On the existence of a generalized solution to the first initial-boundary value problem for a nonlinear parabolic equation

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Summary. - In the paper first the existence of a classical solution to an initial-boundary value problem for the nonlinear parabolic equation $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f(x, t, z)$ is proved under the standard condition on Hölder continuity of f but a quite general condition on the growth of f . Then, by using the possibility of the approximation of a continuous function by means of Hölder continuous functions, the foregoing result is applied to the proof of the existence of a generalized solution to the first boundary value problem for the same equation where only continuity of f and a weak assumption on the growth of f is required.

1. - The existence of a solution to an initial-boundary value problem for the nonlinear parabolic equation $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f(x, t, z)$ (also in more general form) was already proved by many authors and in different ways. The result as well as the method to obtain it mainly depend on the class of functions to which the expected solution has to belong. When a classical solution is searched, then f is subjected to a Hölder condition and to a restriction concerning its growth. (See e.g. [1], [2], [3], [9]). The existence of a generalized solution requires only mild conditions on f concerning its continuity and measurability. Still a restriction on the growth of f remains. (See [5], [6]). The review of the applied methods and the obtained results can be found e.g. in [3], [4], [7], [10]).

The aim of the present paper is to assure the existence of a generalized solution to the mentioned problem assuming the continuity of f and a fairly general condition on the growth of f . The idea of the proof consists of approximating f by a sequence of Hölder continuous functions and thus, of approximating the given initial-boundary value problem by a sequence of similar problems for which the classical solution z_n can be found. Then the uniform limit of z_n exists and gives the searched generalized solution. The structure of that solution is also given.

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We shall start with some properties of Hölder continuous functions. Let (R, ρ) be a metric space, $\emptyset \neq M \subset R$ be a set, and let $d(M)$ be the diameter of M . Let us denote by $H_\alpha(M)$ the set of all Hölder continuous functions on M of exponent α , $0 < \alpha \leq 1$. Thus $f \in H_\alpha(M)$ iff f is defined on M and there exists a constant $L = L(f) > 0$ such that for any two points the inequality

$$(1) \quad |f(y) - f(x)| \leq L[\rho(x, y)]^\alpha$$

is true. Here as well as in what follows only real functions are considered. Further the following remark and theorems will be of use.

REMARK. - If $d(M) < +\infty$, and β satisfies the inequalities $0 < \alpha \leq \beta \leq 1$, then from the relation $[\rho(x, y)]^\beta \leq [\rho(x, y)]^\alpha$ and $[\rho(x, y)]^\beta \leq d^{\beta-\alpha}[\rho(x, y)]^\alpha$, respectively, which is valid for $\rho(x, y) \leq 1$ and $1 < \rho(x, y) \leq d$ (if $d > 1$), respectively, it follows that

$$(2) \quad H_\beta(M) \subset H_\alpha(M).$$

THEOREM 1. - Let M be a compact set, α be a real number satisfying the inequalities $0 < \alpha \leq 1$ and $f \in C_0(M)$. Let further the functions $F_1, F_2 \in H_\alpha(M)$ and be such that $F_1(x) \leq f(x) \leq F_2(x)$ for each $x \in M$. Then there exists a sequence $f_n \in H_\alpha(M)$, $n = 1, 2, 3, \dots$, which is uniformly convergent to f on M whereby

$$(3) \quad F_1(x) \leq f_n(x) \leq F_2(x)$$

for each $x \in M$ and all natural n .

THEOREM 2. - Let R be a locally compact separable metric space and $\{M_m\}_{m=1}^\infty$ an arbitrary increasing sequence of compact sets in R such that $R = \bigcup_{m=1}^\infty M_m$. Let the number α satisfy the inequalities $0 < \alpha \leq 1$. Let the function $f \in C_0(R)$. Then there exists a sequence $\{f_n\}_{n=1}^\infty$ of the functions $f_n \in H_\alpha(R)$ which is uniformly convergent to f on each M_m , $m = 1, 2, 3, \dots$.

If moreover there exist two functions F_1, F_2 such that $F_1, F_2 \in H_\alpha(R)$ (for each $m = 1, 2, 3, \dots$ $F_1, F_2 \in H_\alpha(M_m)$) and $F_1(x) \leq f(x) \leq F_2(x)$ for every $x \in R$, then the sequence $\{f_n\}_{n=1}^\infty$ satisfies the inequalities (3) for each $x \in R$ and all natural n (the sequence $\{f_n\}_{n=1}^\infty$ satisfies the inequalities (3) for each $x \in R$ and all natural n , but instead of $f_n \in H_\alpha(R)$ it is only true that $f_n \in H_\alpha(M_m)$ for each $m = 1, 2, 3, \dots$).

NOTE. - The proof of Theorems 1 and 2 will be given in another paper. By approximating a continuous function by means of polynomials we obtain Theorems 1 and 2 only in a special case when $F_1 \leq f - \varepsilon$, $F_2 \geq f + \varepsilon$, $\varepsilon > 0$.

Although this case is sufficient for the proof of Theorem 3, in Lemma 5 a continuous f satisfying the inequality (10) is approximated by means of $f_n \in H_\alpha$ satisfying (19). Hence Theorem 2 is applied in a more general case which does not follow from the Weierstrass theorem.

2. - We are now going to apply the above mentioned results. As a locally compact separable metric space we shall take the n -dimensional euclidean space E_n . Further for given $a > 0$, $b > 0$ and $0 \leq k \leq l$ integers we define the sets $B_{k,l}$, B in E_2 by the relation $B_{k,l} = \{(t, y) : 0 \leq t \leq b, k \leq y \leq l\}$, $B = \{(t, y) : 0 \leq t \leq b, 0 \leq y < +\infty\}$ and the sets D_k , D in E_3 by $D_k = \{(x, t, y) : 0 \leq x \leq a, 0 \leq t \leq b, -k \leq y \leq k\}$, $D = \{(x, t, y) : 0 \leq x \leq a, 0 \leq t \leq b, -\infty < y < +\infty\}$. Finally, let $E = \{(x, t) : 0 \leq x \leq a, 0 \leq t \leq b\}$ and let E_0 denote the interior of E . In the last part of the paper we shall consider the space $W^{2,1}(E)$ which means the set of all functions u which together with their weak derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial t}$ belong to $L^2(E)$. The norm in this linear space is given by

$$\|u\|_{2,1} = \left\{ \int \int_E \left[u^2(x, t) + \left(\frac{\partial u(x, t)}{\partial t} \right)^2 + \left(\frac{\partial u(x, t)}{\partial x} \right)^2 + \left(\frac{\partial^2 u(x, t)}{\partial x^2} \right)^2 \right] dx dt \right\}^{\frac{1}{2}} \quad (\text{compare with}$$

[4], p. 15). Lemma 6.2 ([4], p. 16) remains to be true in this space.

Similarly as in [9], p. 257, we shall say that the function $p = p(t, y)$ satisfies the condition (A) if $p \in C_0(B)$ and for any $c \geq 0$ the right maximal integral $y(t, c)$ of the differential equation

$$(4) \quad y' = p(t, y)$$

such that

$$(5) \quad y(0, c) = c$$

exists on the whole interval $< 0, b >$.

From the property of the differential equation (4) that their solutions form a lattice it follows that if $0 \leq c_1 < c_2$, then $y(t, c_1) \leq y(t, c_2)$ on the intersection of their intervals of definition. Hence for $n-1 < c < n$ $y(t, n-1) \leq y(t, c) \leq y(t, n)$ on the common part of their intervals of definition and the condition (A) will be satisfied if for each $n=0, 1, 2, \dots$ $y(t, n)$ exists on the whole interval $< 0, b >$. If moreover $p(t, y) \geq 0$ for all $(t, y) \in B$, it suffices to prove the existence of $y(t, n)$ on the interval $< 0, b >$ only for all natural n .

THEOREM 3. - Let p satisfy the condition (A) and $p(t, y) \geq 0$ in B . Then there exists a function q which also satisfies the condition (A) whereby $q \in H_1(B_{0,m})$ for each $m=1, 2, 3, \dots$ and $q(t, y) > p(t, y)$ in B .

PROOF. - For each natural n , on the basis of Theorem 2, [8], p. 83, there exists an $\varepsilon_n > 0$ such that the right maximal solution of the problem $y' = p(t, y) + \varepsilon_n$, $y(0) = n$, exists in $\langle 0, b \rangle$. By using the Comparison theorem 1, [8], p. 82, we may assume that $\varepsilon_n \geq \varepsilon_{n+1}$ for all n . Further from Theorem 1 the existence of a function $q_n \in H_1(B_{n-1, n})$ in $B_{n-1, n}$ follows which satisfies the inequalities

$$(6) \quad p(t, y) + \frac{\varepsilon_n}{4} < q_n(t, y) < p(t, y) + \frac{3}{4} \varepsilon_n, \quad (t, y) \in B_{n-1, n}.$$

By the uniform continuity of p in $B_{n-1, n}$ there exists such y_n , $n-1 < y_n < n$ that

$$(7) \quad p(t, n) - \frac{\varepsilon_{n+1}}{8} < p(t, y) < p(t, n) + \frac{\varepsilon_{n+1}}{8}, \quad 0 \leq t \leq b, \quad y_n \leq y \leq n.$$

Let q be the function defined in B in the following way:

$$q(t, y) = \begin{cases} q_n(t, y), & 0 \leq t \leq b, \quad n-1 \leq y \leq y_n \\ q_n(t, y_n) + Q_n(t)(y - y_n), & 0 \leq t \leq b, \quad y_n \leq y \leq n, \end{cases}$$

where $Q_n(t) = [q_{n+1}(t, n) - q_n(t, y_n)]/(n - y_n)$. By this definition q satisfies the inequalities (6) for $0 \leq t \leq b$, $n-1 \leq y \leq y_n$ (with $q_n(t, y) = q(t, y)$). As to the behaviour of q on the set $0 \leq t \leq b$, $y_n \leq y \leq n$, two cases may happen.

When $q_n(t, y_n) \leq q_{n+1}(t, n)$, in virtue of (7) and (6) we have $p(t, y) < p(t, n) + \frac{\varepsilon_{n+1}}{8} < p(t, y_n) + \frac{\varepsilon_{n+1}}{4} < q_n(t, y_n) \leq q(t, y) \leq q_{n+1}(t, n) < p(t, n) + \frac{3}{4} \varepsilon_{n+1} < p(t, y) + \frac{7}{8} \varepsilon_{n+1}$.

When $q_n(t, y_n) > q_{n+1}(t, n)$, it is true that $p(t, y) + \frac{\varepsilon_{n+1}}{8} < p(t, n) + \frac{\varepsilon_{n+1}}{4} < q_{n+1}(t, n) \leq q(t, y) \leq q_n(t, y_n) < p(t, y_n) + \frac{3}{4} \varepsilon_n < p(t, n) + \frac{3}{4} \varepsilon_n + \frac{1}{8} \varepsilon_{n+1} < p(t, y) + \varepsilon_n$.

Altogether we have got that in $B_{n-1, n}$ for q the following inequalities are true

$$(8) \quad p(t, y) < q(t, y) < p(t, y) + \varepsilon_n.$$

Further for $0 \leq t \leq b$, $y_n \leq \bar{y} < \bar{y} \leq n$, $|q(t, \bar{y}) - q(t, \bar{y})| \leq M |\bar{y} - \bar{y}|$ because $Q_n(t)$ is continuous in $\langle 0, b \rangle$. On the other hand for $0 \leq t_1 < t_2 \leq b$, $y_n \leq y \leq n$, $|q(t_2, y) - q(t_1, y)| \leq \{L_n + (y - y_n)(L_{n+1} + L_n)/(n - y_n)\} |t_2 - t_1|$, where L_n is the Lipschitz constant for the function q_n in $B_{n-1, n}$. This, together with the properties of q_n , gives that q satisfies a Lipschitz condition in $B_{n-1, n}$ and hence also in $B_{0, n}$. With respect to the inequalities (8) on the basis of the Comparison theorem 1, [8], p. 82, there exists the right maximal solution of

the problem $y' = g(t, y)$, $y(0) = n$ for each natural n and thus, using the remark before Theorem 3, the function q satisfies the condition (A).

3. - The next step in proving the existence of a generalized solution will be the extension in a certain sense of the validity of the existence theorem proved in [9]. (Compare also with the existence theorem in [2], p. 314). To that aim (similarly as in [9], p. 259) we shall say that the function $f = f(x, t, y)$ defined in D satisfies the condition (H) if for each point (x, y) , $0 < x < a$, $-\infty < y < +\infty$ there exist a neighborhood $N(x, y)$ of that point and constants $L > 0$, $0 < \alpha < 1$, $0 < \beta < 1$ depending in general on (x, y) such that for $(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}) \in N(x, y)$ and arbitrary $0 \leq t \leq b$ the inequality $|f(\bar{x}, t, \bar{y}) - f(\bar{x}, t, \bar{y})| \leq L(|\bar{x} - \bar{x}|^\alpha + |\bar{y} - \bar{y}|^\beta)$ holds.

Further the solution of the first initial-boundary value problem

$$(9) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < a, \quad 0 < t < b$$

$$u(x, 0) = \chi(x), \quad 0 \leq x \leq a,$$

$$u(0, t) = \varphi(t), \quad u(a, t) = \psi(t), \quad 0 \leq t \leq b$$

where $\chi \in C_0(< 0, a >)$, $\varphi, \psi \in C_0(< 0, b >)$, and $\chi(0) = \varphi(0)$, $\chi(a) = \psi(0)$, will be denoted by $u = u(x, t; \varphi, \chi, \psi)$ (compare with [11]). The following theorem is true.

THEOREM 4. - Suppose that the function $p = p(t, y)$ satisfies the condition (A), the function $f = f(x, t, y)$ is continuous in D , satisfies the condition (H) and

$$(10) \quad |f(x, t, y)| \leq p(t, |y|), \quad (x, t, y) \in D.$$

Let further the functions $\varphi, \psi \in H_\alpha(< 0, b >)$, $\chi \in H_\alpha(< 0, a >)$ for an α , $0 < \alpha < 1$ and $\chi(0) = \varphi(0)$, $\chi(a) = \psi(0)$. Then there exists at least one solution z of the differential equation

$$(11) \quad \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + f(x, t, z) \quad 0 < x < a, \quad 0 < t < b,$$

satisfying the conditions

$$(12) \quad z(0, t) = \varphi(t), \quad z(a, t) = \psi(t), \quad 0 \leq t \leq b,$$

$$z(x, 0) = \chi(x), \quad 0 \leq x \leq a.$$

Moreover there exists a constant $M > 0$ such that for any two points

$(x_1, t_1), (x_2, t_2) \in E$ the solution z satisfies the inequality

$$(13) \quad |z(x_2, t_2) - z(x_1, t_1)| \leq M(|x_2 - x_1|^\alpha + |t_2 - t_1|^{\alpha/2}).$$

The proof of this theorem will be based on Lemmas either stated or proved in the papers [9], [11] and [2].

LEMMA 1 ([9], p. 257). - Assume that the function $p = p(t, y)$ satisfies the condition (A), the function $f = f(x, t, y)$ is defined in D and satisfies the inequality (10) in D . Suppose that $v = v(x, t)$ is continuous in E and possesses the continuous derivative $\frac{\partial^2 v}{\partial x^2}$ in the interior of E . Assume that v satisfies in interior of E the equation (11) and the boundary inequalities

$$|v(0, t)| \leq c, \quad |v(a, t)| \leq c, \quad 0 \leq t \leq b,$$

$$|v(x, 0)| \leq c, \quad 0 \leq x \leq a$$

for some $c \geq 0$. Under these assumptions we have the inequality

$$|v(x, t)| \leq y(t, c), \quad (x, t) \in E,$$

where $y(t, c)$ is the right maximal integral of the differential equation (4) satisfying (5).

LEMMA 2 ([2], p. 301). - For any γ , $0 < \gamma < 1$, there exists a constant $c(\gamma) > 0$ such that for any function $g = g(x, t) \in C_0(E)$ satisfying the inequality $|g(x, t)| \leq M$ for $(x, t) \in E$ it is true: If

$$p(x, t) = \frac{1}{2\sqrt{\pi}} \int_0^t \int_0^a \frac{\exp[-(x-\xi)^2/4(t-\tau)]}{\sqrt{t-\tau}} g(\xi, \tau) d\xi d\tau, \quad 0 \leq x \leq a, \quad 0 < t \leq b,$$

$$p(x, 0) = 0, \quad 0 \leq x \leq a,$$

then for any two points $(x_1, t_1), (x_2, t_2) \in E$ we have

$$|p(x_2, t_2) - p(x_1, t_1)| \leq c(\gamma) M(|x_2 - x_1| + |t_2 - t_1|^{\frac{1+\gamma}{2}}).$$

LEMMA 3 ([2], p. 302). - For any σ , $0 < \sigma < \frac{1}{2}$, there exists a constant $r(\sigma) > 0$ such that if the functions φ and $\psi \in C_0(<0, b>)$, $\varphi(0) = \psi(0) = 0$, satisfy a Hölder condition $|\varphi(t_2) - \varphi(t_1)| \leq L|t_2 - t_1|^\sigma$, $|\psi(t_2) - \psi(t_1)| \leq L|t_2 - t_1|^\sigma$, $t_1, t_2 \in <0, b>$, then the solution $u = u(x, t; \varphi, 0, \psi)$ of the equation (9) satisfies

the inequality

$$|u(x_2, t_2) - u(x_1, t_1)| \leq r(\sigma) L(|x_2 - x_1|^{2\sigma} + |t_2 - t_1|^\sigma)$$

for any two points $(x_1, t_1), (x_2, t_2) \in E$.

LEMMA 4 ([11], p. 363). - Let $\chi \in C_0(<0, a>)$, $\chi(0) = \chi(a) = 0$. Let, further, there exist numbers $0 < \alpha \leq 1$, $M > 0$ such that $|\chi(x_2) - \chi(x_1)| \leq M|x_2 - x_1|^\alpha$ for any two points $x_1, x_2 \in <0, a>$.

Then the solution $u = u(x, t; 0, \chi, 0)$ of the differential equation (9) possesses the following properties:

$$a) \quad |u(x_2, t) - u(x_1, t)| \leq M|x_2 - x_1|^\alpha, \quad (x_1, t), (x_2, t) \in E,$$

$$b) \quad |u(x, t_2) - u(x, t_1)| \leq \frac{2^{2+\alpha} M}{\sqrt{\pi}} \int_0^\infty w^\alpha e^{-w^2} dw |t_2 - t_1|^{\alpha/2}$$

for any two points $(x, t_1), (x, t_2) \in E$.

Proof of Theorem 4. In the Banach space $C_0(E)$ where the norm is, as usually, defined by $\|v\| = \max |v(x, t)|$ we define (similarly as in [9]) the operator T in the following way. For each $v \in C_0(E)$ we consider the function r given by

$$(14) \quad r(x, t) = \frac{1}{2\sqrt{\pi}} \int_0^t \int_0^a \frac{\exp[-(x-\xi)^2/4(t-\tau)]}{\sqrt{t-\tau}} f[\xi, \tau, v(\xi, \tau)] d\xi d\tau$$

$$0 \leq x \leq a, \quad 0 < t \leq b,$$

$$r(x, 0) = 0, \quad 0 \leq x \leq a$$

and the solution q of the equation (9) $q = u(x, t; \varphi - r(0, t), \chi, \psi - r(a, t))$. Then we put

$$(15) \quad T(v) = r + q = w.$$

We also denote the equality (15) by $w = T(v; f, \varphi, \chi, \psi)$, where f, φ, χ, ψ play the role of parameters. From the definition of w it follows that $w \in C_0(E)$ and w satisfies the conditions (12).

To prove that w satisfies a Hölder condition in x of the exponent α uniformly with respect to t in E , let us consider the decomposition (15). The function r shows this property on basis of Lemma 2 and the remark before Theorem 1. When we introduce the functions $\chi_1(x) = \chi(0) + [\chi(a) - \chi(0)] \frac{x}{a}$, $0 \leq$

$x \leq a$, $\varphi_1(t) = \varphi(t) - r(0, t) - \varphi(0)$, $\psi_1(t) = \psi(t) - r(a, t) - \psi(0)$, $0 \leq t \leq b$, we can write

$$q(x, t) = \chi_1(x) + u(x, t; 0, \chi - \chi_1, 0) + u(x, t; \varphi_1, 0, \psi_1).$$

From the assumption $\chi \in H_\alpha(<0, a>)$, in virtue of Lemma 4, it follows that $u(x, t; 0, \chi - \chi_1, 0)$ satisfies a Hölder condition in x of the exponent α uniformly with respect to t in E . Lemma 2 gives that the functions $\varphi_1, \psi_1 \in H_\sigma(<0, b>)$, where $\sigma \leq \alpha$, therefore it may be assumed that $0 < \frac{\alpha}{2} \leq \sigma < \frac{1}{2}$.

Then, by Lemma 3, $u(x, t; \varphi_1, 0, \psi_1)$ satisfies a Hölder condition in x of the exponent 2σ and hence of the exponent α , too, uniformly with respect to t in E . Similarly we get that v satisfies a Hölder condition in t of the exponent $\frac{\alpha}{2}$ uniformly with respect to x in E . Therefore there exists a constant $M > 0$, $M = M(v)$ such that v satisfies the inequality (13) for any two points $(x_1, t_1), (x_2, t_2) \in E$. From that property of v it follows that if $T(z) = z$, then the function $f[x, t, z(x, t)]$ is locally Hölder continuous in x uniformly with respect to t in E and hence, on the basis of Theorem 9, [3], p. 21, z is a solution of the problem (11), (12). Therefore it suffices to prove the existence of a fixed point of the operator T . To that aim let us consider some properties of the operator T .

By the continuity of f in D the function r continuously depends on v . With respect to the continuous dependence of the solution q of the equation (9) on the initial and boundary conditions this implies that the operator T continuously maps $C_0(E)$ into itself. When a sequence $v_n \in C_0(E)$ is bounded, then again by the continuity of f there exists a constant $M > 0$ such that $|f(x, t, v_n(x, t))| \leq M$ for each $(x, t) \in E$ and on basis of Lemma 2 the corresponding sequence of r_n given by (14) where instead of v we put v_n is uniformly bounded and equicontinuous. Hence, there exists a subsequence r_{n_k} , $k = 1, 2, 3, \dots$ which is uniformly convergent on E to a function $r \in C_0(E)$. Let q_n be the solution of the differential equation (9), $q_n = u(x, t; \varphi - r_n(0, t), \chi, \psi - r_n(a, t))$. From the continuous dependence of the solutions of (9) on the initial and the boundary conditions we get that q_{n_k} uniformly converge on E to the solution q of the equation (9) given by the relation $q = u(x, t; \varphi - r(0, t), \chi, \psi - r(a, t))$. Thus the sequence $T(v_{n_k}; \varphi, \chi, \psi)$ is convergent in $C_0(E)$ to $q + r$ and the operator T is completely continuous. Its further property is that for every bounded set $S \subset C_0(E)$ there exists a constant $K = K(S) > 0$ such that

$$16) \quad \|T(v; \varphi, \chi, \psi)\| \leq K$$

for each $v \in S$. This follows from the uniform boundedness of the functions

(14) corresponding to the functions $v \in S$ and from the maximum (and minimum) principle for the solutions of the equation (9).

Let us, now, consider the system of the operators T_k , $0 \leq k \leq 1$, defined in $C_0(E)$ by the relation $T_k(v) = T(v; kf, k\varphi, k\chi, k\psi)$. From the construction of the operator T we see that $T_k(v) = kT(v)$ for each $v \in C_0(E)$ and $k \in (0, 1)$. Then the following statements are true:

a) For each k , $0 \leq k \leq 1$, T_k is a continuous mapping of $C_0(E)$ into itself.

b) If $S \subset C_0(E)$ is an arbitrary bounded set and $v \in S$, then $\|T_{k_2}(v) - T_{k_1}(v)\| = |k_2 - k_1| \cdot \|T(v)\|$ what, on the basis of (16), implies the uniform continuity of T_k in k on S .

c) For any fixed k , $0 \leq k \leq 1$, T_k is a completely continuous operator.

d) If $v = T_k(v)$, then v satisfies the conditions

$$v(0, t) = k\varphi(t), \quad v(a, t) = k\psi(t), \quad 0 \leq t \leq b$$

$$v(x, 0) = k\chi(x), \quad 0 \leq x \leq a$$

and similarly as in the case of the solution of the functional equation $z = T(z)$ it can be shown that v satisfies the differential equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + kf(x, t, v)$$

Since the function kf also satisfies the inequality (10), by Lemma 1 there exists a constant $M_1 > 0$ independent on k such that $\|v\| \leq M_1$.

e) The equation $v = T_0(v)$ has only the trivial solution.

We see that all assumptions of the Leray-Schauder theorem ([3], p. 189) are satisfied and, by that theorem, the operator T has at least one fixed point in $C_0(E)$.

4. - Finally we proceed to the main theorem of that paper. The proof of that theorem will be based on the following lemmas.

LEMMA 5. - Let the function $p = p(t, y)$ satisfy the condition (A), the function $f = f(x, t, y)$ be continuous in D and satisfy the inequality (10). Let further the functions $\varphi, \psi \in C_0(<0, b>)$, $\chi \in C_0(<0, a>)$, $\chi(0) = \varphi(0)$, $\chi(a) = \psi(0)$. Let the number α satisfy the inequalities $0 < \alpha < 1$. Then there exist:

1. A sequence of functions $\{f_n\}_{n=1}^\infty$, $f_n = f_n(x, t, y) \in H_\alpha(D_m)$ for each $m=1, 2, 3, \dots$, which is uniformly convergent to f on each D_m , $m=1, 2, 3, \dots$;

2. Sequences $\varphi_n, \psi_n \in H_\alpha(<0, b>)$, $\chi_n \in H_\alpha(<0, a>)$ such that $\chi_n(0) =$

$\varphi_n(0)$, $\chi_n(a) = \psi_n(0)$ which uniformly converge on the corresponding intervals to the functions φ , ψ and χ , respectively;

3. A sequence of the solutions z_n of the first initial-boundary value problem

$$(17) \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x^2} + f_n(x, t, z), \quad 0 < x < a, \quad 0 < t < b$$

$$(18) \quad z(0, t) = \varphi_n(t), \quad z(a, t) = \psi_n(t), \quad 0 \leq t \leq b, \quad z(x, 0) = \chi_n(x), \quad 0 \leq x \leq a$$

which satisfy the inequality (13) with constants $M = M(n)$ for any two points $(x_1, t_1), (x_2, t_2) \in E$. This sequence is uniformly convergent on E to a function $z \in C_0(E)$ which satisfies the conditions (12) and possesses the continuous $\frac{\partial z}{\partial x}$ in E_0 .

PROOF. - By Theorem 3 there exists q which satisfies the condition (A) whereby $q \in H_1(B_{0,m})$ for each $m = 1, 2, 3, \dots$ and $q(t, y) > p(t, y)$ in B . On the basis of Theorem 2 we get the existence of sequences of functions f_n , φ_n , ψ_n , and χ_n with the properties mentioned in the statement of the last theorem. f_n satisfy the inequality

$$(19) \quad |f_n(x, t, y)| \leq q(t, |y|), \quad (x, t, y) \in D, \quad n = 1, 2, 3, \dots$$

The existence of a solution z_n of the boundary value problem (17), (18) as well as the validity of the inequality (13) for this solution follows from Theorem 4 by using the inequalities (19). From the proof of that theorem we can see that z_n can be got as the solution of the functional equation

$$z_n = r_n + q_n,$$

where q_n is determined by the equality (14) in which instead of f and v we put f_n and z_n , respectively, and $q_n(x, t) = u(x, t; \varphi_n - r_n(0, t), \chi_n, \psi_n - r_n(a, t))$. Since the sequences of functions φ_n , ψ_n and χ_n are uniformly bounded, in view of Lemma 1 the same property is shared by the sequence of z_n on E . Lemma 2 then gives that the sequence of functions r_n is uniformly bounded and equicontinuous. Hence there exists a subsequence r_{n_k} which is uniformly convergent on E . With that sequence q_{n_k} is also uniformly convergent. If we now consider only the subsequences f_{n_k} , φ_{n_k} , ψ_{n_k} , χ_{n_k} , z_{n_k} and denote them by f_n , φ_n , ψ_n , χ_n , and z_n , respectively, we obtain the statement of the theorem. The continuity of $\frac{\partial z}{\partial x}$ follows from the decomposition $z = r + q$ where q is a suitable solution of the equation (9) and r is determined by the equality (14)

whereby instead of v acts z . Since the composite function $f[x, t, z(x, t)] \in C_0(E)$, $\frac{\partial f}{\partial x}$ is continuous in the interior of E ([3], p. 21).

Further properties of the sequences $\{z_n\}$ mentioned in Lemma 5 will be derived from the following lemma based on Theorem 7, [7], p. 33.

LEMMA 6. - If the function $u = u(x, t)$ together with the derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial t}$ is continuous on E and in E_0 satisfies the equation

$$(20) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x, t) \quad (h \in C_0(E))$$

as well as the initial-boundary conditions

$$(21) \quad u(x, 0) = 0, \quad 0 \leq x \leq a, \quad u(0, t) = u(a, t) = 0, \quad 0 \leq t \leq b,$$

then there exists a constant $K > 0$, depending only on a, b such that

$$(22) \quad \int_E \left[u^2(x, t) + \left(\frac{\partial u(x, t)}{\partial t} \right)^2 + \left(\frac{\partial u(x, t)}{\partial x} \right)^2 + \left(\frac{\partial^2 u(x, t)}{\partial x^2} \right)^2 \right] dx dt \leq K \int_E h^2(x, t) dx dt.$$

PROOF. - By the mentioned Theorem 7, there exist two constants $M_1 > 0$, $M_2 > 0$ depending only on a, b , such that for every τ , $0 \leq \tau \leq b$,

$$\int_0^a u^2(x, \tau) dx + \int_0^\tau \int_0^a \left[\frac{\partial u(x, t)}{\partial x} \right]^2 dx dt \leq M_1 \int_0^\tau \int_0^a h^2(x, t) dx dt,$$

$$\int_0^\tau \int_0^a \left[\frac{\partial u(x, \tau)}{\partial x} \right]^2 dx + \int_0^\tau \int_0^a \left[\frac{\partial u(x, t)}{\partial t} \right]^2 dx dt \leq M_2 \int_0^\tau \int_0^a h^2(x, t) dx dt.$$

From these two inequalities, in view of the inequality

$$\int_0^\tau \left[\int_0^a \int_0^u h^2(x, t) dx dt \right] du = \int_0^\tau \int_0^a (\tau - t) h^2(x, t) dx dt \leq b \int_0^\tau \int_0^a h^2(x, t) dx dt$$

and the equation (20), the inequality (22) follows.

Finally, we come to the main theorem.

THEOREM 5. - Let the function $p = p(t, y)$ satisfy the condition (A), let the function $f = f(x, t, y)$ be continuous in D and satisfy the inequality (10).

Let further the functions $\varphi, \psi \in C_0(<0, b>)$, $\chi \in C_0(<0, a>)$, and $\chi(0) = \varphi(0)$, $\chi(a) = \psi(0)$. Denote $f(0, 0, \varphi(0)) = c_1$, $f(a, 0, \psi(0)) = c_2$. Then for every δ , $0 < \delta < \frac{1}{2}$, there exists a function $z \in C_0(E)$ with $\frac{\partial z}{\partial x} \in C_0(E_0)$ and having the weak derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial z}{\partial t}$ such that:

$$1. \quad z = z_1 + z_2,$$

where $z_1 \in \bar{C}_{1+\delta}(E) \cap W^{2,1}(E)$ and z_1 satisfies almost everywhere (a.e.) in E_0 the differential equation

$$(23) \quad \frac{\partial z_1}{\partial t} = \frac{\partial^2 z_1}{\partial x^2} + f(x, t, z_1(x, t)) - \left(c_1 + \frac{c_2 - c_1}{a} x \right)$$

and everywhere the conditions (21); $z_2 \in C_{2+1}(E_0)$ is the (classical) solution of the equation

$$(24) \quad \frac{\partial z_2}{\partial t} = \frac{\partial^2 z_2}{\partial x^2} + c_1 + \frac{c_2 - c_1}{a} x$$

which satisfies the conditions (12).

2. z satisfies the equation (11) in E_0 a.e and the conditions (12) everywhere.

PROOF. - For the given δ , $0 < \delta < \frac{1}{2}$, put $\alpha = \sqrt{2\delta}$. Then $0 < \alpha < 1$, and by Lemma 5, there exists a sequence of solutions z_n of the problem (17), (18) which is uniformly convergent on E to a function $z \in C_0(E)$ with $\frac{\partial z}{\partial x} \in C_0(E_0)$. We shall prove that z has all properties mentioned in this theorem.

Preserving the notations from Lemma 5, for every $n = 1, 2, 3, \dots$ and every $(x, t) \in E$ we denote

$$g_n(x, t) = f_n(x, t, z_n(x, t)), \quad h_n(x, t) = g_n(x, t) - \left[g_n(0, 0) + \frac{g_n(a, 0) - g_n(0, 0)}{a} x \right]$$

$$g(x, t) = f(x, t, z(x, t)), \quad h(x, t) = g(x, t) - \left(c_1 + \frac{c_2 - c_1}{a} x \right).$$

Thus, $h_n(0, 0) = h_n(a, 0) = 0$ and, by Lemma 5, the sequences $\{g_n\}_{n=1}^\infty$, $\{h_n\}_{n=1}^\infty$ uniformly converge on E to g and h , respectively. Also each g_n as well as h_n belongs to $H_\delta(E)$.

Let $z_{1,n}$ be the solution of the differential equation

$$(25) \quad \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + h_n(x, t), \quad 0 < x < a, \quad 0 < t < b,$$

which satisfies (21). By Theorem 6, [3], p. 124, $z_{1,n}$ exists and is unique. Theorem 4, [3], p. 191, assures the existence of a constant $K > 0$ depending only on δ, a, b such that $\overline{z_{1,n}}_{1+\delta}^{E_0} \leq K, \overline{h_n}_0^{E_0} \leq K_1$. Since each Hölder continuous function on E_0 can be extended to E as Hölder continuous with the same Hölder coefficient and the derivative of such a function at a boundary point p of E_0 is equal to the limit at p of that derivative, we also have $\overline{z_{1,n}}_{1+\delta}^E \leq K_1$. By this inequality both sequences $\{z_{1,n}\}, \left\{\frac{\partial z_{1,n}}{\partial x}\right\}$, are uniformly bounded and equicontinuous on E . Hence there exists a function $z_1 \in \overline{C}_{1+\delta}(E)$ satisfying (21) and a subsequence $\{z_{1,n_k}\}_{k=1}^\infty$ which uniformly converges to z_1 , and $\left\{\frac{\partial z_{1,n_k}}{\partial x}\right\}$ uniformly converges to $\frac{\partial z_1}{\partial x}$, both on E . Similarly, as in Lemma 5, we denote this sequence again by $\{z_{1,n}\}_{n=1}^\infty$. The same change of notation will be referred to all other sequences which we are dealing with. From Theorem 7, [3], p. 65, it follows that each $z_{1,n} \in \overline{C}_{2+\delta}(E)$ and thus we can apply Lemma 6 to $z_{1,n}$. By this lemma, there is a constant $K_2 > 0$ such that

$$\int_E \int \left[z_{1,n}^2(x, t) + \left(\frac{\partial z_{1,n}(x, t)}{\partial t} \right)^2 + \left(\frac{\partial z_{1,n}(x, t)}{\partial x} \right)^2 + \left(\frac{\partial^2 z_{1,n}(x, t)}{\partial x^2} \right)^2 \right] dx dt \leq K_2$$

for all $n = 1, 2, 3, \dots$. From that, on the basis of Lemma 6.2, [4], p. 16, we get that $z_1 \in W^{2,1}(E)$ and $\left\{\frac{\partial z_{1,n}}{\partial t}\right\}, \left\{\frac{\partial^2 z_{1,n}}{\partial x^2}\right\}$ converge weakly (in $L^2(E)$) to the weak derivatives $\frac{\partial z_1}{\partial t}$ and $\frac{\partial^2 z_1}{\partial x^2}$, respectively. Hence, by going to the limit in (25), as $n \rightarrow +\infty$, we obtain that z_1 satisfies (23) in E_0 a .e. It also fulfils the conditions (21).

Let $z_{2,n}$ be the function given by

$$z_{2,n}(x, t) = \left(g_n(0, 0) + \frac{g_n(a, 0) - g_n(0, 0)}{a} x \right) t + u(x, t; \varphi_n(t) - g_n(0, 0) t, \\ \chi_n(x), \psi_n(t) - g_n(a, 0) t),$$

that is, the unique solution of

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g_n(x, t) - h_n(x, t), \quad 0 < x < a, \quad 0 < t < b,$$

which satisfies the conditions (18). Corollary to Theorem 15, [3], p. 80, gives that $\{z_{2,n}\}$ is uniformly convergent on E to the function z_2 , given by

$$z_2(x, t) = \left(c_1 + \frac{c_2 - c_1}{a} x \right) t + u(x, t; \varphi(t) - c_1 t, \chi(x), \psi(t) - c_2 t)$$

and hence z_2 is the unique solution of the equation (24) which satisfies the conditions (12). By Theorem 9, [3], p. 69, $z_2 \in C_{2+1}(E_0)$.

Since $z_{1,n} + z_{2,n}$ satisfy the conditions (18) and the equation $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g_n(x, t)$, thus, the same conditions as z_n , we have $z_{1,n} + z_{2,n} = z_n$, and thus, $z_1 + z_2 = z$. Putting the result for z_1, z_2 together, we finally see that z satisfies the equation (11) in E_0 a.e. and the conditions (12).

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