Properties of H_i-Rings (*) (**).

M. EDWARD PETTIT, Jr. (Columbia, South Carolina, U.S.A)

Summary. – *The purpose of this paper is to study the general properties of H_i-rings (a ring R is an H_i-ring in case, for every height i prime ideal p in R, height p + depth p = altitude R) and to link the results to Nagata's chain conjectures. For R a local domain with maximal ideal M equivalences are given to « R is an H_i-ring » in relationship to the rings R/p, where p is a prime ideal in R such that height $p \leqslant i$; $R[X_1, \ldots, X_n]_{MR[X_1,\ldots,X_n]}$, where $X_1, \ldots, X_n$ are indeterminates; and $R[c_1/b, \ldots, c_j/b]_{MR[c_1/b,\ldots,c_j/b]}$, where $b, c_1, \ldots, c_j$ are analytically independent elements in R and $j \leqslant i$. Also, there are equivalences of $R[X]_{(M,X)}$, $R[c/b]_{(M,c/b)}$, the completion of R, the Henselization of R and localities of R being H_i-rings in terms of conditions on R. These results then yield some new equivalences of the chain conjectures.*

1. – Introduction.

The chain conditions (see (2.7) for the definitions) and the chain conjectures ((8.1)-(8.3)) have been studied for a number of years, begining with NAGATA in 1956 in [5]. In addition, a number of related conditions have been studied in connection with the chain conditions, such as unmixed, quasi-unmixed, and the altitude formula.

The major open question concerning the chain conditions is the chain conjecture: « The integral closure of a local domain satisfies the chain condition (c.c). » There are a number of equivalent statements to this and weaker conjectures which have appeared in the literature (for example, in [5] and [15]). None of these conjectures are answered in this paper, but some new equivalences of the conjectures do arise from the results obtained, and these are listed in Section 8 along with a list of open problems on H_i-rings.

In [13] Ratliff defined an H-ring (in the terminology of this paper, an H_1-ring), and in [15] suggested the generalization to H_i-rings. Using this extended concept, the results in this paper generalize most of the known facts about H_1-rings. These results give new statements about the chain conditions and imply some known results, since the condition of a ring being an H_i-ring is weaker than satisfying the chain conditions (see (2.9)); in particular, the first chain condition (f.c.c.). Also, some existing definitions can be shortened using the concept of H_i-rings; for example, a semi-local ring R is quasi-unimixed in case the completion of R is an H_0-ring.

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The « dual » condition of D_i-rings is defined (this is also a weaker condition than satisfying the f.c.c.). A few results were obtained and are stated in Section 7. However, the majority of the results on H_i-rings are not « dualized » to D_i-rings.

R will be assumed to be a local domain with maximal ideal M. It is shown that the following statements are equivalent: (1) R is an H_i-ring. (2) For every height $k(\leqslant i)$ prime ideal p in R, R/p is an H_{i-k}-ring and depth $p =$ altitude $R - k$ or depth $p \leqslant i - k$ (2.11). (3) $R[X_1, \dots, X_n]_{MR[X_1,\dots,X_n]}$ is an H_i-ring, where $X_1, \dots, X_n$ are indeterminates over R (3.1). (4) $R[c/b]_{MR[c/b]}$ is an H_{i-1}-ring, for every pair of elements b, c in R such that height $(b, c) = 2$ (4.2). (5) Altitude $R[c_1/b, \dots, c_i/b]_{MR[c_1/b,\dots,c_i/b]} =$ $=$ altitude $R - i$, for every set of elements $b, c_1, \dots, c_i$ which are analytically independent (a.r.) in R (4.2). (6) Every set of $i + 1$ elements which are a.i. in R can be extended to a set of a ($=$ altitude R) a.i. elements in R (4.3).

Theorems 3.2 and 4.5 show that the following statements are equivalent: (A) R is H_{i-1} and H_i and the integral closure of R/p is a D_0-ring, for every height $i - 1$ prime ideal p in R. (B) $R[X]_{(M,X)}$ is an H_i-ring. (C) $R[c/b]_{(M,c/b)}$ is an H_{i-1}-ring, for each pair of elements b, c in R such that height $(b, c) = 2$.

In addition to these results Section 2 includes the basic definitions and more results on the factor rings R/p; Section 3 discusses further the rings $R[X]_{MR[X]}$ and $R[X]_{(M,X)}$; and Section 4 is concerned with $R[y]_{(M,y)}$, where y is in the quotient field or R, and k-graded extensions (see (4.8) for the definition) besides the rings $R[c_1/b, \dots, c_j/b]_{MR[c_1/b,\dots,c_j/b]}$ and $R[c_1/b, \dots, c_j/b]_{(M,c_1/b,\dots,c_j/b)}$.

In (5.1) (respectively, (5.5)) we give equivalences for the completion (respectively, the Henselization) of R to be an H_i-ring.

Section 6 deals with localities over R; for example, it is shown that R satisfies the second chain condition if and only if every locality over R is an H_i-ring. Then the paper concludes with Sections 7 and 8.

2. – Basic definitions and factor rings of H_i-rings.

In this section the basic definitions to be used in this paper are given. Then the factor rings R/p, where p is a prime ideal in a local domain R, are analyzed in relation to these definitions; in particular, in (2.11) we show that the property of R being an H_i-ring implies that the rings R/p are H_{i-j}-rings, where $j =$ height p. All the results in this section, especially (2.11), are used throughout the remainder of the paper in the proofs of many of the results.

In this paper a number of notational conventions are used. These conventions are summarized in the following remark.

(2.1) Remark. – (2.1.1) In this paper all rings will be commutative rings with an identity element. An ideal will always be assumed to be a proper ideal. $I \subseteq J$ will mean that I is a subset of J, and $I \subset J$ will mean that I is a proper subset of J. Any underfined terminology is as stated in [7].

(2.1.2) If S is a ring, then S' will denote the integral closure of S in its total quotient ring.

(2.1.3) (R, M) will always denote a local domain, $a =$ altitude R, and K is the quotient field of R. It will be assumed that $a > 1$, since all results automatically hold if $a \leqslant 1$.

(2.2) DEFINITION. – A ring S is said to be an *H_i-ring* (or, S is H_i) in case, for every height i prime ideal p in S, depth $p =$ altitude $S - i$.

The above definition is a generalization of the definition of an H-ring given in [13, (4.5)].

It clearly holds that a ring S is an H_i-ring, for all $i < 0$ and all $i >$ altitude S. This fact will be used in various places in this paper without reference or further comment.

(2.3) EXAMPLE. – *For each $i > 0$, there is a local domain R which is not an H_i-ring but is an H_j-ring, for all $j = i + 1, \ldots, a =$* altitude R. Namely, let i be fixed $(i > 0)$. Then in [7, Example 2, pp. 203-205] (for $r > 0$ and $m = i$, and with the notation of [7]) the local domain (R, I) is an H_j-ring if and only if $j \neq 1, \ldots, i$. Since $I = R: R'$ (the conductor of R in R'), by [13, Remark 3.14 (iv)], there exists exactly one prime ideal q in R' lying over any prime ideal $p \neq I$ in R, by [16, Remark, p. 269], and, by [3, Theorems 44, 46 and 47, pp. 29 and 31], height $q =$ height p and depth $q =$ depth p. Hence there is a one-to-one correspondence between the non-maximal prime ideals in R and R'. Also, R' has exactly two maximal ideals MR' and NR', $R_{MR'}$ and $R_{NR'}$ are regular local rings (therefore satisfy the first chain condition (see (2.7.2)), and height $MR' = i + 1$ and height $NR' = r + i + 1$. With these facts it is a straightforward proof that the above statement holds, q.e.d.

The following definition is a «dual» of (2.2), and as with (2.2), a ring S is a D_i-ring, for all $i < 0$ and all $i >$ altitude S.

(2.4) DEFINITION. – A ring S is said to be a *D_i-ring* (or, S is D_i) in case, for every depth i prime ideal p in S, height $p =$ altitude $S - i$.

(2.5) EXAMPLE. – *For each $i > 0$, there is a local domain R which is not a D_i-ring but is a D_j-ring, for all $j = i + 1, \ldots, a =$* altitude R. Namely, in [7, Example 2, pp. 203-205] (for $r > 0$ and $m = i$) the local domain (R, I) is a D_j-ring if and only if $j \neq 1, \ldots, i$, as shown by an argument similar to that used in proving (2.3).

The following remark lists two facts which follow readily from the definitions and which will be used frequently in the paper.

(2.6) REMARK. – Let S be a ring such that altitude $S = a < \infty$.

(i) If (S, N) is a quasi-local ring, then S is D_0, D_a, H_{a-1} and H_a.

(ii) If S is an integral domain, then S is H_0, H_a, D_{a-1} and D_a.

The chain conditions which are defined below in (2.7), will be used throughout this paper. The definitions also appear in [2], [7], [11] and [15]; in particular, a number of properties of the chain conditions are listed in [11, Remarks 2.5-2.7] and [13, Remarks 2.22-2.25].

(2.7) DEFINITION. – Let S be a ring.

(2.7.1) A chain of prime ideals $p_0 \subset p_1 \subset \ldots \subset p_k$ in S is a *maximal chain of prime ideals in* S in case p_0 is a minimal prime ideal in S, p_k is a maximal ideal in S, and, for each $i = 1, \ldots, k$, height $p_i/p_{i-1} = 1$. The *length* of the chain is k.

(2.7.2) S satisfies the *first chain condition for prime ideals* (*f.c.c.*) in case every maximal chain of prime ideals in S has length equal to the altitude of S.

(2.7.3) S is *catenary* (or satisfies the *saturated chain condition for prime ideals*) in case, for every pair of prime ideals $p \subset q$ in S, $(S/p)_{q/p}$ satisfies the f.c.c.

(2.7.4) S satisfied the *second chain condition for prime ideals* (*s.c.c.*) in case, for every minimal prime ideal p in S and for every integral domain T which contains and is integral over S/p, T satisfies the f.c.c. and altitude T = altitude S.

(2.7.5) S satisfies the *chain condition for prime ideals* (*c.c.*) in case, for every pair of prime ideals $p \subset q$ in S, $(S/p)_{q/p}$ satisfies the s.c.c.

The following lemma is used in the proof of (2.10), (2.11), and (5.1).

(2.8) LEMMA. – *Let* (S, N) *be a local ring, and let* $p \subset q$ *be prime ideals in* S *such that* $q \neq N$ *and* height $q/p = 1$. *If* height $q >$ height $p + 1$, *then there exist infinitely many prime ideals* Q *in* S *such that* $p \subset Q$, height $Q/p = 1$, height Q = height $p + 1$ *and* depth Q = depth $q <$ altitude $S -$ height Q.

PROOF. – Apply [13, Corollary 2.3(2)] to $p \subset q$. Thus there exist infinitely many prime ideals Q in S such that $p \subset Q$, height $Q/p = 1$ and depth Q = depth q. Since, by [14, Lemma 2.1], there exist only finitely many prime ideals P in S such that $p \subset P$, height $P/p = 1$ and height $P >$ height $p + 1$, there exist infinitely many prime ideals Q in S such that $p \subset Q$, height $Q/p = 1$, depth Q = depth q and height Q = height $p + 1$. Furthermore, depth $q \leqslant a -$ height $q < a -$ (height $p + 1$) $= a -$ $-$ height Q, q.e.d.

There are a number of results in [11], [13], [14] and [15] which can be restated using the terms H_i- and D_i-rings; for example, see [13, Corollaries 2.4 and 2.7]. Remark 2.9 combines two known results into an equivalence which is used to prove that a local domain is catenary in later theorems.

(2.9) REMARK. – The following statements are equivalent:

(i) R satisfies the f.c.c.

(ii) R is catenary.

(iii) R is an H_i-ring, for all i.

(iv) R is an D_i-ring, for all i.

PROOF. – (i) is equivalent to (ii), by [11, Remark 2.7], and the equivalence of (i), (iii) and (iv) follows from [14, Theorem 2.2] and the definition of an H_i-ring and a D_i-ring, q.e.d.

Lemma 2.10 is part of Theorem 2.11 and is given here in order to include more general rings than local domains.

(2.10) LEMMA. – *Let S be a ring such that* altitude $S = a < \infty$, *and let j be fixed* $(0 \leqslant j \leqslant a)$. *If S is a D_0-ring, then the following statement holds for each k* $(0 \leqslant k \leqslant j)$ *If S/p is an H_{j-k}-ring and either* depth $p \leqslant j - k$ *or* depth $p = a - k$, *for every height k prime ideal p in S, then S is an H_j-ring.*

PROOF. – It may be assumed that $0 < j < a - 1$, since it is clear if $j = 0$ and S is H_{a-1} and H_a. Let p be a height j prime ideal in S. Since S is a D_0-ring, p is not maximal. Also, there exists a prime ideal q in S such that $q \subseteq p$, height $q = k$ and height $p/q = j - k$. By hypothesis, height q + depth $q = a$, since height q + depth $q \geqslant k$ + height p/q + depth $p \geqslant k + (j - k) + 1 = j + 1$. Since S/q is an H_{j-k}-ring and height $p/q = j - k$, depth p = depth p/q = altitude $S/q - j + k$ = depth $q - j + k = a - j$. So S is an H_j-ring, q.e.d.

Theorem 2.11 below shows the relationship between an H_j-ring R and the ring R/p, where p is a prime ideal in R such that height $p \leqslant j$. It and its corollaries will be used frequently throughout the rest of the paper.

(2.11) THEOREM. – *Let j be fixed* $(0 \leqslant j \leqslant a)$, *and let k be fixed* $(0 \leqslant k \leqslant j)$. *Then R is an H_j-ring if and only if, for every height k prime ideal p in R, R/p is an H_{j-k}-ring and either* depth $p = a - k$ *or* depth $p \leqslant j - k$.

PROOF. – The theorem is obvious for $j = 0$, so it may be assumed that $j > 0$.

Assume that R is an H_j-ring, and let p be a prime ideal in R such that height $p = k \leqslant j$. Clearly, we can assume $0 < k < j$. By [13, Corollary 2.7], either depth $p = a - k$ or depth $p \leqslant j - k$. It remains to show that R/p is an H_{j-k}-ring in both cases. If depth $p \leqslant j - k$, then, R/p is clearly an H_{j-k}-ring (this includes the case where $j = a$; so assume $j < a$). If depth $p = a - k$, then altitude $R/p = a - k$. Let q be a prime ideal in R such that $p \subset q$ and height $q/p = j - k < a - k$, so $a >$ height $q \geqslant j$. If height $q > j$, then let height $q = m$. Thus depth $q \leqslant a - m$, say depth $q = n$. By [13, (2.2.1)] (the case $i = j - k - 1$), there exists a prime ideal P in R such that $P \subset q$, height $P = j - 1$ and height $q/P = 1$. Hence, by (2.8), there exists a prime ideal Q in R such that $P \subset Q$, depth $Q = n$ and height $Q = j$. Then $n = a - j$, since R is an H_j-ring. However, $n \leqslant a - m < a - j$. This contradiction implies height $q = j$. Thus depth $q/p =$ depth $q =$ (R is an H_j-ring) $a - j =$ $= (a - k) - (j - k) =$ altitude $R/p -$ height R/p. Hence R/p is an H_{j-k}-ring.

Since R is a D_0-ring, the converse holds by (2.10), q.e.d.

3. – Conditions for $R(X_1, \ldots, X_n)$ and $\mathfrak{T}_n$ to be H_i-rings.

This section describes the relationship, with respect to being H_i-rings, between a local domain R and certain localizations of the polynomial ring $R[X_1, \ldots, X_n]$.

If $X_1, \ldots, X_n$ are indeterminates over a local ring (S, N), we fix the following notation for the remainder of the paper: $S_n = S[X_1, \ldots, X_n]$; $S(X_1, \ldots, X_n) = (S_n)_{NS_n}$; and $\mathfrak{T}_n^*(S) = (S_n)_{(N, X_1, \ldots, X_n)S_n}$. We also let $\mathfrak{T} = \mathfrak{T}_1(R)$.

It is known [15, Theorem 3.1] that R is an H_1-ring if and only if $R(X)$ is an H_1-ring. Theorem 3.1 extends this result to H_i-rings.

(3.1) THEOREM. – *Let i be fixed. Then R is an H_i-ring if and only if, for any $n \geq 1$, $R(X_1, \ldots, X_n)$ is an H_i-ring.*

PROOF. – It suffices to show that R is an H_i-ring if and only if $R(X)$ is an H_i-ring, since $R(X_1, \ldots, X_n) \cong R(X_1, \ldots, X_{n-1})(X_n)$. Also, altitude $R(X) = a$, so we can assume $0 < i < a - 1$, since R and $R(X)$ are H_0, H_{a-1} and H_a, by (2.6).

First, assume that R is an H_i-ring and let q be a height i prime ideal in $R(X)$. Then $p = q \cap R$ is either a height $i - 1$ or height i prime ideal in R. If height $p = i$, then $q = pR(X)$ and $a - i =$ depth $p =$ altitude $R/p =$ altitude $R(X)/q =$ depth q, as desired. If height $p = i - 1$, then $q \supset pR(X)$ and height $q/pR(X) = 1$. Also, R/p is an H_1-ring and either depth $p = a - i + 1$ or depth $p = 1$, by (2.11). Thus $(R/p)(X) \cong R(X)/pR(X)$ is an H_1-ring, by [15, Theorem 3.1], and hence depth $q =$ $=$ depth $q/pR(X) =$ altitude $R(X)/pR(X) - 1 =$ altitude $R/p - 1 =$ depth $p - 1$. So depth $q = 0$ or $a - i$; thus depth $q = a - i$, since q is not maximal. Hence $R(X)$ is an H_i-ring.

Conversely, let $R(X)$ be an H_i-ring and let p be a height i prime ideal in R. Then height $pR(X) = i$. Therefore $a - i =$ depth $pR(X) =$ (as in the previous paragraph) depth p, and hence R is an H_i-ring, q.e.d.

It is known [15, Theorem 3.2 and Remark 3.4] that $\mathfrak{T}$ is an H_1-ring if and only if R is an H_1-ring and R' is a D_0-ring. Theorem 3.2 generalizes this result to H_i-rings.

(3.2) THEOREM. – *Let i be fixed. $\mathfrak{T}$ is an H_i-ring if and only if R is H_{i-1} and H_i and $(R/p)'$ is a D_0-ring, for every height $i - 1$ prime ideal p in R.*

PROOF. – The case $i \leq 0$ is trivial, by (2.6) (ii), and it is known that the theorem holds for the case $i = 1$, by [15, Theorem 3.2 and Remark 3.4], so assume $i > 1$. Altitude $\mathfrak{T} = a + 1$, so we can assume $i < a$; since, by (2.6) (i), $\mathfrak{T}$ is H_a and H_{a+1} and R is H_{a-1} and H_a, and if p is a height $a - 1$ prime ideal in R, then altitude $R/p = 1$, so $(R/p)'$ is a D_0-ring, as is well known.

First, assume $\mathfrak{T}$ is an H_i-ring. $R \cong \mathfrak{T}/X\mathfrak{T}$ and height $X\mathfrak{T} = 1$, so, by (2.11), R is an H_{i-1}-ring. Let P be a height i prime ideal in R. Then height $P\mathfrak{T} = i$; therefore, since $\mathfrak{T}$ is an H_i-ring, depth $P\mathfrak{T} = a + 1 - i$, and clearly depth $P\mathfrak{T} \leqslant$ depth $PR[X] =$ = depth $P + 1$. On the other hand, depth $P\mathfrak{T} =$ height $(M, X)/PR[X] \geqslant$ height $MR[X]/PR[X] + 1 =$ depth $P + 1$. Thus depth $P = a - i$. Hence R is an H_i-ring. Let p be a height $i - 1$ prime ideal in R, so height $p\mathfrak{T} = i - 1$. Since $\mathfrak{T}$ is an H_i-ring, $\mathfrak{T}/p\mathfrak{T}$ $\big(\cong (R/p)[X]_{(M/p, X)}\big)$ is an H_1-ring, by (2.11). Therefore, by the case $i = 1$, $(R/p)'$ is a D_0-ring.

Conversely, let R be H_{i-1} and H_i, and let $(R/p)'$ be a D_0-ring, for every height $i - 1$ prime ideal p in R. Let q be a height i prime ideal in $\mathfrak{T}$, and let $p = q \cap R$. Then, either height $p = i - 1$ or i. If height $p = i$, then $q = p\mathfrak{T}$. Since R is an H_i-ring, depth $p = a - i$. As above, depth $p\mathfrak{T} =$ depth $p + 1$, and hence depth $q = a - i + 1$, as desired. If height $p = i - 1$, then $p\mathfrak{T} \subset q$ and height $q/p\mathfrak{T} = 1$. Since R is an H_{i-1}-ring, depth $p = a - i + 1$. Again, depth $p\mathfrak{T} =$ depth $p + 1 =$ $= a - i + 2$. Now $(R/p)'$ is a D_0-ring and R is an H_i-ring, by hypothesis, so R/p is an H_1-ring, by (2.11). Thus, by the case $i = 1$, $\mathfrak{T}/p\mathfrak{T}$ is an H_1-ring, and so depth $q =$ depth $q/p\mathfrak{T} =$ altitude $\mathfrak{T}/p\mathfrak{T} - 1 =$ depth $p\mathfrak{T} - 1 = a - i + 1$. Hence $\mathfrak{T}$ is an H_i-ring, q.e.d.

(3.3) EXAMPLE. – *For each $i > 0$, there exists a local domain R such that $\mathfrak{T}$ is an H_j-ring id and only if $j \neq 1, \ldots, i$;* namely, [7, Example 2, pp. 203-205] (for $r > 0$ and $m = i - 1$). To see this, use the facts from (2.3) and also that R' is a special extension of R (see [15, Theorem 4.7 and Remark 4.8]).

The next result is an immediate corollary to (3.2).

(3.4) COROLLARY. – *Let i and j be fixed $(i, j > 0)$, and assume that $\mathfrak{T}_j$ is an H_i-ring. Then the following statements hold:*

(1) *R is $H_i, \ldots, H_{i-j}$.*

(2) *$(R/p)'$ is a D_0-ring, for every prime ideal p in R such that $i - j \leqslant$* height *$p \leqslant i - 1$.*

(3) *For all k $(1 \leqslant k \leqslant j)$, $\mathfrak{T}_k$ is $H_i, \ldots, H_{i-j+k}$.*

PROOF. – Since $\mathfrak{T}_1(\mathfrak{T}_k) \cong \mathfrak{T}_{k+1}$, it follows from (3.2), by induction on k, that (3) holds. Then (1) and (2) follow from (3) $(k = 1)$, by (3.2), q.e.d.

(3.5) EXAMPLES. – (1) *If R satisfies the s.c.c.* (for example, if R is a complete local domain [7, Theorem 34.4, p. 124]), *then $\mathfrak{T}_j$ is an H_i-ring, for all $i \geqslant 0$ and $j > 0$.*

(2) *For i and j fixed $(i, j > 0)$, there exists a local domain R such that $\mathfrak{T}_j$ is an H_k-ring if and only if $k \neq 1, \ldots, i + j$.* Using a similar argument as in (3.3), [7, Example 2, pp. 203-205] (for $r > 0$ and $m = i$) provides this example.

4. – Certain algebraic localities of H_i-rings.

This section deals with three different types of algebraic localities and their connection with R and $\mathcal{T}_n(R)$ relative to being H_i-rings. In (4.2) an equivalence is given for R to be an H_i-ring in terms of conditions on the localities $R(c_1/b, \ldots, c_j/b)$ (see below for the definition). Then the condition that all the localities $R[y]_{(M,y)}$, where y is in the quotient field of R, be H_i-rings, gives rise to a number of statements concerning $\mathcal{T}(R)$ in (4.5). These results are followed by two lemmas which are needed for the proof of (4.10) and which give general information on the polynominal rings of R. To conclude this section, k-graded extensions are defined, and then (4.10) relates k-graded extensions of R to $\mathcal{T}_k(R)$.

A number of results concerning the ring $R[c/b]$, where b and c are analytically independent elements in R, are stated below; but first, the definition of analytically independent elements will be given.

(4.1) DEFINITION. – Let (S, N) be a quasi-local ring. The elements $c_0, \ldots, c_n$ in N are *analytically independent in S* (*a.i. in S*) in case the following condition is satisfied: if $F(X_0, \ldots, X_n)$ is a form in $S[X_0, \ldots, X_n]$ of arbitrary degree such that $F(c_0, \ldots, c_n) = 0$, then all the coefficients of F are in N.

[12, pp. 126-128] and [13, Remark 4.4] contain a number of known facts about analytically independent elements that will be used in the following results.

Let $R(c_1/b, \ldots, c_n/b) = R[c_1/b, \ldots, c_n/b]_{MR[c_1/b,\ldots,c_n/b]}$, where $b, c_1, \ldots, c_n$ are a.i. in R. (Note that by [13, Remark 4.4 (i)], $MR[c_1/b, \ldots, c_n/b]$ is a proper ideal, so the definition of $R(c_1/b, \ldots, c_n/b)$ makes sense.) Also, recall from (2.1.3) that K denotes the quotient field of R.

[13, Proposition 4.7] gives statements which are equivalent to « R is an H_1-ring ». Theorem 4.2 below contains statements which are equivalent to « R is an H_i-ring, for some $i \geqslant 2$ » and is an application of (3.1).

(4.2) THEOREM. – *Let i be fixed ($i \geqslant 2$). Then the following statements are equivalent:*

(1) *R is an H_i-ring.*

(2) *For each pair of elements b, c such that height $(b, c) = 2$, $R(c/b)$ is an H_{i-1}-ring.*

(3(*For each fixed j ($1 \leqslant j \leqslant i-1$) and for each set of elements $b, c_1, \ldots, c_j$ which are a.i. in R, $R(c_1 b, \ldots, c_j/b)$ is an H_{i-j}-ring.*

(4) *For each set of elements $b, c_1, \ldots, c_i$ which are a.i. in R,* altitude *$R(c_1/b, \ldots, c_i/b) = a - i$.*

PROOF. – The equivalence of (1), (2) and (3) will be shown first.

For (1) implies (3), assume $1 \leqslant j \leqslant i-1$ and let $b, c_1, \ldots, c_j$ be an arbitrary set of elements which are a.i. in R. Then $R(X_1, \ldots, X_j)/P \cong R(c_1/b, \ldots, c_j/b)$, where P is a prime ideal in $R(X_1, \ldots, X_j)$ such that $P \cap R = (0)$. It is well known (using [17, Proposition 2, p. 326]) that since $R(X_1, \ldots, X_j)/P$ is algebraic over R and $P \cap R = (0)$, height $P = j$. Since R is an H_i-ring, $R(X_1, \ldots, X_j)$ is an H_i-ring, by (3.1), and therefore $R(c_1/b, \ldots, c_j/b)$ is an H_{i-j}-ring, by (2.11). Hence (1) implies (3).

It will now be shown that (2) implies (1). Let p be a height i prime ideal in R. We can assume $1 < a-1$, by (2.6) (i). Since $i \geqslant 2$, there exists a pair of elements $b, c \in p$ such that height $(b, c) = 2$, by [7, Theorem 9.5, pp. 26-27]. Then $R(c/b)$ is an H_{i-1}-ring, by hypothesis, and altitude $R(c/b) = a-1$, by [12, Lemma 4.3] (b, c are a subset of a system of parameters, as in the first paragraph of this proof). Also, $pR[c/b]$ is a height $i-1$ prime ideal, since height $(b, c) = 2$ implies height $(b/1, c/1) = 2$ in R_p, so $b/1$, $c/1$ are a subset of a system of parameters in R_p (as above), and so $p^* = pR_p[c/b]$ is a height $i-1$ prime ideal and the p^*-residue class of c/b is transcendental over R_p/pR_p [12, Lemma 4.3], and $pR[c/b] = p^* \cap R[c/b]$ [12, Lemma 4.2]. Further $pR[c/b] \subset MR[c/b]$, so $pR(c/b)$ is a height $i-1$ prime ideal. Therefore depth $pR(c/b) = (a-1)-(i-1) = a-i$. Since the $pR_p[c/b]$-residue class of c/b is transcendental over $L = R_p/pR_p$ [13, Remark 4.4 (i)], and since L is isomorphic to the quotient field of R/p [2, Corollary 5.9, p. 57], $R[c/b]/pR[c/b] \cong (R/p)[X]$, where X is an indeterminate. Hence depth p = height M/p = height $(M/p)R/p[X] =$ = height $MR[c/b]/pR[c/b]$ = depth $pR(c/b) = a-i$, and thus (1) holds.

For (3) implies (2), the case $j = 1$ follows from [7, (9.8), p. 27] and [8, Theorems 2 and 3, pp. 64, 68], since if b, c are elements in R such that height $(b, c) = 2$, then b, c are a subset of a system of parameters and so are a.i. in R.

If $1 < j \leqslant i-1$, to prove that (2) holds, it will be shown that if (3) holds for j, then (3) holds for all k $(1 \leqslant k \leqslant j)$, by induction on k. It is clearly true for $k = j$. Assume it holds for $k+1$ $(2 \leqslant k+1 \leqslant j)$, let $b, c_1, \ldots, c_k$ be a.i. in R, and let $C = R(c_1/b, \ldots, c_k/b)$. If altitude $C = 1$, then C is an H_{i-k}-ring (in fact, is catenary). If altitude $C > 1$, then there exist d, e which are a.i. in C. Then d, $e \in K$, and so $e/d \in K$. Therefore, $e/d = s/r$ where $r, s \in R$ and $r \neq 0$. If $r \notin M$, then $s/r \in C$, which contradicts $C \subset C[s/r]$. If $s \notin M$, then $r/s \in MC$, and so $1 = r/s \cdot s/r \in MC[s/r]$, which contradicts $MC[s/r]$ is a proper ideal [13, Remark 4.4 (i)]. Since $C(s/r) =$ $= R(c_1/b, \ldots, c_k/b, s/r) = R(c_1r/br, \ldots, c_kr/br, bs/br)$, it follows from [13, Remark 4.4(i)] that $br, c_1r, \ldots, c_kr, bs$ are a.i. in R. Thus, by the induction hypothesis, $C(s/r)$ is an H_{i-k-1}-ring. Apply the case $j = 1$ to the local domain C, and since (2) implies (1), C is an H_{i-k}-ring. Thus, by induction, the above statement holds; in particular, for $j = 1$. So (2) holds.

The proof that (1) implies (4) is similar to the proof that (1) implies (3), so it will be omitted.

To complete the proof it must be shown that (4) implies (1). Again, we can assume that $i < a-1$, by (2.6) (i). Let p be a prime ideal in R such that height $p = i$. Then there exists $b, c_1, \ldots, c_{i-1} \in p$ such that height $(b, c_1, \ldots, c_{i-1}) = i$ [7, Theorem 9.5, pp. 26-27], and so (as above), $b, c_1, \ldots, c_{i-1}$ are a subset of a system of parameters. Let $C = R(c_1/b, \dots, c_{i-1}/b)$. Then, by [12, Lemma 4.3], altitude $C = a - i + 1$ (> 2). Also, as in the proof that (2) implies (1), $b/1, c_1/1, \dots, c_{i-1}/1$ are a system of parameters in R_p, pC is prime, height $pC =$ height $pR_p - (i-1) = 1$ and $C/pC \cong (R/p)(x_1, \dots, x_{i-1})$, where the x_k are algebraically independent over the quotient field of R/p. Let d, e be elements which are a.i. in C. Then, as in the proof that (3) implies (2), there exist elements $b', c_1, \dots, c_i'$, which are a.i. in R, such that $C(e/d) \cong R(c_1'/b', \dots, c_i'/b')$. Therefore, by (4), altitude $C(e/d) = a - i$, and thus C is an H_1-ring [13, Proposition 4.7]. Hence, since height $pC = 1$, $a - i =$ altitude $C - 1 =$ depth $pC =$ altitude $C/pC =$ altitude $R/p(x_1, \dots, x_{i-1}) =$ depth p and so (1) holds, q.e.d.

Theorem 4.2 has some interesting implications which are stated in (4.3) and (4.4).

(4.3) REMARK. – (i) R is an H_i-ring, for some i ($1 \leqslant i \leqslant a-1$), if and only if every set of $i+1$ elements which are a.i. in R can be extended to a set of a a.i. elements in R.

(ii) If R is an H_i-ring, for some i ($1 \leqslant i \leqslant a-1$), then every set of k elements ($k \leqslant i$) which are a.i. in R can either be extended to a set of a a.i. elements in R, or is contained in a maximal set of at most i elements which are a.i. in R.

PROOF. – (i) follows from (4.2) (4) and [13, Corollary 4.19], and (ii) readily follows from (i), q.e.d.

(4.4) EXAMPLE. – *With R as in* [7, Example 2, pp. 203-205] (*for $r > 0$ and $m = i > 1$*), *there exists, for each $j (1 \leqslant j \leqslant i)$, a set of elements $b, c_1, \dots, c_j$ which are a.i. in R such that $R(c_1/b, \dots, c_j/b)$ is an H_k-ring if and only if $k \neq 1, \dots, i-j$.* By (2.3) and (4.2), $R(c_1/b, \dots, c_j/b)$ is an H_k-ring, for all $k > i - j$ and $k = 0$. The proof is completed by showing that $R\big(y_1/(x^2 - x), \dots, y_j/(x^2 - x)\big)$ is not $H_1, \dots, H_{i-j}$ ($j \leqslant i$), since $(x^2 - x), y_1, \dots, y_j$ are a subset of a system of parameters in R, q.e.d.

Theorem 3.2 stated an equivalence for $\mathfrak{T}$ to be an H_i-ring ($i \geqslant 2$). There are also statements concerning $R[y]_{(M,y)R[y]}$, where $y \in K$, which are equivalent to « $\mathfrak{T}$ is an H_i-ring », and they are stated in (4.5).

(4.5) THEOREM. – *Let i be fixed* ($1 < i \leqslant a + 1$). *Then the following statements are equivalent*:

(1) *$\mathfrak{T}$ is an H_i-ring.*

(2) *For every $y \in K$ such that $1 \notin (M, y)R[y]$, $R[y]_{(M,y)}$ is an H_{i-1}-ring.*

(3) *For each pair of elements b, c which are a.i. in R, $R[c/b]_{(M,c/b)}$ is an H_{i-1}-ring.*

(4) *For each pair of elements b, c in R such that height $(b, c) = 2$, $R[c/b]_{(M,c/b)}$ is an H_{i-1}-ring.*

PROOF. – It is clear that we can assume $a > 2$, since if $a = 2$, then $\mathfrak{T}$ is H_2 and H_3 and $R[y]_{(M,y)}$ is H_1 and H_2, for all $y \in K$ such that $1 \notin (M, y)R[y]$ (altitude $R[y]_{(M,y)} \leqslant 2$), by (2.6) (i). We can also assume $i < a+1$, since $\mathfrak{T}$ is an H_{a+1}-ring and the rings $R[y]_{(M,y)}$ are H_a.

For (1) implies (2), let $y \in K$ such that $1 \notin (M, y)R[y]$ and let $B = R[y]_{(M,y)}$. If $y \in R$, then $B \cong R$ and so is an H_{i-1}-ring, by (3.2). If $y \not\cong R$, then $B \cong \mathfrak{T}/P$ where P is a height one prime ideal in $\mathfrak{T}$ such that $P \cap R = (0)$. So, by (2.11), B is an H_{i-1}-ring, and hence (2) holds.

It is clear that (2) implies (3), by [13, Remark 4.4 (i)], and (3) implies (4), as in the first paragraph of the proof of (4.2), so it only remains to show that (4) implies (1).

For this, it will be shown that if (4) holds, then R is H_{i-1} and H_i and $(R/p)'$ is a D_0-ring, for every height $i-1$ prime ideal p in R, and hence (1) holds, by (3.2). To show that R is an H_{i-1}-ring, let p be a height $i-1$ prime ideal in R. Then there exist b, c elements in R such that $c \in p$, $b \in M$, $b \notin p$ and height $(b, c) = 2$ [7, Theorem 9.5, pp. 26-27]. Let $A = R[c/b]$, let $N = (M, c/b)A$, and let $q = pR[1/b] \cap A$. Since $c \in p$ and $1/b \in R[1/b]$, $c/b \in q$, and so $A/q \cong R/p$, since $q \cap R = p$. Thus $q \subset N$, N is proper, and depth p = depth q = height N/q. Also, height $q = i-1$, since height $pR[1/b] = i-1 =$ height p, and hence height $N/q =$ height $N - i + 1$, since by hypothesis, A_N is an H_{i-1}-ring. Since, as in the first paragraph of the proof of (4.2), b, c are a subset of a system of parameters, height $MA = a-1$ [12, Lemma 4.3]. Also $N \supset MA$, since the MA residue class of c/b is transcendental over R/M [13, Remark 4.4 (i)], so height $N \geqslant a$, but height $N \leqslant a$, since altitude $A \leqslant a$ (since $\mathfrak{T}/P \cong A$ and height $P = 1$). Therefore $a - i + 1 =$ height $N/q =$ depth p. Hence R is an H_{i-1}-ring.

To show that R is an H_i-ring, let p be a height i prime ideal in R. Then there exist elements b, c in R such that $b, c \in p$, height $(b, c) = 2$ and height $pR[c/b] = i-1$ (as in the proof of (4.2) (2) implies (1)). As in the previous paragraph, with $A = R[c/b]$ and $N = (M, c/b)A$, $N \supset MA$ $(\supset pA)$ and height $N = a$. Since A_N is an H_{i-1}-ring, height $N/pA =$ height $N -$ height $pA = a - i + 1$, and so, depth $pA \geqslant a - i + 1$. Therefore, depth $pA = a - i + 1$, since height $pA +$ depth $pA \leqslant$ altitude $A \leqslant a$. Also, as in the proof of (4.2) (2) implies (1), $A/pA \cong R/p[X]$, where X is an indeterminate; thus depth $p + 1 =$ altitude $A/pA =$ depth $pA = a - i + 1$. Hence R is an H_i-ring.

To show that $(R/p)'$ is a D_0-ring, for all height $i-1$ prime ideals p in R, let p be a height $i-1$ prime ideal in R. We must consider two cases. If $i > 2$, then as in the previous paragraph and with its notation, there exist $b, c \in p$ such that height $(b, c) = 2$, height $pA = i-2$, $pA \subset MA \subset N$, and $A/pA \cong R/p[X]$. Thus, with $B = A_N$, $B/pB \cong (R/p[X])_{(M/p,X)}$ $(= \mathfrak{T}(R/p))$. Since B is an H_{i-1}-ring and height $pB = i-2$, B/pB is an H_1-ring, by (2.11), and therefore, by (3.2) $(i=1)$, $(R/p)'$ is a D_0-ring.

It remains to show, for the case $i = 2$, that $(R/p)'$ is a D_0-ring, for every height $i-1$ prime ideal p in R. Let p be a height one prime ideal in R. We have proved R is H_1 and H_2, and hence R/p is an H_1-ring, by (2.11), and altitude $R/p = a-1$. By [15, Remark 3.4], either $(R/p)'$ is a D_0-ring, as desired, or there exists a height one maximal ideal in $(R/p)'$. Assume the latter, and let v be an element in $(R/p)'$ such that v is in every height one maximal ideal in $(R/p)'$, and $1-v$ is in all the other maximal ideals in $(R/p)'$, by [16, Theorem 31, p. 177]. Then $C=(R/p)[v]$ has exactly two maximal ideals, namely, $N_1=(M/p, v)C$ and $N_2=(M/p, 1-v)C$. Height $N_1=$ $=1$ (since $(R/p)'$ is integral over C, there exists a prime ideal Q in $(R/p)'$ such that height $Q=$ height N_1 and $Q\cap C=N_1$ [3, Theorems 44 and 46, pp. 29 and 31]; this implies $v\in Q$, and so height $Q=1$) and height $N_2=a-1$ (since C is integral over R/p, $a-1=$ altitude $R/p=$ altitude C). Since v is in the quotient field of R/p, $v=z/w$ with $w, z\in R/p$ and $w, z\neq 0$. Let b, c be elements in R such that $b+p=w$ and $c+p=z$, so $b, c\notin p$. Now it will be shown that there exists an element $d\in p$ such that $b+d$ is not in any height one prime divisor of cR.

For this, let $P_1, \ldots, P_n$ be the height one prime divisors of cR, and assume that $b\in P_1, \ldots, P_m$ and $b\notin P_{m+1}, \ldots, P_n$, where $1\leqslant m\leqslant n$. $\Big($If $b\notin\bigcap_1^n P_i$, then let $d=0$, and so we can assume $m\geqslant 1$; if $m=n$, then ignore all the following expressions which involve $m+1.\Big)$ Pick $d\in p$ such that $d\notin\bigcup_{j=1}^m P_j$ and $d\in\bigcap_{i=m}^n P_i$. $\Big($Suppose $p\cap\bigcap_{i=m}^n P_i\subseteq$ $\subseteq\bigcup_{j=1}^m P_j$. Then there exists some P_j $(1\leqslant j\leqslant m)$ such that either $p\subseteq P_j$ or some $P_k\subseteq P_j$ $(m+1\leqslant k\leqslant n)$, by [1, Proposition 1.11, p. 8]. Since $k\neq j$, $P_k\nsubseteq P_j$, therefore $p\subseteq P_j$. Since height $p=1=$ height P_j, $p=P_j$. But $c\notin p$ and $c\in P_j$; contradiction. Thus $p\cap\bigcap_{i=m}^n P_i\nsubseteq\bigcup_{j=1}^n P_j$, so there exists such an element $d\Big)$. It follows from the choice of d that $b+d$ is not in any of the P_i $(i=1, \ldots, n)$. Therefore height $(c, b+d)=2$, and $b+p=w=b+d+p$ (since $d\in p$).

Let $A=R[c/(b+d)]$, $N=\big(M, c/(b+d)\big)A$ and $B=A_N$. Then, by (4) $(i=2)$, B is an H_1-ring. Let $q=pR[1/b+d]\cap A$, thus height $q=1$. Also, $A/q\cong C$, so $N/q\cong N_1$. Hence $q\subset N$, and since B is an H_1-ring, height $N/q=a-1$. Since $B/qB\cong C_{N_1}$, $1=$ height $N_1=$ height $N/q=a-1$. This contradicts the fact that $a>2$. Hence there does not exist a height one maximal ideal in $(R/p)'$, so $(R/p)'$ is a D_0-ring. Thus (1) holds, q.e.d.

The following definition and three lemmas will be used in the proof of Theorem 4.10.

(4.6) LEMMA. – *Let (S, N) be a quasi-local integral extension domain of R. Then $\mathfrak{T}_n(S)$ is a quasi-local integral extension domain of $\mathfrak{T}_n(R)$.*

PROOF. – Since $\mathfrak{T}_2(S)=\mathfrak{T}_1\big(\mathfrak{T}_1(S)\big)$, it is clearly sufficient to prove the lemma for the case $n=1$. It follows from [2, Theorem 10.7, p. 96] that $S[X]$ is an integral domain which is integral over $R[X]$. By [7, (10.6), pp. 29-30], $S[X]_{R[X]-(M,X)R[X]}$ is integral over $\mathfrak{T}$ and, as is easily seen, $(N, X)S[X]$ is the only prime ideal in $S[X]$ which lies over $(M, X)R[X]$. Therefore $S[X]_{R[X]-(M,X)R[X]}=\mathfrak{T}(S)$, q.e.d.

Recall from the second paragraph of section three that $S_n=S[X_1, \ldots, X_n]$, where S is a ring and $X_1, \ldots, X_n$ are indeterminates over S.

(4.7) LEMMA. – *Let S be a Noetherian integral domain such that* altitude $S<\infty$, *and let k be fixed* ($k \geqslant 1$). *Then, for every prime ideal q in S_k such that height $q > k$, there exists a prime ideal Q in S_k such that $Q \subset q$, $Q \cap S = (0)$,* height $Q = k$ *and* height $q/Q =$ height $q - k$.

PROOF. – The proof will be by induction on k. Let q be a prime ideal in S_k such that height $q = k + i$ $(i > 0)$.

For the case $k = 1$, let $p = q \cap R$. Then either $pS_1 \subset q$ or $pS_1 = q$. If $pS_1 \subset q$, then there exists an element $f \in q$ such that $f \notin pS_1$. Since $(S_1)_q$ is a local domain and f is a parameter, there exists a height one prime divisor Q of fS_1 such that $Q \subset q$ and height $q/Q = i$, by [7, (9.2) and (9.7), pp. 26 and 27]. Suppose $Q \cap S \neq (0)$. Then height $Q \cap S = 1 =$ height Q, and so $Q = (Q \cap S)S_1 \subseteq pS_1$. But this contradicts the fact that $f \notin pS_1$, so $Q \cap S = (0)$.

If $pS_1 = q$, then height $p = i + 1 \geqslant 2$. Hence there exists a pair of elements $b, c \in p$ such that height $(b, c) = 2$, by [7, Theorem 9.5, pp. 26-27]. Let $I = (bX + c)S_1$, and let Q be a minimal prime divisor of I contained in q such that height $q/Q = i$ (as in the previous paragraph). Suppose $Q \cap S \neq (0)$. Then $(Q \cap S)S_1 = Q$, and so $b, c \in Q \cap S$ (it is well known that $g \in JS_1$, J an ideal in S, if and only if all the coefficients of g are in J). This contradicts the fact that height $(b, c) = 2$ and height $(Q \cap S) = 1$. So in both cases the prime ideal Q satisfies the conditions in the lemma.

Assume the lemma holds for $k - 1$ $(k \geqslant 2)$, and let $p = q \cap S_{k-1}$. Then either height $q - 1 =$ height p $(pS_k \subset q)$ or height $q =$ height p $(pS_k = q)$. If height $p =$ height $q - 1 = k + i - 1$, then by the induction hypothesis, there exists a prime ideal P in S_{k-1} such that $P \subset p$, $P \cap S = (0)$, height $P = k - 1$ and height $p/P = k + i - 1 - (k - 1) = i$. It follows from [9, (5.4.6), p. 262] and [3, Theorem 149, pp. 108-109] that $PS_k \subset pS_k \subset q$, height $PS_k = k - 1$ and height $pS_k/PS_k = i$, and so height $q/PS_k \geqslant i + 1$. Hence height $q/PS_k = i + 1$, since height $q = k + i$ and height $PS_k = k - 1$. Thus, by the case $k = 1$ for S_{k-1}/P and $(S_{k-1}/P)[X_k] \cong$ (by [9, (5.4.6), p. 262]) S_k/PS_k, there exists a prime ideal Q in S_k such that $PS_k \subset Q \subset q$, $(Q/PS_k) \cap (S_{k-1}/P) = P/P$ (so $Q \cap S_{k-1} = P$), height $Q/PS_k = 1$ and height $(q/PS_k)/(Q/PS_k)$ ($=$ height q/Q) $= i$. Since height $PS_k = k - 1$ and height $Q/PS_k = 1$, height $Q \geqslant k$. Hence height $Q = k$, since height $q = k + i$ and height $q/Q = i$. Also, $Q \cap S = (Q \cap S_{k-1}) \cap S = P \cap S = (0)$. So Q satisfies the conditions in the lemma.

The case where height $p =$ height $q = k + i$ $(pS_k = q)$ remains. By the induction hypothesis, there exists a prime ideal P in S_{k-1} such that $P \subset p$, $P \cap S = (0)$, height $P = k - 1$ and height $p/P = i + 1$. Hence $PS_k \subset pS_k = q$ and height $q/PS_k =$ height $p/P = i + 1$. Then, by considering $(S_{k-1}/P)[X_k] \cong S_k/PS_k$, the second case in the case $k = 1$ shows there exists a prime ideal Q in S_k such that the conditions in the lemma are satisfied, q.e.d.

The following definition is needed for the statement of (4.10).

(4.8) DEFINITION. – Let k be a positive integer. A local domain T is called a *k-graded extension* of R if $T = S[u_1, \ldots, u_k]_{(N, u_1, \ldots, u_k)}$, where (S, N) is a local domain generated by k elements which are integral over R and $u_1, \ldots, u_k$ are in the quotient field of S such that $(N, u_1, \ldots, u_k) S[u_1, \ldots, u_k]$ is a proper ideal.

(4.9) LEMMA. – *Let $T = R[u_1, \ldots, u_k]_{(M, u_1, \ldots, u_k)}$, where $u_1, \ldots, u_k$ are algebraic over R such that $(M, u_1, \ldots, u_k)$ is a proper ideal. Then T is a k-graded extension of R.*

PROOF. – By [2, Lemma 9.1, p. 84], there exist non-zero elements $r_1, \ldots, r_k$ in R such that $r_1 u_1, \ldots, r_k u_k$ are integral over R. Let $r = r_1 \ldots r_k$, so $ru_1, \ldots, ru_k$ are integral over R. We assume $r \in M$ and $r \neq 0$, since, for $0 \neq m \in M$, $mr \in M$ and $mru_1, \ldots, mru_k$ are integral over R. Then $S = R[r^2 u_1, \ldots, r^2 u_k]$ is a local domain with the only maximal ideal being $(M, r^2 u_1, \ldots, r^2 u_k)S$, and $r^2 u_1, \ldots, r^2 u_k$ are integral over R. Also, $u_1, \ldots, u_k$ are in the quotient field of S, and hence T is a k-graded extension of R, q.e.d.

Theorem 4.10 is an extension of (4.5) to an equivalence of $\mathcal{T}_k$ being an H_{k+i}-ring.

(4.10) THEOREM. – *Let i and k be fixed $(i, k \geqslant 1)$. $\mathcal{T}_k$ is an H_{k+i}-ring if and only if every k-graded extension of R is an H_i-ring.*

PROOF. – Assume $\mathcal{T}_k$ is an H_{k+i}-ring, and let T be a k-graded extension of R, say $T = S[u_1, \ldots, u_k]_{(N, u_1, \ldots, u_k)}$ with (S, N) and $u_1, \ldots, u_k$ as in (4.8). By (4.6), $\mathcal{T}_k(S)$ is a quasi-local integral extension domain of $\mathcal{T}_k(R)$, and hence $\mathcal{T}_k(S)$ is an H_{k+i}-ring, by [13, Corollary 2.16]. Essentially, the same proof as in the proof of (4.2) (1) implies (3) shows that $T \cong \mathcal{T}_k(S)/Q$, where Q is a prime ideal of $\mathcal{T}_k(S)$ such that $Q \cap R = (0)$ and height $Q = k(u_1, \ldots, u_k)$ are algebraic over S. Thus, by (2.11), T is an H_i-ring, and so every k-graded extention of R is an H_i-ring.

The converse will be proved by induction on k. For the case $k = 1$, assume every 1-graded extension of R is an H_i-ring. Then, in particular, for each y in the quotient field of R such that $1 \notin (M, y)R[y]$, $R[y]_{(M,y)}$ is an H_i-ring, and hence $\mathcal{T}$ is an H_{i+1}-ring, by (4.5).

Now let $k > 1$ and assume the conclusion holds for $k - 1$. Also, assume every k-graded extension of R is an H_i-ring. To prove that $\mathcal{T}_k$ is an H_{k+i}-ring it suffices, by (3.2), to show that $\mathcal{T}_{k-1}$ is H_{k+i-1} and H_{k+i} and that $(\mathcal{T}_{k-1}/Q)'$ is a D_0-ring, for every prime ideal Q in $\mathcal{T}_{k-1}$ such that height $Q = k + i - 1$. Let T be a $k-1-$ graded extension of R with maximal ideal N. Then, for all u in the quotient field of T such that $(N, u)T[u]$ is a proper ideal, $T[u]_{(N,u)}$ is a k-graded extension of R, by the definition (4.8), and hence $T[u]_{(N,u)}$ is an H_i-ring. Therefore, since T is a local domain, $\mathcal{T}(T)$ is an H_{i+1}-ring, by (4.5). Thus, by (3.2), T is H_i and H_{i+1} and $(T/q)'$ is a D_0-ring, for every height i prime ideal q in T. Thus, since T is an arbitrary $k-1$-graded extension of R, the induction hypothesis implies that $\mathcal{T}_{k-1}$ is H_{k-1+i} and H_{k+i}: So it only remains to show that $(\mathcal{T}_{k-1}/Q)'$ is a D_0-ring, for every height $k + i - 1$ prime ideal Q in $\mathcal{T}_{p-1}$.

For this, let Q be a height $k + i - 1$ prime ideal in $\mathcal{T}_{k-1}$. It follows from (4.7) that there exists a prime ideal P in $\mathfrak{I}_{k-1}$ such that $P \subset Q$, $P \cap R = (0)$, height $P = k-1$ and height $Q/P = i$. Let $P' = P \cap R_{k-1}$, and let $U = (R_{k-1})_{(R-(0))}$. Then $P' \cap R = (0)$ and height $P' = k-1$, and thus height $P'U = k-1$. Since $U \cong K[X_1, ..., X_{k-1}]$ [9, (5.4.7), p. 262], altitude $U = k-1$, and hence $P'U$ is a maximal ideal of U. Let N be the maximal ideal in $K[X_1, ..., X_{k-1}]$ corresponding to $P'U$. So, $U/P'U$ is a field and is isomorphic to $K[u_1, ..., u_{k-1}]$, where $u_j = X_j + N$. By [17, Lemma, p. 165], $u_1, ..., u_{k-1}$ are algebraic over K and hence are algebraic over R. Also, it follows from [9, Proposition 9 and (5.4.7), pp. 153, 262] that $R_{k-1}/P' \cong R[u_1, ..., u_{k-1}]$, and thus $\mathfrak{I}_{k-1}/P \cong R[u_1, ..., u_{k-1}]_{(M, P_1, ..., P_{k-1})}$, which is a $(k-1)$-graded extension of R, by (4.9). Therefore, since height $Q/P = i$, $(\mathfrak{I}_{k-1}/Q)' \cong \big((\mathfrak{I}_{k-1}/P)/(Q/P)\big)'$ is a D_0-ring, by the statement in the previous paragraph, q.e.d.

5. – Conditions for R^* and R^H to be H_i-rings.

The main objective of this section is to give necessary and sufficient conditions for the completion R^* (respectively, the Henselization R^H) of R to be an H_i-ring. (5.1) and (5.2) give equivalences for R^* to be an H_1-ring and an H_0-ring, respectively. To close the section, (5.3) shows that certain conditions on R are equivalent to « R^H is an H_i-ring ».

It is known that R^* is a local ring [8, p. 92], that the theorem of transition holds for R and R^* [7, Corollary 17.11 and Theorem 19.1, pp. 57, 64-65], and that altitude $R^* = a$ [7, (17.12), p. 57].

Theorem 5.1 gives an equivalence of « R is catenary and R/p satisfies the s.c.c., for every height one prime ideal p in R ».

(5.1) THEOREM. – *R^* is an H_1-ring if and only if R is catenary and R/p satisfies the s.c.c., for every height one prime ideal p in R.*

PROOF. – First, assume that R is catenary and R/p satisfies the s.c.c., for every height one prime ideal p in R, and let p^* be a height one prime ideal in R^*. Then there exists a minimal prime ideal q^* in R^* such that $q^* \subset p^*$ and height $p^*/q^* = 1$, and it follows from [11, Proposition 2.16 (2)] $(i = 1)$ that either depth $q^* = 1$ or a. Since R^* is a local ring and $a \geqslant 2$, depth $q^* = a$. By [8, Proposition 4, p. 86], R^*/q^* is a complete local domain, and hence, by [7, Theorem 34.4, p. 124], R^*/q^* satisfies the f.c.c. Since height $p^*/q^* = 1$, depth $p^* =$ depth $p^*/q^* =$ altitude $R^*/q^* - 1 = a - 1$, and thus R^* is an H_1-ring.

Conversely, assume R^* is an H_1-ring. It will now be shown that, for every minimal prime ideal p^* in R^*, either depth $p = 1$ or a.

For this, let p^* be a minimal prime ideal in R^* (p^* is not a maximal ideal since R^* is local). Assume depth $p^* > 1$. Then there exists a prime ideal P^* in R^* such that $p^* \subset P^*$, height $P^*/p^* = 1$ and depth $p^* =$ depth $P^* + 1$ (so depth $P^* > 0$). Thus height $P^* = 1$ or height $P^* > 1$. In either case $\big($height $P^* > 1$, by (2.8)$\big)$, there exists a height one prime ideal q^* in R^* such that $p^* \subset q^*$, height $q^*/p^* = 1$, and depth $q^* =$ depth $p^* - 1$. Since, by assumption, R^* is an H_1-ring, depth $q^* = a - 1$, and hence depth $p^* = a$. Thus, for every minimal prime ideal p^* in R^*, either depth $p^* = 1$ or a.

It will now be shown that R is catenary. Let i be fixed $(0 < i < a - 1)$, let P be a height i prime ideal in R, and let P^* be a minimal prime divisor of PR^* in R^*. Then it follows from [7, Theorem 22.9, p. 75] (since the theorem of transition holds for R and R^*) that height $P^* =$ height $P = i$. Let p^* be a minimal prime ideal in R^* such that $p^* \subset P^*$ and height $P^*/p^* = i$. Thus depth $p^* > 1$, and so, by the previous paragraph, depth $p^* = a$. Hence $a - i =$ altitude $R^*/p^* -$ height $P^*/p^* =$ $=$ (since R^*/p^* satisfies the f.c.c.) depth $P^*/p^* =$ depth $P^* \leqslant$ altitude $R^*/PR^* =$ $=$ (by [7, (17.12) and Corollary 17.9, p. 57]) altitude $R/P =$ depth $P \leqslant a - i$. So R is an H_i-ring, for all i $(0 < i < a - 1)$, and hence R is catenary, by (2.9).

Finally, it will be shown that R/p satisfies the s.c.c., for every height one prime ideal p in R. Let p be a height one prime ideal in R. Then, by the same argument as in the previous paragraph (for $i = 1$), for every minimal prime divisor p^* of pR^*, depth $p^* = a - 1 =$ depth $p =$ altitude R/p. Therefore, since R^*/pR^* is the completion of R/p, [7, Corollary 17.9, p. 57], [11, Theorem 3.1] and the definition of quasi-unmixed [7, p. 124] imply that R/p satisfies the s.c.c., q.e.d.

Theorem 5.2 adds five more equivalences to « R satisfies the s.c.c. » to those in [12, Theorem 2.21]. More equivalent statements are in a similar theorem (7.6), using the concept of D_i-rings.

(5.2) THEOREM. – *The following statements are equivalent:*

(1) *R satisfies the s.c.c.*

(2) *R^* is an H_0-ring.*

(3) *R^* is an H_i-ring, for all i.*

(4) *$R^{*\prime}$ is an H_1-ring.*

(5) *$R^{*\prime}$ is an H_i-ring, for all i.*

(6) *$R^{*\prime}$ satisfies the f.c.c.*

PROOF. – It will first be shown that (1), (2) and (3) are equivalent. If R satisfies the s.c.c., then R^* satisfies the f.c.c., by [12, Theorem 2.21], and thus (3) holds, by [13, Remark 2.22 (i)]. It is clear that (3) implies (2), and it follows from [11, Theorem 3.1] and the definition of quasiunmixed [7, p. 124] that (2) implies (1).

The equivalence of (1), (4), (5) and (6) will now be shown. For (1) implies (6), let q be a minimal prime ideal in $R^{*\prime}$, and let $p = q \cap R^*$. Then $R^{*\prime}/q$ is integral over R^*/p, by [1, Proposition 5.6, p. 61], and since R^* and $R^{*\prime}$ have the same total quotient ring, p is a minimal prime ideal in R^*. Since R satisfies the s.c.c., R^* is a

H_0-ring ((1) implies (2)), and so altitude $R^* = a =$ depth $p =$ altitude $R^*/p =$ $=$ altitude $R^{*\prime}/q =$ depth q. By [11, Remark 2.6 (ii)] and [12, Theorem 2.21], R^*/p satisfies the s.c.c., and hence, by definition (2.7.4), $R^{*\prime}/q$ satisfies the f.c.c. Therefore $R^{*\prime}$ satisfies the f.c.c., by [13, Remark 2.23 (ii)], that is, (6) holds.

(6) implies (5), by [13, Remark 2.22 (i)], and it is clear that (5) implies (4). So it remains to show that (4) implies (1).

Assume (4) holds. Since $R^{*\prime}$ is integral over R^*, it is easy to show, using [3, Theorems 44, 46-48, pp. 29, 31 and 32], that R^* is an H_1-ring. Hence, by (5.1), R is catenary and R/p satisfies the s.c.c., for every height one prime ideal p in R. Since $R^{*\prime}$ is an H_1-ring, there does not exist a height one maximal ideal in R', by [11, Proposition 3.5]. Thus, by [11, Theorem 3.1 and Proposition 3.3], R satisfies the s.c.c., and so (1) holds, q.e.d.

It follows from the definition of R^H that R^H/PR^H is the Henselization of R/P, for every prime ideal P in R, and, by [7, Theorem 19.1 and 43.8, pp. 64-65 and 182], the theorem of transition holds for R and R^H.

Theorem 5.3 states an equivalence for « R^H is an H_i-ring » and, using (3.2), shows that if $\mathfrak{F}$ is an H_{i+1}-ring $(0 \leqslant i \leqslant a)$, then R^H is an H_i-ring.

(5.3) THEOREM. – *Let i be fixed $(0 \leqslant i \leqslant a)$. R^H is an H_i-ring if and only if R is an H_i-ring and $(R/p)'$ is a D_0-ring, for every height i prime ideal p in R.*

PROOF. – Assume R^H is an H_i-ring, and let p be a height i prime ideal in R. Since R and R^H satisfy the theorem of transition, it follows from [7, Theorem 22.9, p. 75] that height $q = i$, for every minimal prime divisor q of pR^H, and it follows from [7, Theorem 43.20, p. 187] that every prime divisor of pR^H is a minimal prime divisor. Hence if q is a prime divisor of pR^H, then depth $q = a - i$, since R^H is an H_i-ring. By [7, Theorem 43.20 and Exercise 2, pp. 187, 188], there is a one-to one correspondence between maximal ideals of $(R/p)'$ and prime divisors of pR^H, and if M' corresponds to q, then $(R^H/q)'$ is the Henselization of $(R/p)_{M'}$. Thus $a - i =$ depth $q =$ altitude $R^H/q =$ altitude $(R^H/q)' =$ (by [7, Theorem 22.9, p. 75]) altitude $(R/p)'_{M'} =$ $=$ height M'. Hence the heights of the maximal ideals of $(R/p)'$ are the same, and so $(R/p)'$ is a D_0-ring. Also, $a - i =$ altitude $(R/p)' =$ altitude $R/p =$ depth p, and thus R is an H_i-ring.

Conversely, assume R is an H_i-ring and $(R/p)'$ is a D_0-ring, for every height i prime ideal p in R. Let q be a height i prime ideal in R^H, and let $p = q \cap R$ Then, since $R^H/pR^H = (R/p)^H$, it follows from [7, Theorems 43.20 and 22.9, pp. 187 and 75] that q is a minimal prime divisor of pR^H and height $p = i$. Thus depth $q =$ (as in the previous paragraph) height M' (where M' is the maximal ideal of $(R/p)'$ associated with q) $=$ (since $(R/p)'$ is a D_0-ring) altitude $(R/p)' =$ depth $p =$ $=$ (since R is an H_i-ring) $a - i$. Hence R^H is an H_i-ring, q.e.d.

6. – Conditions for certain sets of localities to consist of H_i-rings.

In this section certain sets of localities over R are discussed relative to the condition that every ring in such a set is an H_i-ring. (6.1) shows that every locality over R is an H_1-ring (or is an H_i-ring, for some fixed i $(0 < i < a)$) if and only if R satisfies the s.c.c. (6.2) uses a set $\mathfrak{B}$ of localities contained in the quotient field of R to get an equivalence of « R is catenary and R' satisfies the c.c. »

Theorem 6.1 is an extension of [10, Corollaries 2.5 and 2.8] and lists a number of statements equivalent to « R satisfies the s.c.c. » Other equivalent statements are in (5.2).

(6.1) THEOREM. – *The following statements are equivalent:*

(1) *R satisfies the s.c.c.*

(2) *For each fixed i $(0 < i < a)$, every locality S over R which dominates R is an H_i-ring.*

(3) *Every S, as in* (2), *is catenary.*

(4) *Every S, as in* (2), *satisfies the s.c.c.*

PROOF. – It follows from [11, Theorem 3.1] and [10, Corollary 2.8] that (1) implies (4). By [11, Remark 2.7], (4) implies (3), and by (2.9), (3) implies (2).

For (2) implies (1), we first show that if (2) holds for any fixed i $(0 < i < a)$, then (2) holds for $i = 1$. Then it is proved that (2) $(i = 1)$ implies (1). Let S be a locality over R which dominates R, let N be the maximal ideal of S, and let $L = \mathfrak{T}_{i-1}(S)$. Then $S \cong L/(X_1, \dots, X_{i-1})L$. It follows from the definition of locality that L is a locality over R which dominates R (since S is such a locality), and thus, by (2), is an H_i-ring. Since height $(X_1, \dots, X_{i-1})L = i - 1$, S is an H_1-ring, by (2.11).

To show that (2) $(i = 1)$ implies (1), again let S be a locality over R which dominates R. By the definition of locality, $S = A_Q$, where Q is a prime ideal in a finitely generated integral domain A over R such that $Q \cap R = M$. Let T be a locality over S which dominates S. Then, as is readily seen, T is a locality over R which dominates R. Therefore S is an H_1-ring and every locality over S is an H_1-ring. Hence, since $S(c_1/b, \dots, c_j/b)$ is a locality over S, where $b, c_1, \dots, c_j$ are a.i. in S, for all j $(1 \leqslant j \leqslant \text{altitude } S - 1)$, S is catenary, by (2.9) and (4.2). In particular, since $\mathfrak{T}$ is a locality over R which dominates R, $\mathfrak{T}$ is catenary. Thus, by [12, Theorem 2.21], R satisfies the s.c.c. that is, (1) holds, q.e.d.

Theorem 6.2 gives an equivalence to « R is catenary and R' satisfies the c.c. ». Some other equivalences are stated in (4.10) (for $k = a - 2$ and $i = 1$) and (5.1).

It should be noted that some of the localities in $\mathfrak{B}$ (defined below in (6.2)) are k-graded extensions of R, namely, those $R[u_1, \dots, u_k]_Q$, where $Q = (M, u_1, \dots, u_k) \cdot R[u_1, \dots, u_k]$ is a proper ideal $(u_1, \dots, u_k \in K)$.

(6.2) THEOREM. – *Let $\mathfrak{B}$ be the set of localities, B, over R such that $B \subseteq K$, $B = R[u_1, \dots, u_n]_Q$, where $0 \leqslant n \leqslant a - 2$ and Q is a prime ideal in $R[u_1, \dots, u_n]$ such that $Q \cap R = M$. Every $B \in \mathfrak{B}$ is an H_1-ring if and only if R is catenary and R' satisfies the c.c.*

PROOF. – Assume R is catenary and R' satisfies the c.c. Since, for all n $(0 \leqslant n \leqslant a-2)$ and every set of elements $u_1, \ldots, u_n \in K$, $R[u_1, \ldots, u_n]$ is a finitely generated R-algebra and is an integral domain, every $B \in \mathfrak{B}$ is catenary, by [15, Theorem 4.3] and [11, Remark 2.6 (ii)]. Hence every $B \in \mathfrak{B}$ is an H_1-ring.

To prove the converse it will be shown, using (2.9) and (4.2), that $R[y]_{(M,y)}$ is catenary, for every $y \in K$ such that $1 \notin (M, y)R[y]$. Let $B = R[y]_{(M,y)}$, for some $y \in K$ such that $1 \notin (M, y)R[y]$.

Since $B \in \mathfrak{B}$, B is an H_1-ring, and thus if altitude $B \leqslant 3$, B is catenary, by (2.8) (i) and (ii). So we can assume altitude $B > 3$. By definition, $B(c_1/b, \ldots, c_j/b)$ is a locality over B, for all j $(1 \leqslant j \leqslant \text{altitude } B - 3 \leqslant a - 3)$ and for each set of elements $b, c_1, \ldots, c_j$ which are a.i. in B. Thus, as in the proof of (6.1) (2) implies (1), every such $B(c_1/b, \ldots, c_j/b)$ is a locality over R and is in $\mathfrak{B}$, and hence, by hypothesis, is an H_1-ring. Therefore B is caternary, by (4.2) (3) implies (1) and (2.9).

Since all such rings B are catenary, $\mathfrak{T}(R)$ is $H_2, \ldots, H_{a+1}$, by (4.5) and (2.9). It follows from (2.11) that, if p is a height one prime ideal in R, $\mathfrak{T}(R/p) \cong \mathfrak{T}/p\mathfrak{T}$ is catenary, and thus R is catenary and R/p satisfies the s.c.c., for every height one prime ideal p in R, by [12, Theorem 2.21]. Hence R is catenary and R' satisfies the c.c., by [15, Theorem 4.3], q.e.d.

7. – D_i-rings.

In Sections 2-6 we analyzed H_i-rings. In this section we consider the « dual » concept of D_i-rings and state some results « dual » to those in the previous sections; in particular, to (2.11), (3.1), (3.2) and (5.2). (7.1) considers the localizations R_p; (7.2) and (7.3) examine the rings $R(X_1, \ldots, X_n)$; (7.4) looks at the ring $\mathfrak{T}$; and (7.6) deals with the completion of R.

The following theorem is a « dual » to (2.10) and therefore is part of a « dual » to (2.11).

(7.1) THEOREM. – *Let S be a ring such that* altitude $S = a < \infty$, *and let j be fixed* $(0 \leqslant j \leqslant a)$. *Assume that S is an H_0-ring. Then the following statement holds for all k* $(0 \leqslant k \leqslant j)$: *If, for every depth $j-k$ prime ideal p in S, S_p is a D_k-ring and either* height $p \leqslant k$ *or* height $p = a - j + k$, *then S is a D_j-ring.*

PROOF. – We can assume $0 < j < a - 1$, by (2.6).

Let q be a depth j prime ideal in S. Then there exists a prime ideal p in S such that $q \subset p$, height $p/q = j - k$, and depth $p = k$. By assumption, S_p is a D_{j-k}-ring and either height p + depth $p = a$ or height p + depth $p \leqslant j$. But height p + + depth $p \geqslant$ height q + height $p/q + k \geqslant$ (since $j < a$ and S is an H_0-ring) $1 + (j-k) + k = j + 1$; thus height p + depth $p = a$, that is, height $p = a - k$. Since S_p is a D_{j-k}-ring and depth qS_p = height $p/q = j - k$, height q = height qS_p = altitude $S_p - (j-k) = a - k - j + k = a - j$. So S is a D_j-ring, q.e.d.

Theorem 7.2 states part of a « dual » to (3.1).

(7.2) THEOREM. – *If $R(X_1, \ldots, X_n)$ is a D_i-ring, for some $n \geq 1$, then R is a D_i-ring.*

PROOF. – It suffices to show that if $R(X)$ is a D_i-ring, then R is a D_i-ring, since $R(X_1, \ldots, X_n) \cong R(X_1, \ldots, X_{n-1})(X_n)$. Since altitude $R(X) = a$, R and $R(X)$ are D_0, D_{a-1}, and D_a, by (2.6) (i) and (ii), so we can assume $0 < i < a - 1$.

Let p be a depth i prime ideal in R. As in (3.1), $i =$ depth $p =$ altitude $R/p =$ $=$ altitude $R(X)/pR(X) =$ depth $pR(X)$. Since $R(X)$ is a D_i-ring, $a - i =$ height $pR(X) =$ height p, and hence R is a D_i-ring, q.e.d.

(7.3) REMARK. – The converse of (7.2) holds if $i = 1$, by [13, Corollary 2.4 (2)], (2.9) and (3.1).

A portion of a « dual » to (3.2) is given in Theorem 7.4.

(7.4) THEOREM. – *If $\mathfrak{T}$ is a D_i-ring, then R is a D_{i-1}-ring.*

PROOF. – We can assume $1 < i < a$, as before. Let p be a depth $i - 1$ prime ideal in R. As in (3.2), depth $p\mathfrak{T} =$ depth $p + 1 = i$, and thus, since $\mathfrak{T}$ is a D_i-ring, height $p =$ height $p\mathfrak{T} =$ altitude $\mathfrak{T} - i = a + 1 - i = a -$ depth p. Hence R is a D_{i-1}-ring, q.e.d.

(7.5) EXAMPLE. – *For all $i > 0$, there exists a local domain R such that $\mathfrak{T}$ is a D_j-ring if and only if $j \neq 1, \ldots, i + 1$*; namely [7, Example 2, pp. 203-205] (for $r > 0$ and $m = i$). This follows from the facts in (2.3), (2.5) and (3.3), and since $\mathfrak{T}'$ is a special extension of $\mathfrak{T}$ (see [15, Remark 4.8]).

The following theorem lists equivalent and « dual » statements to those in (5.2). Also, the notation R^* for the completion of R is used, as in (5.2).

(7.6) THEOREM. – *The following statements are equivalent:*

(1) *R satisfies the s.c.c.*

(2) *R^* is a D_1-ring.*

(3) *R^* is a D_i-ring, for all i.*

(4) *$R^{*\prime}$ is a D_1-ring.*

(5) *$R^{*\prime}$ is a D_i-ring, for all i.*

PROOF. – (1), (3) and (5) are equivalent, by (5.2) and (2.9). Clearly, (5) implies (4). (4) implies (2) by a straightforward argument using [3, Theorems 44 and 46-48, pp. 29, 31 and 32]. (2) implies (3) is proved by showing that if a local ring S is a D_i-ring ($i > 0$) then S is a D_{i+1}-ring. This is accomplished by applying (2.8) to S, where p is a depth $i + 1$ prime ideal and q is a depth i prime ideal in S, q.e.d.

8. – Open problems.

This section is primarily a list of the open problems related to H_i- and D_i-rings. (8.1)-(8.3) are the chain conjectures. In (8.4) and (8.5) some equivalences of the chain conjectures are given, and then (8.6)-(8.11) state a number of questions about H_i- and D_i-rings which arise from the work in this paper.

(8.1)-(8.3) give the statements of the three main chain conjectures. They are contained in [5], [6] and [15] along with a number of equivalent statements.

(8.1) CHAIN CONIECTURE. – The integral closure R' of a local domain R satisfies the c.c.

(8.2) H-CONJECTURE. – If a local domain R is an H_1-ring, then R is catenary.

(8.3) CATENARY CHAIN CONIECTURE. – If R is a catenary local domain, then R' satisfies the c.c.

The concept of H_i-rings allows us to state in (8.4) a new equivalence of the H-conjecture.

(8.4) THEOREM. – *The H-conjecture holds if and only if the following condition holds: If R is an H_1-ring, then R is an H_2-ring.*

PROOF. – Assume the H-conjecture holds, and let R be an H_1-ring. Then R is catenary and so is an H_2-ring.

Conversely, assume that R is an H_2-ring whenever R is an H_1-ring, and let R be an H_1-ring. It will be shown, by induction on i $(1 \leqslant i \leqslant a-2)$, that R is an H_i-ring. Since R is an H_1-ring, R is an H_2-ring, by assumption. Assume R is an H_j-ring, for all $j \leqslant i$. Let p be a height $i-1$ prime ideal in R. Since R is H_{i-1} and H_i, depth $p = a - i + 1$ and R/p is an H_1-ring, by (2.11). Thus, by assumption, R/p is an H_2-ring. Hence R is an H_{i+1}-ring, by (2.11). So R is $H_1, \ldots, H_{a-2}$, and therefore R is catenary, by (2.9). Thus the H-conjecture holds, q.e.d.

Most of the theorems in the previous sections can be used to give at least one new equivalence of the catenary chain conjecture. Some of those new equivalences are listed below in (8.5).

(8.5) THEOREM. – *The following conditions are equivalent:*

(1) *The catenary chain conjecture holds.*

(2) *If R is catenary, then R/p satisfies the s.c.c., for every height one prime ideal p in R.*

(3) *If R is catenary, then $\mathcal{T}_{a-2}$ is an H_{a-1}-ring.*

(4) *If R is catenary, then $\mathcal{T}$ is $H_2, \ldots, H_{a-1}$.*

(5) *If R is catenary, then $\mathfrak{T}/p\mathfrak{T}$ is catenary, for every height one prime ideal p in R.*

(6) *If R is catenary, then, for every pair of elements b, c in R such that height $(b, c) = 2$, $R[c/b]_{(M,c/b)}$ is $H_1, \ldots, H_{a-2}$ (equivalently, is catenary).*

(7) *If R is catenary, then every $(a-2)$-graded extension of R is an H_1-ring.*

(8) *If R is catenary, then R^* is an H_1-ring.*

(9) *If R is catenary, then every $B \in \mathfrak{B}$ (as in (6.2)) is an H_1-ring.*

PROOF. – (1) is equivalent to (2), by [15, Theorem 4.3]; (5) is equivalent to (2), by [12, Theorem 2.21] (since $\mathfrak{T}(R/p) \cong \mathfrak{T}/p\mathfrak{T}$); (8) is equivalent to (2), by (5.1); and (9) is equivalent to (1), by (6.2).

Since (6) is equivalent to (4), by (4.5), and (7) is equivalent to (3), by (4.10); it remains to show that (2), (3) and (4) are equivalent.

For (2) implies (3), assume that (2) holds. Then, by [11, Theorem 3.1 and Proposition 3.3], R/p satisfies the s.c.c., for every prime ideal p in R such that $p \neq (0)$. Let q be a height $a-1$ prime ideal in $\mathfrak{T}_{a-2}$, and let $p = q \cap R$. Then height $p = m \in \{1, \ldots, a-1\}$, and so, by the above statement, R/p satisfies the s.c.c. Therefore, by [10, Corollary 2.8] and [11, Theorem 3.1], every locality over R/p satisfies the s.c.c. and thus is catenary [11, Remark 2.7]. In particular, $\mathfrak{T}_{a-2}/p\mathfrak{T}_{a-2} \cong \mathfrak{T}_{a-2}(R/p)$ is locality over R/p and thus is catenary, and so depth q = depth $q/p\mathfrak{T}_{a-2}$ = altitude $\mathfrak{T}_{a-2}/p\mathfrak{T}_{a-2}$ − height $q/p\mathfrak{T}_{a-2}$ = depth $p\mathfrak{T}_{a-2}$ − height $q/p\mathfrak{T}_{a-2}$. Also, depth $p\mathfrak{T}_{a-2}$ = (as in the second paragraph of the proof of (3.2)) depth $p + a - 2$ = (since R is catenary) $a - m + a - 2 = 2a - m - 2$. If follows from [2, Theorem 30.18, p. 368] that height q = height p + height $q/p\mathfrak{T}_{a-2}$. Therefore height $q/p\mathfrak{T}_{a-2} = a - 1 - m$ and thus depth $q = (2a - m - 2) - (a - 1 - m) = a - 1$ = altitude $\mathfrak{T}_{a-2} - (a-1)$. Hence $\mathfrak{T}_{a-2}$ is an H_{a-1}-ring.

(3) implies (4), by (3.4). For (4) implies (2), let p be a height one prime ideal in R. Then height $p\mathfrak{T} = 1$ and so $\mathfrak{T}_1/p\mathfrak{T}_1$ is $H_{a-2}, \ldots, H_1$, by (2.11). Since altitude $\mathfrak{T}_1/p\mathfrak{T}_1 \leq a$, $\mathfrak{T}_1/p\mathfrak{T}_1 \cong \mathfrak{T}_1(R/p)$ is catenary, by (2.9). Therefore R/p satisfies the s.c.c., by [12, Theorem 2.21], and hence (2) holds, q.e.d.

A list of questions which arise from the work in the other sections of the paper are given below. (8.6) and (8.7) are basic statements which are related to some of the results in Section 2.

(8.6) If R is an H_i-ring, is the ring R_p an H_i-ring, for all prime ideals p in R such that height $p \geq i$?

(8.7) If R is H_i and H_{i+2}, is R necessarily an H_{i+1}-ring?

(8.8) REMARK. – (8.7) is false for $i = 0$, as [7, Example 2, pp. 203-205] (for $r > 0$ and $m = 1$) shows. In (2.3) it is shown that the local domain (R, I) is an H_j-ring if and only if $j \neq 1$.

(8.9) Consider $\mathfrak{T}_k'$ and $R[u_1, \ldots, u_k]_{(M,u_1,\ldots,u_k)} \subseteq K$. Can (4.5) be generalized in terms of conditions on these two rings?

(8.10) Do the converses hold for (7.1), (7.2) and (7.4), or in other words, can (2.11), (3.1) and (3.2) be « dualized »?

(8.11) In (2.8) we used [13, Proposition 2.2 and Corollary 2.3]. It would be useful, in working with D_i-rings, to have a « dual » of [13, Proposition 2.2] (of [13, (2.2.1)], or of [13, Corollary 2.3]) like:

Let $p' \subset p \subset P$ be prime ideals in a Noetherian ring A, let height $p/p' = h$ and height $P/p = d$. Then, for each $i = 0, \ldots, h-1$, there exist infinitely many prime ideals q in A such that $p' \subset q \subset P$, height $q/p' = h - i$ and height $P/q = d + i$.

But it is not known whether this statement is true of false.

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