Continuity and Set Function Summability.

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Summary. - *Two characterization theorems for functions from $\mathcal{E}^N$ into* $\boldsymbol{R}$ *are given in terms of set function integrabiuty and set function summability.*

1. - Introduction.

The statement that $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space means that U is a set, $\boldsymbol{F}$ is a field of subsets of U, and μ is a real-nonnegative-valued finitely additive function on $\boldsymbol{F}$. If $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space, then $\mathfrak{p}_F$ denotes the set of all real-valued functions on $\boldsymbol{F}$, $\mathfrak{p}_F^+$ denotes the set of all nonnegative-valued elements of $\mathfrak{p}_F$, $\mathfrak{p}_{FA}$ denotes the set of all bounded finitly additive elements of $\mathfrak{p}_F$, and $\mathfrak{p}_{FA}^+$ denotes $\mathfrak{p}_F^+ \cap \mathfrak{p}_{FA}$.

In a previous paper [1] the author proved the following theorem:

THEOREM 1.A.1. - If N is a positive integer and $\{[t_k, u_k]\}_{i=1}^N$ is a sequence of number intervals and τ is a real-valued function on $[t_1, u_1]X \ldots X[t_N, u_N]$, then the following two statements are equivalent:

1) If γ is a real-valued function on the number interval $[a, b]$ having bounded variation and $\{\mathrm{H}_k\}_{k=1}^N$ is a sequence of functions of subintervals of $[a, b]$ such that for each positive integer $k \leq N$, the range of H_k is a subset of $[t_k, u_k]$ and the «interval function integral» (see section 2)

$$\int_{[a, p]} \mathrm{H}_k(I) d\gamma$$

exists, then the «interval function integral»

$$\int_{[a, b]} \tau[\mathrm{H}_1(I), \ldots, \mathrm{H}_N(I)] d\gamma$$

exists, and

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2) τ is continuous.

In section 2 we shall discuss how the following extension of Theorem 1.A.1 can be obtained:

THEOREM 2.2. - Under the hypothesis of Theorem 1.A.1 the following two statements are equivalent:

1) If $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space and $\{H_k\}_{i=1}^{N}$ is a sequence of elements of $\mathfrak{p}_F$ such that for each positive integer $k \leq N$, the range of H_k is a subset of $[t_k, u_k]$ and the integral (section 2)

$$\int_U H_k(I)\mu(I)$$

exists, then

$$\int_U \tau[H_1(I), \ldots, H_N(I)]\mu(I)$$

exists, and

2) τ is continuous.

The purpose of this paper is to give the following two extensions of Theorem 2.2:

THEOREM 4.1 (section 4). - If N is a positive integer and σ is a function from $\mathcal{E}^N$ into the real numbers, then the following two statements are equivalent:

1) If $\{U, \boldsymbol{F}, \mu\}$ is finitely additive measure space and $\{H_k\}_{k=1}^{N}$ is a sequence of μ-summable (section 2) elements of $\mathfrak{p}_F$, then the integral

$$\int_U \sigma[H_1(I), \ldots, H_N(I)]\mu(I)$$

exists, and

2) σ is continuous and bounded.

THEOREM 5.1 (section 5). - If N is a positive integer and Γ is a function from $\mathcal{E}^N$ into the real numbers, then the following two statements are equivalent:

1) If $\{U, \boldsymbol{F}, \mu\}$ is finitely additive measure space and $\{H_k\}_{k=1}^{N}$ is a sequence of μ-summable elements of $\mathfrak{p}_F$, then $\Gamma[H_1, \ldots, H_N]$ is μ-summable, and

2) Γ is continuous and

$$\{ | \Gamma[x_1, \ldots, x_N] | \,/\, [\sum_{k=1}^{N} | x_k |] \mid 1 \leq \sum_{k=1}^{N} | x_k | \}$$

is bounded.

2. - Preliminary theorems and definition.

For the basic facts, notations and conventions concerning subdivision, refinement and integral, we refer the reader to section 2 of [1] and sections of 2, 3 and 4 of [2] for these notions as they pertain to intervals and fields, respectively. We also refer the reader to [2] for a statement of KOLMOGOROFF'S [3] differential equivalence theorem and its implications about the existence and equivalence of the integrals that we shall use. When the existence of an integral or its equivalence to an integral is an easy consequence of the above mentioned material, the integral need only be written, and the proof of existence or equivalence left to the reader.

We now consider a particular « extension » of an interval function. Suppose $a < b$. It is well known that if $\boldsymbol{G}_{(a,b]} = \{ (p, q] \mid a \leq p < q \leq b \}$, then the smallest field $\boldsymbol{F}_{(a,b]}$ of sets including $\boldsymbol{G}_{(a,b]}$ is the collection of all unions of finite subcollections of $\boldsymbol{G}_{(a,b]}$. Further, if V is in $\boldsymbol{F}_{(a,b]}$, then the collection $\mathfrak{C}_V$ of all components of V is a finite subset of $\boldsymbol{G}_{(a,b]}$. Suppose τ is a function of subintervals of $[a, b]$. We define τ^* as follows:

For each subinterval $[p, q]$ of $[a, b]$ we let $\tau'((p, q]) = \tau([p, q])$, and for each $\boldsymbol{F}_{(a,b]}$ we let

$$\tau^*(V) = \sum_{\mathfrak{C}_V} \tau'(I).$$

We have the following theorem which the reader can easily prove:

THEOREM 2.1. - *If $a < b$ and υ is a function of subintervals of $[a, b]$ $a \leq p < q \leq b$, then the « interval function integral »*

$$\int_{[p,q]} \upsilon(I)$$

exists iff the « set function integral » (with respect to $\boldsymbol{F}_{(a,b]}$)

$$\int_{(p,q]} \upsilon^*(I)$$

exists, in which case equality holds.

We now prove Theorem 2.2, as stated in the introduction.

PROOF OF THEOREM 2.2. - It is easy to see that the argument that 2) implies 1) can be formulated with only minor modifications to the corresponding argument in [1], by virtue of analogous theorems in [2].

We show that 1) implies 2). Suppose 1) is true and 2) is not true. There is a positive number c and a sequence of points $\{(x_1^{(v)}, \ldots, x_N^{(v)})\}_{v=0}^{\infty}$ of $[t_1, u_1]X \ldots X[t_N, u_N]$ such that

$$\sum_{k=1}^{N} |x_k^{(m)} - x_k^{(0)}| \to 0 \quad \text{as} \quad m \to \infty,$$

but for each positive integer m,

$$|\tau(x_1^{(m)}, \ldots, x_N^{(m)}) - \tau(x_1^{(0)}, \ldots, x_N^{(0)})| \geq c.$$

There is a sequence $\{H_k\}_{k=1}^{N}$ of functions of subintervals of [0, 1] such that if $[p, q]$ is a subinterval of [0, 1], then

$$\sum_{k=1}^{N} |H_k([p, q]) - x_k^{(m)}| = 0$$

if for some integer m, q is $1/m$; and

$$\sum_{k=1}^{N} |H_k([p, q]) - x_k^{(0)}| = 0$$

otherwise.

There is a function γ, defined on the subintervals of [0, 1], such that

$$\gamma([p, q]) = \begin{cases} 1 & \text{if} \quad p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\{(0, 1], \boldsymbol{F}_{(0,1]}, \gamma^*\}$ is a finitely additive measure space.

For each positive interger $k \leq N$, let $H_k'' = (H_k\gamma)^*/\gamma^*$. We see that for each such k and subdivision $\mathfrak{D}$ of (0, 1],

$$\sum_{\mathfrak{D}} H_k''(I)\gamma^*(I) = \sum_{\mathfrak{D}} \sum_{C_I} H_k'(J)\gamma'(J) = H_k'(J_{\mathfrak{D}}),$$

where $J_{\mathfrak{D}}$ is the «left most» J in $\cup_{\mathfrak{D}}\mathfrak{C}_I$, which implies

$$\int_{(0,1]} H_k''(I)\gamma^*(I) = x_k^{(0)}.$$

Now suppose $\mathbf{E}$ is a subdivision of $(0, 1]$.
There are refinements $\mathbf{E}_1$ and $\mathbf{E}_2$ of $\mathbf{E}$, each containing only elements of $\boldsymbol{G}_{(0,1]}$, such that for some integer w, $(0, 1/w]$ is in $\mathbf{E}_1$, and for no integer m is $(0, 1/m]$ in $\mathbf{E}_2$. We see that

$$\Big| \sum_{\mathbf{E}_1} \tau(\mathrm{H}_1''(I), \dots, \mathrm{H}_N''(I))\gamma^*(I) - \sum_{\mathbf{E}_2} \tau(\mathrm{H}_1'(I), \dots, \mathrm{H}_N'(I))\gamma^*(I) \Big| =$$

$$| \tau(x_1^{(w)}, \dots, x_N^{(w)}) - \tau(x_1^{(0)}, \dots, x_N^{(0)}) | \geq c,$$

so that

$$\int_{[0,1]} \tau(\mathrm{H}_1''(I), \dots, \mathrm{H}_N(I))\gamma^*(I)$$

does not exist, a contradiction.

Therefore 1) implies 2).

Therefore 1) and 2) are equivalent.

3. - Summable set functions.

In this section we discuss some previous results [2] concerning set function summability and prove a representation theorem.

Suppose $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space.

We let W_μ denote the set to which H belongs iff H is an element of $\mathfrak{p}_F^+$ such that for some number z and all nonnegative numbers K,

$$\int_U \min\{\mathrm{H}(I), K\}\mu(I)$$

exists and does not exceed z.

We observe that for each H in W_μ, the element $\boldsymbol{s}_\mu(\mathrm{H})$ of $\mathfrak{p}_F$ defined by

$$\boldsymbol{s}_\mu(\mathrm{H})(V) = \sup\left\{ \int_V \min\{\mathrm{H}(I), \mu(I) ; 0 < K \right\}$$

is in $\mathfrak{p}_{FA}^+$. We also observe that, trivially,

$$\int_U \left| \boldsymbol{s}_\mu(\mathrm{H})(I) - \int_I \min\{\mathrm{H}(J), K\}\mu(J) \right| \to 0, \ K \to \infty,$$

so that $\boldsymbol{s}_\mu(\mathrm{H})$ is absolutely continuous with respect to μ.

We state a previous theorem of the author [2].

THEOREM 3.A.1. - If Π is in W_μ and H is in $\mathfrak{p}_F^+$, then $\Pi + \mathrm{H}$ is in W_μ iff H is, in which case

$$\boldsymbol{s}_\mu(\Pi + \mathrm{H}) = \boldsymbol{s}_\mu(\Pi) + \boldsymbol{s}_\mu(\mathrm{H}).$$

We let W_μ^* denote the set to which H belongs iff H is « μ-summable », i.e., for some Π and Z, each in W_μ,

$$\mathrm{H} = \Pi - \mathrm{Z}.$$

It is easily shown in [2] that Theorem 3.A.1 implies that if each of Π, Π', Z and Z' is in W_μ and

$$\Pi - \mathrm{Z} = \Pi' - \mathrm{Z}',$$

then

$$\boldsymbol{s}_\mu(\Pi) - \boldsymbol{s}_\mu(\mathrm{Z}) = \boldsymbol{s}_\mu(\Pi') - \boldsymbol{s}_\mu(\mathrm{Z}').$$

This consistency enables us to define a « summability operator », $\boldsymbol{s}_\mu^*$ on W_μ^*, as follows: If H is in W_μ^*, then

$$\boldsymbol{s}_\mu^*(\mathrm{H}) = \boldsymbol{s}_\mu(\Pi) - \boldsymbol{s}_\mu(\mathrm{Z}),$$

where

$$\mathrm{H} = \Pi - \mathrm{Z},$$

and each of Π and Z is in W_μ.

We also easily see that $\boldsymbol{s}_\mu \subseteq \boldsymbol{s}_\mu^*$, so that we can write $\boldsymbol{s}_\mu$ for $\boldsymbol{s}_\mu^*$. It further follows that

$$W_\mu = W_\mu^* \cap \mathfrak{p}_F^+.$$

We note that if Υ is in W_μ^*, then $\boldsymbol{s}_\mu(\Upsilon)$ is absolutely continuous with respect to μ.

We now state a condensation of some results of [2] that we shall need in subsequent sections.

THEOREM 3.A.2. - Suppose each of X and Υ is in W_μ^* and c is a number. Then W_μ^* contains each of $\mathrm{X} + \Upsilon$, $c\mathrm{X}$, $\min\{\mathrm{X}, \Upsilon\}$, and $\max\{\mathrm{X}, \Upsilon\}$. Furthermore,

$$\boldsymbol{s}_\mu(\mathrm{X} + \Upsilon) = \boldsymbol{s}_\mu(\mathrm{X}) + \boldsymbol{s}_\mu(\Upsilon).$$

Also, if X is bounded, then for each V in $\boldsymbol{F}$,

$$\int_V \mathrm{X}(I)\mu(I) = \boldsymbol{s}_\mu(\mathrm{X})(V).$$

Now suppose Z is in $\mathfrak{p}_F$. We see that

$$Z = \max\{Z, \boldsymbol{O}\} + \min\{Z, \boldsymbol{O}\},$$

and that Z is in W_μ^* iff each of $\max\{Z, \boldsymbol{O}\}$ and $-\min\{Z, \boldsymbol{O}\}$ is in W_μ.
We now prove a representation theorem.

THEOREM 3.1. - If H is in $\mathfrak{p}_F$ and η is in $\mathfrak{p}_{FA}$, then the following two statements are equivalent:

1) H is in W_μ^* and $\eta = \boldsymbol{s}_\mu(\mathrm{H})$, and

2) if $p \leq 0 \leq q$, then

$$\int_U \min\{\max\{\mathrm{H}(I), p\}, q\}\mu(I)$$

exist and

$$\int_U \left| \eta(I) - \int_I \min\{\max\{\mathrm{H}(J), p\}, q\}\mu(J) \right| \to 0, \ \min\{-p, q\} \to \infty.$$

PROOF. - We observe that if $a \leq 0 \leq b$ and x is a number, then

$$\min\{\max\{x, 0\}, b\} - \min\{-\min\{x, 0\}, -a\} = \min\{\max\{x, a\}, b\}.$$

Now suppose 1) is true.
We see that each of $\max\{\mathrm{H}, \boldsymbol{O}\}$ and $-\min\{\mathrm{H}, \boldsymbol{O}\}$ is in W_μ, so that, if $p \leq 0 \leq q$, then by the above equality we have the following equality and consequent existence:

$$\int_U \min\{\max\{\mathrm{H}(I), 0\}, q\}\mu(I) - \int_U \min\{-\min\{\mathrm{H}(I), 0\}, -p\}\mu(I) =$$
$$\int_U \min\{\max\{\mathrm{H}(I), p\}, q\}\mu(I).$$

We also see that

$$\eta = \boldsymbol{s}_\mu(\mathrm{H}) = \boldsymbol{s}_\mu(\max\{\mathrm{H}, \boldsymbol{O}\}) - \boldsymbol{s}_\mu(-\min\{\mathrm{H}, \boldsymbol{O}\}),$$

so that

$$\int_U \left| \eta(I) - \int_I \min\{\max\{H(J), p\}, q\}\mu(J) \right| \leq$$

$$\int_U \left| s_\mu(\max\{H, \boldsymbol{O}\})(I) - \int_I \min\{\max\{H(J), 0\}, q\}\mu(J) \right| +$$

$$\int_U \left| s_\mu(-\min\{H, \boldsymbol{O}\})(I) - \int_I \min\{-\min\{H(J), 0\}, -p\}\mu(J) \right| \to 0$$

$$\min\{-p, q\} \to \infty.$$

Therefore 1) implies 2).

Now suppose 2) is true.

First, we see that if $0 \leq \min\{K, K'\}$ and V is in $\boldsymbol{F}$, then, since $0 \leq 0 \leq K$,

$$\int_V \min\{\max\{H(I), 0\}, K\}\mu(I)$$

exists, so that, since $-K' \leq 0 \leq K$, we have the following equality and consequent existence:

$$\int_V \min\{\max\{H(I), -K'\}, K\}\mu(I) - \int_V \min\{\max\{H(I), 0\}, K\}\mu(I) =$$

$$\int_V - \min\{ \quad \min\{H(I), 0\}, K'\}\mu(I).$$

Therefore

$$\int_U \left| \eta(I) - \left[\int_I \min\{\max\{H(J), 0\}, K\}\mu(J) - \right.\right.$$

$$\left.\left. \int_I \min\{-\min\{H(J), 0\}, K'\}\mu(J) \right] \right| =$$

$$\int_U \left| \eta(I) - \int_I \min\{\max\{H(J), -K'\}, K\}\mu(J) \right| \to 0, \ \min\{K', K\} \to \infty.$$

We see that there are numbers r and s such that $r \leq 0 \leq s$ and such that if $p \leq r$ and $s \leq q$, then

$$\int_U \left| \eta(I) - \left[\int_I \min \{ \max \{ H(J), 0 \}, q \} \mu(J) - \int_I \min \{ -\min \{ H(J), 0 \}, -p \} \mu(J) \right] \right| < 1,$$

so that

$$\int_U \left| \eta(I) - \left[\int_I \min \{ \max \{ H(J), 0 \}, q \} \mu(J) - \int_I \min \{ -\min \{ H(J), 0 \}, -r \} \mu(J) \right] \right| < 1$$

and

$$\int_U \left| \eta(I) - \left[\int_I \min \{ \max \{ H(J), 0 \}, s \} \mu(J) - \int_I \min \{ -\min \{ H(J), 0 \} - p \} \mu(J) \right] \right| < 1,$$

which implies that

$$\sup \left\{ \int_U \min \{ \max \{ H(I), 0 \}, K \} \mu(I) \,|\, 0 \leq K \right\} < \infty$$

and

$$\sup \left\{ \int_U \min \{ \min \{ H(I), 0 \}, K' \} \mu(I) \,|\, 0 \leq K' \right\} < \infty .$$

This implies that each of $\max \{ H, \boldsymbol{O} \}$ and $-\min \{ H, \boldsymbol{O} \}$ is in W_μ so that H is in W_μ^* and, since

$$\boldsymbol{s}_\mu(H) = \boldsymbol{s}_\mu(\max \{ H, \boldsymbol{O} \}) - \boldsymbol{s}_\mu(-\min \{ H, \boldsymbol{O} \}),$$

it follows from then end of the preceding paragraph that $\eta = \boldsymbol{s}_\mu(H)$.

Therefore 2) implies 1).

Therefore 1) and 2) are equivalent.

4. - An integrability theorem.

In this section we prove Theorem 4.1, as stated in the introduction.

PROOF OF THEOREM. - First, suppose 2) is true, $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space and $\{\mathrm{H}_k\}_{k=1}^{N}$ is a sequence of elements of W_μ^*.

Suppose $0 < c$. There is a number M such that $|\sigma(x_1, \ldots, x_N)| \leq M$ for all $(x_1, \ldots, x_N)$ in $\mathcal{E}^N$. By Theorem 3.1 there is a positive number T such that if k is a positive integer $\leq N$ and each of r, s, t and $u \geq T$, then

$$\int_U |\min\{\max\{\mathrm{H}_k(I), -r\}, s\} - \min\{\max\{\mathrm{H}_k(I), -t\}, u\}|\,\mu(I) < c/[8N(M+1)].$$

It follows that for each positive integer positive integer $k \leq N$ there is a subdivision $\mathfrak{D}_k$ of U such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}_k$, then

$$c/[8N(M+1)] < \sum_{\mathfrak{E}} |\min\{\max\{\mathrm{H}_k(I), -T-1\}, T+1\} -$$

$$\min\{\max\{\mathrm{H}_k(I), -T\}, T\}|\,\mu(I) \geq \sum_{\mathfrak{E}^*} \mu(I),$$

where $\mathfrak{E}^*$ is the set (if any) of all I in $\mathfrak{E}$ such thnt $\mathrm{H}_k(I) > T+1$ or $\mathrm{H}_k(I) < -T-1$.

Let $\mathrm{X}(I)$ denote $\mathrm{H}_1(I), \ldots, \mathrm{H}_N(I)$ and $\mathrm{Z}(I)$ denote $\min\{\max\{\mathrm{H}_1(I), -T-1\}, T+1\}, \ldots, \min\{\max\{\mathrm{H}_N(I), -T-1\}, T+1\}$.
There is a common refinement $\mathfrak{D}$ of all the above $\mathfrak{D}_k$. If $\mathfrak{E}$ is a refinement of $\mathfrak{D}$, then

$$\sum_{\mathfrak{E}} |\sigma(\mathrm{X}(I)) - \sigma(\mathrm{Z}(I))|\,\mu(I) = \sum_{\mathfrak{E}'} |\sigma(\mathrm{X}(I)) - \sigma(\mathrm{Z}(I))|\,\mu(I) \leq$$

$$2MNc/[8N(M+1)] = 2Mc/[8(M+1)] < c/4,$$

where $\mathfrak{E}'$ is the set (if any) of all I in $\mathfrak{E}$ such that for some positive integer $k \leq N$, $\mathrm{H}_k(I) > T+1$ or $\mathrm{H}_k(I) < -T-1$.

By Theorem 2.2, $\int_U \sigma(\mathrm{Z}(I))\mu(I)$ exists, so that there is a subdivision $\mathfrak{D}^*$ of U such that if $\mathfrak{E}$ is a refinement of $\mathfrak{D}^*$, then

$$\sum_{\mathfrak{E}} \left| \sigma(\mathrm{Z}(I))\mu(I) - \int_I \sigma(\mathrm{Z}(J))\mu(J) \right| < c/4.$$

There is a common refinement $\mathfrak{D}^{**}$ of $\mathfrak{D}$ and $\mathfrak{D}^*$.
If $\mathfrak{E}$ is a refinement of $\mathfrak{D}^{**}$, then

$$\left| \sum_{\mathbf{E}} \sigma(X(I))\mu(I) - \int_U \sigma(Z(J))\mu(J) \right| \leq$$

$$\sum_{\mathbf{E}} | \sigma(X(I)) - \sigma(Z(I)) | \mu(I) + \sum_{\mathbf{E}} \left| \sigma(Z(I))\mu(I) - \int_I \sigma(Z(J))\mu(J) \right| < c/4 + c/4.$$

Therefore, if each of $\mathbf{E}_1$ and $\mathbf{E}_2$ is a refinement of $\mathfrak{D}^{**}$, then

$$| \sum_{\mathbf{E}_1} \sigma(X(I))\mu(I) - \sum_{\mathbf{E}_2} \sigma(X(I))\mu(I) | < c/2 < c.$$

Therefore $\int_U \sigma(H_1(I), \ldots, H_N(I))\mu(I)$ exists, so that 2) implies 1).

We now that 1) implies 2). Suppose 1) is true. It is easy to see that the continuity of σ follows from Theorems 2.2 and 3.A.2.

We show σ is bounded. Suppose, on the contrary, that there is a sequence $\{ (x_1^{(m)}, \ldots, x_N^{(m)}) \}_{m=1}^{\infty}$ of points of $\mathcal{E}^N$ such that

$$| \sigma(x_1^{(m)}, \ldots, x_N^{(m)}) | \to \infty, \ m \to \infty.$$

For each positive integer $k \leq N$, there is a function Π_k defined on the subintervals of $[0, 1]$ by

$$\Pi_k([p, q]) \begin{cases} x_k^{(1/p)} \text{ if } p \neq 0 \text{ and } 1/p \text{ is a positive integer} \\ 0 \text{ otherwise} \end{cases}$$

There is a function, η, defined on the subintervals of $[0, 1]$ by $\eta([p, q]) = q - p$.

We see that $\{ (0, 1], \boldsymbol{F}_{(0,1]}, \eta^* \}$ is a finitely additive measure space. For each positive integer $k \leq N$, we let $\Pi_k'' = (\Pi_k \eta)^* / \eta^*$.
Suppose $0 < c$ and $0 < S$. There is a number w such that $0 < w < 1$ and $w < c/(4S)$. There is a subdivision $\mathfrak{D}$ of $(0, 1]$ containing only $(0, w]$ and elements of $\boldsymbol{G}_{(0,1]}$ such that if Q is the number of positive integers whose reciprocals $\geq w$, and $(p, q]$ is in $\mathfrak{D}$ and $w \leq p$, then $q - p < c/(4SQ)$. Suppose $\mathbf{E}$ is a refinement of $\mathfrak{D}$. We see that there are at most Q elements V of $\mathbf{E}$ included in $(w, 1]$ such that $\Pi_k''(V) \neq 0$. Therefore

$$\sum_{\mathbf{E}} \min \{ | \Pi_k''(V) |, \ S \} \eta^*(V) \leq Sc/(4S) + QSc/(4SQ) = c/2.$$

Therefore

$$\int_U \min \{ | \Pi_k''(V) |, \ S \} \eta^*(V) = 0,$$

which implies that

$$\int_U \min\{\max\{\Pi''(V),\ 0\},\ S\}\,\eta^*(V) = 0 = \int_U \min\{-\min\{\Pi_k''(V),\ 0\},\ S\}\,\eta^*(V),$$

so that Π_k'' is in $W_{\eta^*}^*$. Now suppose $1 < M$ and $\mathfrak{D}$ is a subdivision of $(0,\ 1]$. There is a refinement $\mathfrak{D}'$ of $\mathfrak{D}$ such that for some $w > 0$, $(0,\ w]$ is in $\mathfrak{D}'$. There is a positive integer Q such that if m is a positive integer $\geq Q$, then $w - 1/m > w/2$ and

$$|\sigma(x_1^{(m)},\ \ldots,\ x_N^{(m)})| > M(1 + T)(2/w),$$

where

$$T = |\sigma(0,\ \ldots,\ 0)| + \sum_{\mathfrak{D}^*} |\sigma(\Pi_1''(I),\ \ldots,\ \Pi_N''(I))|\,\eta^*(I),$$

and

$$\mathfrak{D}^* = \{I \mid I \text{ in } \mathfrak{D}',\ w < x \text{ for all } x \text{ in } I\}.$$

Let $\mathfrak{E} = \{(0,\ 1/Q],\ (1/Q,\ w]\} \cup \mathfrak{D}^*$. $\mathfrak{E}$ is a refinement of $\mathfrak{D}'$ and

$$\Big|\sum_{\mathfrak{E}} \sigma(\Pi_1''(I),\ \ldots,\ \Pi_N''(I))\eta^*(I)\Big| \geq |\sigma(x_N^{(1/Q)},\ \ldots,\ x_N^{(1/Q)})|\,(w - 1/Q) -$$

$$\Big[|\sigma(0,\ \ldots,\ 0)|\,(1/Q) + \sum_{\mathfrak{D}^*} |\sigma(\Pi_1'(I),\ \ldots,\ \Pi_N''(I))|\,\eta^*(I)\Big] >$$

$$M(1 + T)(2/w)(w - 1/Q) - T > M(1 + T)(2/w)(w/2) - T =$$

$$M + MT - T > M.$$

Therefore

$$\int_U \sigma(\Pi_1''(I),\ \ldots,\ \Pi_N''(I))\eta^*(I)$$

does not exist, a contradiction. Therefore 1) implies 2).

Therefore 1) and 2) are equivalent.

5. - A summability theorem.

In this section we prove Theorem 5.1, as stated in the introduction.

We begin with a lemma.

LEMMA 5.1. - If $\{U,\ \boldsymbol{F},\ \mu\}$ is a finitely additive measure space and H is in $\mathfrak{p}_F$, then the following two statements are equivalent:

1) H is μ-summable, and

2) $|\mathrm{H}|$ is μ-summable, and if $p \leq 0 \leq q$, then $\int_U \min\{\max\{\mathrm{H}(I), p\}, q\}\mu(I)$ exists.

PROOF. - From the theorems of section 3 it is obvious that 1) implies 2). Suppose 2) is true and $0 \leq K$.

$$\int_U \min\{\max\{\mathrm{H}(I), 0\}, K\}\mu(I) \leq \int_U \min\{|\mathrm{H}(I)|, K\}\mu(I),$$

which implies that $\max\{\mathrm{H}, \boldsymbol{O}\}$ is μ-summable. Further,

$$\min\{-\min\{\mathrm{H}, \boldsymbol{O}\}, \boldsymbol{K}\} = \min\{\max\{-\mathrm{H}, \boldsymbol{O}\}, \boldsymbol{K}\} = \max\{\min\{-\mathrm{H}, \boldsymbol{K}\}, \boldsymbol{O}\} =$$
$$\max\{-\max\{\mathrm{H}, -\boldsymbol{K}\}, \boldsymbol{O}\} = -\min\{\max\{\mathrm{H}, -\boldsymbol{K}\}, \boldsymbol{O}\}.$$

Now, we see that

$$\int_U -\min\{\max\{\mathrm{H}(I), -K\}, 0\}\mu(I)$$

exists, so that

$$\int_U \min\{-\min\{\mathrm{H}(I), 0\}, K\}\mu(I)$$

exists. Also

$$\int_U \min\{-\min\{\mathrm{H}(I), 0\}, K\}\mu(I) \leq \int_U \min\{|\mathrm{H}(I)|, K\}\mu(I),$$

which implies that $-\min\{\mathrm{H}, \boldsymbol{O}\}$ is μ-summable, so that $\min\{\mathrm{H}, \boldsymbol{O}\}$ is μ-summable. Since $\mathrm{H} = \max\{\mathrm{H}, \boldsymbol{O}\} + \min\{\mathrm{H}, \boldsymbol{O}\}$, it follows that H is μ-summable. Therefore 2) implies 1).

Therefore 1) and 2) are equivalent.

PROOF OF THEOREM 5.1. - Suppose 2) is true and $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space and $\{\mathrm{H}_k\}_{k=1}^N$ is a sequence of μ-summable elements of $\mathfrak{p}_F$. Let $\mathrm{X}(I)$ denote $\mathrm{H}_1(I), \ldots, \mathrm{H}_N(I)$. If $p \leq 0 \leq q$ and $0 \leq K$, then the functions $\min\{\max\{\Gamma, \boldsymbol{p}\}, \boldsymbol{q}\}$ and $\min\{|\Gamma|, \boldsymbol{K}\}$ are continuous and bounded, so that by Theorem 4.1, each of

$$\int_U \min\{\max\{\Gamma(\mathrm{X}(I)), p\}, q\}\mu(I) \text{ and } \int_U \min\{|\Gamma(\mathrm{X}(I))|, K\}\mu(I)$$

exists. Now, there are nonnegative numbers M and M' such that if $(x_1, \ldots, x_N)$ is in $\mathcal{E}^N$, then

$$|\Gamma(x_1, \ldots, x_N)| \leq \begin{cases} M \text{ if } \sum_{k=1}^{N} |x_k| < 1 \\ M' \sum_{k=1}^{N} |x_k| \text{ otherwise} \end{cases}$$

$M' \sum_{k-1}^{N} |\mathrm{H}_k|$ is μ-summable. If $\mathfrak{D}$ is a subdivision of U and $0 \leq K$, letting $\mathfrak{D}^* = \{I \mid I \text{ in } \mathfrak{D}, \sum_{k=1}^{N} |\mathrm{H}_k(I)| < 1\}$ (if any), we have

$$\sum_{\mathfrak{D}} \min\{|\Gamma(\mathrm{X}(I))|, K\}\mu(I) = \sum_{\mathfrak{D}^*} \min\{|\Gamma(\mathrm{X}(I))|, K\}\mu(I) +$$

$$\sum_{\mathfrak{D}-\mathfrak{D}^*} \min\{|\Gamma(\mathrm{X}(I))|, K\}\mu(I) \leq M \sum_{\mathfrak{D}^*} \mu(I) +$$

$$\sum_{\mathfrak{D}-\mathfrak{D}^*} \min\{M' \sum_{k=1}^{N} |\mathrm{H}_k(I)|, K\}\mu(I) \leq M\mu(U) +$$

$$\sum_{\mathfrak{D}} \min\{M' \sum_{k=1}^{N} |\mathrm{H}_k(I)|, K\}\mu(I).$$

This implies that

$$\int_U \min\{\Gamma(\mathrm{X}(I))|, K\}\mu(I) \leq M\mu(U) + \int_U \min\{M' \sum_{k=1}^{N} \mathrm{H}_k(I)|, K\}\mu(I),$$

which implies that $|\Gamma(\mathrm{H}_1, \ldots, \mathrm{H}_N)|$ is μ-summable. Therefore, by Lemma 5.1. 5.1, $\Gamma(\mathrm{H}_1, \ldots, \mathrm{H}_N)$ is μ-summable. Therefore 2) implies 1).

Now suppose 1) is true.

Suppose $p \leq 0 \leq q$. If $\{U, \boldsymbol{F}, \mu\}$ is a measure space and $\{\mathrm{H}_k\}_{k=1}^{N}$ is a sequence of μ- summable elements of $\mathfrak{p}_F$, then

$$\int_U \min\{\max\{\Gamma(\mathrm{H}_1(I), \ldots, \mathrm{H}_N(I)), p\}, q\}\mu(I)$$

exists. Theorem 4.1 implies that $\min\{\max\{\Gamma, \boldsymbol{p}\}, \boldsymbol{q}\}$ is continuous. We easily see that the continuity of $\min\{\max\{\Gamma, \boldsymbol{p}\}, \boldsymbol{q}\}$ for all p and q such that $p \leq 0 \leq q$ implies that Γ is continuos.

Now suppose that

$$\{|\Gamma(x_1, .., x_N)|/[\sum_{k=1}^{N} |x_k|] \mid 1 \leq \sum_{k=1}^{N} |x_k|\}$$

is not bounded, so that there is a sequence $\{(x_1^{(w)}, \ldots, x_N^{(w)})\}_{w=1}^{\infty}$ of points of $\mathcal{E}^N$ such that

$$1 \leq \sum_{k=1}^{N} |x_k^{(w)}|$$

and

$$w^2 \leq |\Gamma(x_1^{(w)}, \ldots, x_N^{(w)})| / [\sum_{k=1}^{N} |x_k^{(w)}|]$$

for all positive integers w. For each positive integer w, let

$$a_w = [w^2 \sum_{k=1}^{N} |x_k^{(w)}|]^{-1}.$$

We see that

$$\sum_{w=1}^{\infty} a_w < \infty.$$

Further,

$$\sum_{w=1}^{\infty} \{\sum_{k=1}^{N} |x_k^{(w)}|\} a_w = \sum_{w=1}^{\infty} w^{-2}.$$

Also

$$|\Gamma(x_1^{(w)}, \ldots, x_N^{(w)})| a_w \geq 1$$

for all positive integers w. Set σ be a function defined on $[0, 1]$ by

$$\sigma(x) = \begin{cases} \sum_{w=1}^{n} a_w \text{ if } (n-1)/n < x \leq n/(n+1), \\ \sum_{w=1}^{\infty} a_w \text{ if } x = 1, \\ 0 \text{ if } x = 0. \end{cases}$$

Let, for each subinterval $[x, y]$ of $[0, 1]$, $\tau([x, y]) = \sigma(y) - \sigma(x)$, and for each positive integer $k \leq N$,

$$H_k([x, y]) = \begin{cases} x_k^{(w)} \text{ if } x = (w-1)/w \text{ and } y < w/(w+1), \\ 0 \text{ otherwise.} \end{cases}$$

$\{(0, 1], \boldsymbol{F}_{(0,1]}, \tau^*\}$ is a finitely additive measure space. For each positive integer $k \leq N$, let $H_k'' = (H_k\tau)^*/\tau^*$. Suppose k is a positive integer $\leq N$. We show that H_k'' is τ^*-summable. We observe that if n is a positive integer and I is in a subdivision of $((n-1)/n, n/(n+1)]$, then

$$\mathrm{H}_k''(I)\tau^*(I) = (\mathrm{H}_k\tau^*)(I) = \sum_{\mathfrak{C}_I} \mathrm{H}_k'(J)\tau'(J),$$

which is $x_k^{(n)}a_n$ if I includes $((n-1)/n,\ y]$ for some y, and is 0 otherwise; and that $\tau^*(I) = \sum_{\mathfrak{C}_I} \tau'(J)$, which is a_n if I includes $((n-1)/n,\ y]$ for some y, and is 0 otherwive. We see that

$$\sum_{w=1} |x_k^{(w)}|\, a_w < \infty.$$

Suppose $p \leq 0 \leq q$ and $0 < c$. From the above we see that

$$\sum_{w=1}^{\infty} |\min\{\max\{x_k^{(w)},\ p\},\ q\}|\, a_w < \infty.$$

There is a positive integer $Q > 1$ such that

$$\sum_{w=Q}^{\infty} |x_k^{(w)}|\, a_w < c/4.$$

Let $\mathfrak{D}$ denote

$$\{((n-1)/n,\ n/(n+1)] \mid n \text{ a positive integer} \leq Q+1\} \cup \{((Q+1)/(Q+2),\ 1]\}.$$

$\mathfrak{D}$ is a subdivision of $(0,\ 1]$. Suppose $\mathfrak{E}$ is a refinement of $\mathfrak{D}$. Let

$$\mathfrak{E}_Q = \{I \mid I \text{ in } \mathfrak{E},\ x \leq (Q+1)/(Q+2) \text{ for all } x \text{ in } I\}.$$

We see that

$$\sum_{\mathfrak{E}_Q} \min\{\max\{\mathrm{H}_k''(I),\ p\},\ q\}\tau^*(I) =$$

$$\sum_{\mathfrak{E}_Q} \min\{\max\{\mathrm{H}_k''(I)\tau^*(I),\ p\tau^*(I)\},\ q\tau^*(I)\} =$$

$$\sum_{w=1}^{Q+1} \min\{\max\{x_k^{(w)}a_w,\ pa_w\},\ qa_w\} = \sum_{w=1}^{Q+1} \min\{\max\{x_k^{(w)},\ p\},\ q\}\, a_w.$$

Therefore

$$\left|\sum_{w=1}^{\infty} \min\{\max\{x_k^{(w)},\ p\},\ q\}\, a_w - \sum_{\mathfrak{E}} \min\{\max\{\mathrm{H}_k''(I),\ p\},\ q\}\tau^*(I)\right| =$$

$$\Big|\sum_{w=Q+2}^{\infty} \min\{\max\{x_k^{(w)}a_w,\ pa_w\},\ qa_w\} -$$

$$\sum_{\mathfrak{D}-\mathfrak{E}_Q} \min \{ \max \{ H_k''(I)\tau^*(I),\ p\tau^*(I) \},\ q\tau^*(I) \} | \leq$$

$$2 \sum_{w=Q+2}^{\infty} | x_k^{(w)} | a_w < c/2.$$

Therefore

$$\int_{(0,1]} \min \{ \max \{ H_k''(I),\ p \},\ q \} \tau^*(I)$$

exists and is

$$\sum_{w=1}^{\infty} \min \{ \max \{ x_k^{(w)},\ p \},\ q \} a_w .$$

By observations and proceedures similar to the proceding ones, we see that if $0 \leq K$, then

$$\int_{(0,1]} \min \{ | H_k''(I) |,\ K \} \tau^*(I)$$

exists and is

$$\sum_{w=1}^{\infty} \min \{ | x_k^{(w)} |,\ K \} a_w ,$$

so that $| H_k'' |$ is τ^*-summable. Therefore, by Lemma 5.1, H_k'' is τ^*-summable.

It therefore follows, since we are assuming 1), that $\Gamma(H_1', \ldots, H_N'')$ is τ^*-summable, which implies that $| \Gamma(H_1'', \ldots, H_N'') |$ is τ^*-summable. However, suppose $0 < M$. There is a positive integer $Q > M$. There is a $K \geq 0$ such that $| \Gamma(x_1^{(w)}, \ldots, x_N^{(w)} | \leq K$ for all positive integers $w \leq Q$. Suppose $\mathfrak{D}$ is a subdivision of U. There is refinement $\mathfrak{E}$ of $\mathfrak{D}$ and $\{(x, y] | (x, y] = ((n-1)/n, n/(n+1)]$ for some positive integer $n \leq Q \} \cup \{ (Q/(Q+1), 1] \}$, such that $\mathfrak{E}$ contains only elements of $\boldsymbol{G}_{(0,1]}$. Let

$\mathfrak{E}^* = \{ I | I$ in $\mathfrak{E}$, $I = ((n-1)/n, y]$ for some y and positive integer $n \leq Q \}$.

Now

$$\sum \min \{ | \Gamma(H_1''(I), \ldots, H_N'(I)) |,\ K \} \tau^*(I) \geq$$

$$\sum_{\mathfrak{E}^*} \min \{ | \Gamma(H_1''(I), \ldots, H_N''(I)) |,\ K \} \tau^*(I) =$$

$$\sum_{w=1}^{Q} \min \{ | \Gamma(x_1^{(w)}, \ldots, x_N^{(w)}) |,\ K \} a_w =$$

$$\sum_{w=1}^{Q} | \Gamma(x_1^{(w)}, \ldots, x_N^{(w)}) | a_w \geq Q > M.$$

This implies that

$$\int_{[0,1]} \min\{|\Gamma(H_1''(I), \ldots, H_N''(I))|, K\}\tau^*(I) \geq M.$$

Therefore $|\Gamma(H_1'', \ldots, H_N'')|$ is not τ^*-summable, a contradiction. Therefore 1) implies 2).

Therefore 1) and 2) are equivalent.

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