) for the kinetic energy. In this sense, the only reasonable guide for a correct definition of the differential form (2.30) is a *correspondence principle*, namely

$$dL = \boldsymbol{F}\cdot d\boldsymbol{x} + dL^*$$

where dL^* depends only on the anisotropy of the inertial mass, or, more generally, on the fact that the « particle » in study is not a point particle in the usual sense.

Returning to our problem (i.e. to the case of a relativistic particle with proper energy E^0, momentum $\boldsymbol{p}$ and velocity $\boldsymbol{v}$), the only natural candidate for dL^* is a multiple of the differential form

$$[\tilde{p}_\alpha - (1 - v^2)^{-\frac{1}{2}} E^0 \tilde{v}_\alpha]\, d\tilde{v}^\alpha .$$

The latter is dimensionally correct, and is identically zero whenever the particle reduces to an ordinary (isotropic) one. We are therefore led to accept a definition of the elementary work dL of the form

$$dL = \boldsymbol{F}\cdot d\boldsymbol{x} + \lambda[\boldsymbol{p} - (1 - v^2)^{-\frac{1}{2}} E^0 \boldsymbol{v}]\cdot d\boldsymbol{v} \tag{2.31}$$

λ being a dimensionless constant.

The rest is now easy: comparison of eq. (2.31) with eq. (2.27*a*) shows that, with the ansatz $\lambda = 1$, the quantity dL is identical to the total *mechanical* work done on the globule Δ in the frame of reference $[\Gamma]$. Therefore, setting explicitly $\lambda = 1$ in eq. (2.31), we obtain a complete agreement between the point particle model and the physical interpretation of the energy equations (2.23) discussed in (i).

As a concluding remark we observe that, in the case $E^0 = \text{const.}$ (absence of dissipative effects on Δ), eq. (2.31) (with $\lambda = 1$) may be written in the equivalent form

$$dL = \frac{\delta^* \boldsymbol{p}}{\delta T}\cdot dx + [\boldsymbol{p} - (1 - v^2)^{\frac{1}{2}} E^0 \boldsymbol{v}]\, d\boldsymbol{v} = d[\boldsymbol{p}\cdot\boldsymbol{v} + (1 - v^2)^{\frac{1}{2}} E^0] . \tag{2.32}$$

If we now define the kinetic energy $\mathcal{T}$ of Δ by the usual conditions $d\mathcal{T} = dL$, $\lim_{v\to 0} \mathcal{T} = 0$, eq. (2.32) gives

$$\mathcal{T} = \boldsymbol{p}\cdot\boldsymbol{v} + (1 - v^2)^{\frac{1}{2}} E^0 - E^0 \tag{2.33}$$

whence $E = \mathcal{T} + E^0 = \boldsymbol{p}\cdot\boldsymbol{v} + (1 - v^2)^{\frac{1}{2}} E^0$, in complete agreement with eq. (2.24*b*).

Conversely, one could use eq. (2.33) and the conditions $dE^0 = 0$, $d\mathcal{T} = dL$ as an axiomatic definition of the mechanical work dL, thus avoiding most of the previous calculations. In this way, however, one would not get an equally direct evidence of the fundamental role played by the relativistic mass tensor $\tilde{m}^{\alpha\beta}$ in the interpretation of the energy equations.

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