Calderón algebras of smoothing operators (*)

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Summary. - *Certain classes of integral operators between generalized Sobolev spaces are shown to form algebras, endowed with an approximate functional calculus, having properties similar to those of pseudo-differential operators.*

Introduction.

Over the last few years CALDERÓN (see [1] and [2]) has developed an algebra of integral operators which refines the algebra of singular integral operators in [3] without imposing unduly restrictive assumptions on the regularity of the symbols. The aim of this paper is to discuss this algebra of operators from a different point of view, providing new and simpler proofs of the fundamental properties and describing some additional results which are indispensable for the study of these operators on manifolds.

The paper is divided into three sections. In the first, we give the definition of the operators and of their symbols, and, after establishing a few preliminary properties, we state the main results. Section II is devoted to the proofs of these main results. Finally, in sections III, we present the additional results mentioned above.

Our notation is fairly standard. We denote by $x=(x_1, \ldots, x_n)$, y, z points of Euclidean space E^n, $n \geq 2$, and by $\alpha=(\alpha_1, \ldots, \alpha_n)$ and β the multi-indices. Points in the dual Euclidean space are denoted by $\zeta=(\zeta_1, \ldots, \zeta_n)$ and $<z, \zeta>=z_1\zeta_1+\ldots+z_n\zeta_n$ is the dual pairing. As usual, $|x|=(x_1^2+\ldots+x_n^2)^{1/2}$, $|\alpha|=\alpha_1+\ldots+\alpha_n$, $\alpha!=\alpha_1!\ldots\alpha_n!$ and $D=(2\pi i)^{-1}(\partial/\partial x)$, where $(\partial/\partial x)$ denotes the gradientoperator on E^n. We write $S^{n-1}=\{\zeta : |\zeta|=1\}$, $E_0^n=E^n-\{0\}$, and $\partial_x^\alpha f=(\partial/\partial x)^\alpha f$. The space of smooth, rapidly decreasing functions

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is denoted by $\mathcal{S}$ and its dual, the space of temperate distributions, by $\mathcal{S}^*$. The Fourier transform of a function f is denoted by $\hat{f}$ or $\mathcal{F}(f)$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are respectively the FOURIER transformation and its inverse. When taking the (inverse) FOURIER transform only with respect to certain variables, we use abbreviations of the form

$$\mathfrak{F}_\zeta^{-1}[p(x, \zeta)](z) = \int e^{2\pi i < z, \zeta >} p(x, \zeta) d\zeta.$$

Finally, with $1 < p < \infty$ and s real, we denote by $L_s^p = L_s^p(E^n)$ the generalized SOBOLEV spaces (e.g. [4], Chapter II) and by $\| f \|_{p,s}$ the norm of f in L_s^p. When $p = 2$, $L_s^2 = H^s$ and the norm is simply denoted by $\| \quad \|_s$. If $s = k$ is a positive integer, L_s^p coincides with the SOBOLEV space $L_k^p(E^n) = \{ f : D^\alpha f = L^p, 0 \leq |\alpha| \leq k \}$, where L^p is the usual LEBESGUE space and the derivatives are taken in the sense of distributions.

I.

Let us recall some convenient terminology. A function $\theta \in C^\infty(E^n)$ is called a *patch-function* if $\theta(\zeta) = 0$ in a neighborhood of the origin and $\theta(\zeta) = 1$ on a neighborhood of infinity. Let $[\rho]$ denote the integral part of a real number ρ, and let $m \geq 1$ be an integer.

In what follows, we shall always have $0 \leq \rho < m$.

DEFINITION 1.0. - A function $a(x, \zeta)$, on $E^n \times E_0^n$, is said to be a B_m^∞ homogeneous symbol of degree-ρ if, for all $\lambda > 0$,

$$a(x, \lambda\zeta) = \lambda^{-\rho} a(x, \zeta)$$

and if, for every β and each α with $0 \leq |\alpha| \leq 2m - [\rho]$, the functions $\partial_x^\alpha \partial_\zeta^\beta a(x, \zeta)$ are continuous on $E^n \times E_0^n$ and bounded on $E^n \times S^{n-1}$.

For brevity we denote by $\mathfrak{L}(p)$ the algebra of all bounded linear operators on L^p, $1 < p < \infty$. Let us define certain classes $\mathfrak{I}_m$ in this algebra.

DEFINITION 1.1. - A linear operator $S : \mathcal{S} \to L_m^p$ belongs to $\mathfrak{I}_m$ if the operators S, $(\partial/\partial x)^\alpha S$ and $S(\partial/\partial x)^\alpha$ are bounded in L^p norm for all α with $|\alpha| = m$.

Clearly, we can regard $\mathfrak{I}_m$ as a subset of $\mathfrak{L}(p)$ by extending (by continuity) each S in $\mathfrak{I}_m$ to an element of $\mathfrak{L}(p)$. We set $\mathfrak{I}_0 = \mathfrak{L}(p)$.

LEMMA 1.2. - Operators in $\mathfrak{I}_m$ can be extended to bounded operators from L_s^p to L_{s+m}^p for all real s such that $-m \leq s \leq 0$. Conversely, every linear operator $S : \mathcal{S} \to L_m^p$ with this property belongs to $\mathfrak{I}_m$.

PROOF. - Let $S \in \mathfrak{I}_m$ and $s = 0$. Then, as is well known, the boundedness in L^p norm of the operators S and $(\partial/\partial x)^\alpha S$, for all $|\alpha| = m$, implies that S extends by continuity to a bounded operator from L^p to L^p_m. Now, we let $s = -m$ and we recall that L^p_{-m} can be also characterized as the space of all distributions of the form

$$f_0 + \sum_{|\alpha|=m} (\partial/\partial x)^\alpha f_\alpha$$

where f_0 and the f_α are functions in L^p, endowed with the corresponding norm. Hence, the boundedness in L^p norm of the operators S and $S(\partial/\partial x)^\alpha$, far all $|\alpha| = m$, implies that S also extends to a bounded operator from L^p_{-m} to L^p. Therefore, by interpolation, S can be extended to a bounded operator from L^p_s to L^p_{s+m} for all real s such that $-m \leq s \leq 0$.

The converse follows easily from the fact that, for every α, the linear map $(\partial/\partial x)^\alpha : L^p_t \to L^p_{t-|\alpha|}$ is continuos for $1 < p < \infty$ and every real t. Q.E.D.

The CALDERÓN algebra $\mathcal{S}_m$ of integral operators is defined as follows.

DEFINITION 1.3. - For an integer $m \geq 1$, we denote by $\mathcal{S}_m$ the class of linear operators which are finite sums of the form

$$A = [\Sigma A_j] + S \tag{1.0}$$

where $S \in \mathfrak{I}_m$ and, for all $f \in \mathcal{S}$,

$$[A_j f](x) = \int e^{2\pi i \langle x, \zeta \rangle} p_j(x, \zeta) \hat{f}(\zeta) d\zeta \tag{1.1}$$

where, for some patch-function θ,

$$p_j(x, \zeta) = a_j(x, \zeta)\theta(\zeta) \tag{1.2}$$

and $a_j(x, \zeta)$ is a B^∞_m homogeneous symbol of degree $-\rho_j$, with $0 \leq \rho_j < \rho_{j+1} < m$

The equivalence of this definition with the one given by CALDERÓN is an immediate consequence of the following Lemma the proof of which, as given in [1], applies unchanged to our L^p setting.

LEMMA 1.4. - If two operators in $\mathcal{S}_m$ have corresponding functions $p_j(x, \zeta)$ which coincide for all (x, ζ) with $|\zeta| > c$, then their difference belongs to $\mathfrak{I}_m$.

For any real number s, we denote by I^s the operator given by

$$[I^s f]^\frown = d(\zeta)^{-s} \hat{f}$$

where $d \in C^\infty(E^n)$ is a strictly positive radial function which coincides with $|\zeta|$ on $|\zeta| \geq 1$. See [2], [5]; the basic properties of I^s are also discussed in Chapter II of [4], where this operator was denoted by J^s). We recall that, for $1 < p < \infty$ and any real t, $I^s : L^p_t \to L^p_{t+s}$ is an isometric isomorphism.

LEMMA 1.5. - Let A_j be an operator of the form (1.1)
Then:

(a) A_j can be extended to a bounded operator from L^p_s to L^p_{s+t} provided that $-m \leq s \leq s+t \leq m$ and $t \leq \rho_j$.

(b) If $t > \rho_j$, no such extension exists unless the corresponding symbol a_j in (1.2) is identically zero.

PROOF. - (a) It follows from Definition 1.3 and Lemma 1.4 that we can represent A_j in the form

$$A_j = A_0 I^{\rho_j} + S \tag{1.3}$$

where $S \in \mathfrak{J}_m$, and hence by Lemma 1.2 has the desired continuity property, and where A_0 is given by the principal value integral

$$[A_0 f](x) = p.\, v. \int e^{2\pi i < x,\, \zeta >} a_0(x,\, \zeta) \widehat{f}(\zeta) d\zeta, \qquad f \in \mathcal{S},$$

with $a_0(x, \zeta)$ a homogeneous symbol of degree zero. Consequently, by a well know result of CALDERÓN and ZYGMUND ([3], or Chapter IV of [4]),

$$[A_0 f](x) = a(x) f(x) + p.v. \int k(x,\, x-y) f(y) dy$$

is a singular integral operator which gives a bounded linear map of L^p_k into itself, for every integer k with $|k| \leq m$ and $1 < p < \infty$. Moreover, letting

$$M_{\alpha,\,\beta} = \sup |\partial_x^\alpha \partial_\zeta^\beta a_0(x,\, \zeta)|, \qquad \text{over all } x \in E^n \text{ and } |\zeta| = 1,$$

and

$$\| A_0 \|_m = \max \{ M_{\alpha,\,\beta} : 0 \leq |\alpha| \leq m \text{ and } 0 \leq |\beta| \leq 2n \}$$

we have that

$$\| A_0 f \|_{p,\,k} \leq C \| A_0 \|_m \| f \|_{p,\,k} \tag{1.4}$$

for all f in L^p_k.

Now, by interpolation, A_0 maps L^p_s continuously into itself for all real s with $|s| \leq m$. Therefore, combining this result with the continuity property of I^{ρ_j}, the desired conclusion follows directly from (1.3).

(b) It suffices to consider the L^2 case. We write p_j in the form $p_j(x, \zeta) = a_0(x, \zeta)\theta(\zeta)|\zeta|^{-\rho_j}$, where a_0 is a homogeneous symbol of degree zero and θ is a patch-function. If $\tau \in C_0^\infty(E^n)$, $0 \leq \tau(\zeta) \leq 1$, and $\tau(\zeta) \equiv 1$ on a neighborhood of the set where $\theta(\zeta) = 0$, then, for all $\zeta \in E^n$, the function $\beta(\zeta) = \tau(\zeta) + \theta(\zeta)|\zeta|^{\rho_j}$ satisfies estimates

$$C_1 d(\zeta)^{\rho_j} \leq |\beta(\zeta)| \leq C_2 d(\zeta)^{\rho_j}$$

and hence the operator $L_j : \mathcal{S} \to \mathcal{S}$. given by $[L_j f]^\frown = \beta \widehat{f}$, extends to a linear isomorphism from L_s^2 to $L_{s-\rho_j}^2$, for all real s.

Now, $A_j L_j = T_0 + S$ where

$$[T_0 f](x) = \int e^{2\pi i <x, \zeta>} a_0(x, \zeta)\theta^2(\zeta)\widehat{f}(\zeta) d\zeta$$

and $S \in \mathfrak{J}_m$ since $\tau(\zeta)$ has bounded support. If we suppose that, for some $\varepsilon > 0$ with $\rho_j + \varepsilon < m$, $\| A_j u \|_0 \leq C \| u \|_{-\rho_j - \varepsilon}$ for all $u \in \mathcal{S}$, then, with $u = L_j f$, we would have that

$$\| A_j L_j f \|_0 \leq C \| L_j f \|_{-\rho_j - \varepsilon} \leq C' \| f \|_{-\varepsilon}$$

and hence

$$\| T_0 f \|_0 \leq C \| f \|_{-\varepsilon}$$

for all $f \in \mathcal{S}$. However, it is an immediate consequence of a well known Lemma of GOHBERG (e.g.[7]) that this last inequality cannot hold for any $\varepsilon > 0$ unless $a_0(x, \zeta)$, and hence $a_j(x, \zeta)$, vanishes identically. Q.E.D.

REMARKS 1.6. - (a) Any operator A_j of the form (1.1) belongs to $\mathfrak{J}_k$ for every integer k such that $0 \leq k \leq [\rho_j]$. This follows immediately from Lemmas 1.5 (a) and 1.2.

(b) For every integer $m \geq k \geq 1$, the class $\mathcal{S}_m$ is contained in $\mathcal{S}_k$, and every operator in $\mathcal{S}_m$ extends to a bounded operator on L_s^p, for $1 < p < \infty$ and $0 \leq |s| \leq m$. In fact, with $m > k$ and $A = \Sigma A_j + S$ in $\mathcal{S}_m$, if the homogeneous symbol a_j associated to A_j is of degree $-\rho_j$ with $\rho_j \geq k$, then, by 1.6(a), the corresponding operator A_j belongs to $\mathfrak{J}_k$. Thus $A \in \mathcal{S}_k$. Then, Lemma 1.5 (a) gives the remaining conclusion.

(c) Lemma 1.5 (b) implies, as is easily seen, that for operators A in $\mathcal{S}_m$ the expressions (1.0) are uniquely determined. In fact, if $A = 0$ in (1.0), then all the $A_j = 0$, and hence $S = 0$, because the corresponding homogeneous symbols $a_j(x, \zeta)$ in (1.2) must be identically zero. Consequently, on $\mathcal{S}_m$, we have a well-defined linear map $A \to \sigma(A)$, given by

$$\sigma(A) = \Sigma\, a_j(x, \zeta), \qquad 0 \leq \rho_j < \rho_{j+1} < m, \tag{1.5}$$

with null space equal to $\mathfrak{J}_m$. Conversely, each function σ of the form (1.5) determines an operator $A \in \mathfrak{S}_m$, unique modulo $\mathfrak{J}_m$, such that $\sigma(A) = \sigma$.

DEFINITION 1.7. - The function $\sigma(A)$ is called *the* (full) *symbol* of the operator A in $\mathfrak{S}_m$. The first term $a_j(x, \zeta)$ in (1.5) which is not identically equal to zero is called the *principal symbol* of A, and A is said to be a *smoothing operator* of degree ρ_j, $0 \leq \rho_j < m$, (and class $\mathfrak{S}_m$) with *principal part* A_j.

Of course, the principal part of an operator A in $\mathfrak{S}_m$ is only uniquely defined modulo $\mathfrak{J}_m$. If we denote by Σ_m the vector space of all finite ordered sums of the form (1.5), then Remark 1.6 (c) states that the symbol map, $\sigma : \mathfrak{S}_m \longrightarrow \Sigma_m$, is a well-defined linear map for which

$$\{0\} \longrightarrow \mathfrak{J}_m \xrightarrow{i} \mathfrak{S}_m \xrightarrow{\sigma} \Sigma_m \longrightarrow \{0\}$$

is an exact sequence, where $i : \mathfrak{J}_m \longrightarrow \mathfrak{S}_m$ is the inclusion map.

As it is done for pseudo-differential operators, we can define a «product» and an «adjoint» in Σ_m by means of the formulas:

$$\sigma(A) \circ \sigma(B) = \Sigma(1/\alpha!)(2\pi i)^{-|\alpha|}[\partial_\zeta^\alpha \sigma(A)][\partial_x^\alpha \sigma(B)] \tag{1.6}$$

$$\sigma(A)^{\#} = \Sigma(1/\alpha!)(2\pi i)^{-|\alpha|}[\partial_\zeta^\alpha \partial_x^\alpha \overline{\sigma(A)}] \tag{1.7}$$

where these (finite, ordered) sums extend to all terms whose degree of homogeneity in ζ is $> -m$. In other words, the formal sums $\sum\limits_{|\alpha| \geq 0}$ are reduced modulo terms homogeneous of degree $\leq -m$.

We need not verify a-priori, as it was done in [1], that with respect to these operations Σ_m becomes an associative star-algebra. Since the homomorphic image of a star-algebra is again such an algebra, this result will be a direct consequence of Theorem 1.8 below. On the other hand, in the style of [3], we can define in $\mathfrak{S}_m$ a *pseudo-product* $A \circ B$ and a *pseudo-adjoint* $A^{\#}$ given by

$$\sigma(A \circ B) = \sigma(A) \circ \sigma(B) \tag{1.8}$$

$$\sigma(A^{\#}) = \sigma(A)^{\#}. \tag{1.9}$$

Again, we should note that $A \circ B$ and $A^{\#}$ are uniquely defined modulo $\mathfrak{J}_m$.

THEOREM 1.8 - For any A and B in $\mathfrak{S}_m$, the operators $AB - A \circ B$ and $A^* - A^{\#}$ belong to $\mathfrak{J}_m$ and hence AB and A^* are again in $\mathfrak{S}_m$ with

$$\sigma(AB) = \sigma(A) \circ \sigma(B) \tag{1.8'}$$

$$\sigma(A^*) = \sigma(A)^{\#}. \tag{1.9'}$$

COROLLARY 1.9 - The class $\mathcal{S}_m$ is a self-adjoint associative algebra of operators. The map $\sigma : \mathcal{S}_m \longrightarrow \Sigma_m$ is a star-algebra homomorphism with kernel $\mathfrak{J}_m$. Moreover, for each integer k with $0 < k \leq m$, $\mathcal{S}_m$ is a subalgebra of $\mathcal{S}_k$. $\mathfrak{J}_m$ is a 2-sided self-adjoint ideal in $\mathcal{S}_m$.

The proof of Theorem 1.8 is given in the next section. Corollary 1.9 is is a direct consequence of 1.8, Lemma 1.2 and Remark 1.6 (a).

II.

A smooth function φ is said to be a *cut-off function* if $1 - \varphi$ is a patch-function. Given a positive integer r, we say that a function $b(x, z)$, on $E^n \times E_0^n$, belongs to $B_r^\infty(E^n \times S^{n-1})$ if for every β and for each α with $0 \leq |\alpha| \leq r$, the functions $\partial_z^\beta \partial_x^\alpha b(x, z)$ are continuous in $E^n \times E_0^n$ and bounded on $E^n \times S^{n-1}$. For instance, a B_m^∞ symbol of degree-ρ belongs to $B_r^\infty(E^n \times S^{n-1})$ with $r = 2m - [\rho]$. Throughout this section, we fix $r = 2m - [\rho]$, where $0 \leq \rho < m$ as always.

PROPOSITION 2.0. - Let $A = A_j$ be an operator of the form (1.1) with, $p_j = p$ and $\rho_j = \rho > 0$. Then, for all $f \in \mathcal{S}$,

$$[Af](x) = \int k(x, x-y) f(y) dy \tag{2.0}$$

where the kernel $k(x, z) = \mathfrak{F}_\zeta^{-1}[p(x, \zeta)](z)$ together with its derivatives $\partial_x^\alpha k(x, z)$ for all $|\alpha| \leq r$, is integrable with respect to z, uniformly in x. Moreover, for any cut-off function $\varphi(z)$ there is a corresponding function $\psi(x, z)$ in $B_r^\infty(E^n \times S^{n-1})$ and rapidly decreasing with respect to z, uniformly in x, such that

$$k(x, z) = [h(x, z) + P(x, z) \log |z|] \varphi(z) + \psi(x, z) \tag{2.1}$$

where $h \in B_r^\infty(E^n \times S^{n-1})$ is homogeneous in z of degree $\rho - n$, and $P(x, z)$ is a polynomial in z homogeneous of degree $\rho - n$ if $\rho - n$ is a non-negative integer, $P(x, z) \equiv 0$ otherwise.

This result is a direct consequence of the Theorem in [5] and of Definition 1.3 here. (See also [6], Theorem 1.5).

Although the eventual presence of a logarithmic term in (2.1) causes no serious harm in the proofs that follow, we may suppose, for the sake of simplicity, that $m \leq n$. Thus $0 < \rho < n$ and (2.1) reduces to

$$k(x, z) = h(x, z) \varphi(z) + \psi(x, z) . \tag{2.1'}$$

In addition, on $E^n \times E^n_0$, for every $N > 0$ there exist constants C_N such that

$$|\partial_z^\beta \partial_x^\alpha k(x, z)| \leq C_N |z|^{\rho-n-|\beta|}(1+|z|)^{-N} \tag{2.2}$$

for all β and all α with $0 \leq |\alpha| \leq r$. (The constants C_N depend also on $|\beta|$).

The first step in the proof of Theorem 1.8 is as fallows.

LEMMA 2.1 - If A is in $\mathcal{S}_m$, then $A^* - A^\#$ belongs to $\mathfrak{J}_m$.

PROOF - If $A \in \mathfrak{J}_m$, then $A = S$ and $A^\# = 0$. A^* is again bounded in L^p norm, $1 < p < \infty$, and the same holds for $(\partial/\partial x)^\alpha A^*$ and $A^*(\partial/\partial x)^\alpha$, for all $|\alpha| = m$. Hence $A^* - A^\# = A^* \in \mathfrak{J}_m$.

Let us consider operators $A = A_j$ of the form (1.1), with $p_j = p$, $\rho_j = \rho < m$, and suppose that $\rho > 0$. We may also suppose that on $|\zeta| \geq 1$, $p(x, \zeta) = a(x, \zeta)$ where $a = a_j$ is the corresponding homogeneous symbol of degree $-\rho$. Then, for all $f \in \mathcal{S}$, expressing A in the form (2.0) and passing to the adjoint, we find that

$$[A^*f](x) = \int \bar{k}(y, y-x)f(y)dy \tag{2.3}$$

where $\bar{k}$ is the complex conjugate of the kernel k.

Consider now the following TAYLOR expansion in y of $\bar{k}(y, z)$ at the point $y = x$:

$$\bar{k}(y, z) = \sum_{|\alpha|<r} (1/\alpha!)\bar{k}_\alpha(x, z)(y-x)^\alpha + R(x; y, z)$$

where $\bar{k}_\alpha = \partial_x^\alpha \bar{k}(x, z)$. Letting $z = y - x$ and substituting in (2.3), we have

$$[A^*f](x) = \sum_{|\alpha|<r} (1/\alpha!) \int \bar{k}_\alpha(x, y-x)(y-x)^\alpha f(y)dy + [S_2 f](x) \tag{2.4}$$

where the term

$$[S_2 f](x) = \int R(x; y, y-x)f(y)dy \tag{2.5}$$

will be handled later.

It follows from the definition of k that

$$k_\alpha(x, z) = \mathfrak{F}_\zeta^{-1}[\partial_x^\alpha p(x, \zeta)](z)$$

and thus

$$k_\alpha(x, z)z^\alpha = \mathfrak{F}_\zeta^{-1}\left[\left(\frac{i}{2\pi}\right)^{|\alpha|} \partial_\zeta^\alpha \partial_x^\alpha p(x, \zeta)\right](z).$$

Hence, taking complex conjugates and recalling that $z = y - x$, we obtain

$$\bar{k}_\alpha(x,\ y-x)(y-x)^\alpha = (2\pi i)^{-|\alpha|}\mathfrak{F}_\zeta^{-1}[\partial_\zeta^\alpha \partial_x^\alpha \bar{p}(x,\ \zeta)](x-y)$$

where $p(x,\ \zeta)$ is conjugate to $\bar{p}(x,\ \zeta)$, and $p(x,\ \zeta) = a(x,\ \zeta) = \sigma(A)$ on $|\zeta| \geq 1$. We observe that all integrals in (2.4) are absolutely convergent since, by Proposition 2.0, the functions $\bar{k}_\alpha(x,\ z)z^\alpha$ are integrable with respect to z, for all $0 \leq |\alpha| \leq r$. Moreover, for all y, $f(y) = \mathfrak{F}_\zeta^{-1}[\hat{f}(\zeta)](y)$ since $f \in \mathfrak{S}$. Hence, using the formula $(\mathfrak{F}^{-1}u)*(\mathfrak{F}^{-1}v) = \mathfrak{F}^{-1}(uv)$, we obtain that, for all $0 \leq |\alpha| \leq r$,

$$\int \bar{k}_\alpha(x,\ y-x)(y-x)^\alpha f(y)dy = (2\pi i)^{-|\alpha|}\int e^{2\pi i<x,\,\zeta>}[\partial_\zeta^\alpha \partial_x^\alpha \bar{p}(x,\ \zeta)]\hat{f}(\zeta)d\zeta.$$

Consequently, substituting this expression in (2.4), we see that for all $f \in \mathfrak{S}$

$$[A^*f](x) = \sum_{|\alpha|<r}(1/\alpha\,!)(2\pi i)^{-|\alpha|}\int e^{2\pi i<x,\,\zeta>}[\partial_\zeta^\alpha \partial_x^\alpha \bar{p}(x,\ \zeta)]\hat{f}(\zeta)d\zeta + [S_2 f](x). \tag{2.6}$$

Grouping together all terms in (2.6) with $|\alpha| = j$, $0 \leq j \leq r-1$, we see that, modulo S_2, A^* is a finite sum of operators B_j of the form (1.1) with homogeneous symbols

$$\sigma(B_j) = \sum_{|\alpha|=j}(1/\alpha\,!)(2\pi i)^{-|\alpha|}[\partial_\zeta^\alpha \partial_x^\alpha \overline{\sigma(A)}]$$

of degree $-\rho_j \equiv -\rho - j$. Moreover, $\sigma(B_j) \in B_{r-j}^\infty(E^n \times S^{n-1})$ where $r - j = 2m - [\rho] - j = 2m - [\rho_j]$. Therefore, according to Definition 1.0, $\sigma(B_j)$ is a B_m^∞ homogeneous symbol of degree $-\rho_j$ if and only if $\rho_j = \rho + j < m$. In other words, by Definition 1.3, B_j belongs to $\mathfrak{S}_m$ for all $j < m - [\rho]$, and hence $A^\# = \sum_{0\leq j<m-[\rho]} B_j$ and formula (2.6) becomes

$$A^*f = A^\# f + S_1 f + S_2 f \tag{2.7}$$

where

$$[S_1 f](x) = \sum_{|\alpha|=m-[\rho]}^{r-1}(1/\alpha\,!)(2\pi i)^{-|\alpha|}\int e^{2\pi i<x,\,\zeta>}[\partial_\zeta^\alpha \partial_x^\alpha \bar{p}(x,\ \zeta)]\hat{f}(\zeta)d\zeta$$

and S_2 is given by (2.5). Consequently, it remains to show that S_1 and S_2 belong to $\mathfrak{J}_m$.

By Remark 1.6 (a), S_1 belongs to $\mathfrak{J}_m$, being a finite sum of operators B_j of the for (1.1) with $[\rho_j] = [\rho] + j \geq [\rho] + m - [\rho] = m$. Moreover, since A^*, $A^\#$ and S_1 are all bounded in L^p norm, $1 < p < \infty$, the same is true for S_2 by formula (2.7). Let us show next that, for all β with $|\beta| = m$, also $S_2(\partial/\partial x)^\beta$

is bounded in L^p norm. In order to do this, we need suitable estimates for $|\partial_y^\beta R(x; y, y-x)|$, where

$$R(x; y, z) = \bar{k}(y, z) - \sum_{|\alpha|<r} (1/\alpha!)\, \bar{k}_\alpha(x, z)\,(y-x)^\alpha$$

$\bar{k}_\alpha(x, z) = \partial_x^\alpha \bar{k}(x, z)$, $z = y - x$ and x is fixed.

First of all, using (2.2) and a standard estimate on TAYLOR remainders, we have that for all y and all $|\beta| \leq r$

$$|\partial_y^\beta R(x; y, z)| \leq C\,|z|^{\rho-n}\,|x-y|^{r-|\beta|}\,(1+|x-y|)^{-1}$$

on $|z| \leq 1$. Similarly, but keeping y fixed, we have that on $|z| \leq 1$

$$|\partial_z^\beta R(x; y, z)| \leq C\,|z|^{\rho-n-|\beta|}\,|x-y|^{r}\,(1+|x-y|)^{-1}.$$

Therefore, letting now $z = y - x$, $|\beta| = m$ and combining the preceding estimates, we see that

$$|\partial_y^\beta R(x; y, y-x)| \leq C\,|x-y|^{r+\rho-n-m}\,(1+|x-y|)^{-1}$$

on $|x-y| \leq 1$. Hence, using again (2.2) and substituting $r = 2m - [\rho]$, it follows that for any $N \geq 1$ there is a constant C_N such that

$$|\partial_y^\beta R(x; y, y-x)| \leq C_N\,|x-y|^{m+\rho-[\rho]-n}(1+|x-y|)^{-N} \tag{2.8}$$

for all β, with $|\beta| = m \geq 1$, and all x, y in E^n.

Now, on account of (2.5), we see that for all $f \in \mathcal{S}$

$$[S_2\,\partial_x^\beta f](x) = \int R(x; y, y-x)\,\partial_y^\beta f(y)\,dy\,.$$

Consequently, integrating by parts and applying estimate (2.8) with $N = m + \rho - [\rho] + 1$, we conclude by YOUNG's theorem that the operators $S_2(\partial/\partial x)^\beta$ are bounded in L^p norm, for all $|\beta| = m$ and $1 \leq p \leq \infty$.

Let us consider at this point the case $\rho = 0$. By formula (1.3) with $\rho_j = 0$, we see that, modulo $\mathfrak{J}_m$, $A = A_0$ where

$$[A_0 f](x) = a(x) f(x) + \text{p.v.}\int k(x, x-y)\,\varphi(x-y)\,f(y)\,dy + [Sf](x)$$

with φ being a cut-off function and with S in $\mathfrak{J}_m$. Now, the Lemma is clearly true for the operator $[M_a f](x) = a(x) f(x)$, and also holds for $S \in \mathfrak{J}_m$, as we saw earlier. The remaining term is a principal value operator of the form (2.0), with kernel satisfying estimates (2.2) for $\rho=0$. Consequently, interpreting the approppriate integrals in the principal value sense, we see that the proof of the Lemma, as far as it goes, remains valid also when $\rho=0$.

Summing up, we have shown that if $A\in\mathfrak{J}_m$, or else if $A=A_j$ is an operator of the form (1.1) with $0\leq\rho=\rho_j<m$, then $A^*-A^{\#}=S_1+S_2$ where $S_1\in\mathfrak{J}_m$ and the operators S_2 and $S_2(\partial/\partial x)^\beta$ are bounded in L^p norm for all $1<p<\infty$ and $|\beta|=m$. Thus it remains to be proved that $(\partial/\partial x)^\beta S_2$ or, equivalently, $S_2^*(\partial/\partial x)^\beta$ also has this property.

CLAIM: if $\sigma(A)=a$ is a homogeneous symbol of degree $-\rho$, where $0\leq\rho<m$, then $[\sigma(A)^{\#}]^{\#}=\sigma(A)$.

Taking this momentarily for granted, let us consider the operator $B=A^{\#}+S_1$ in $\mathfrak{S}_m$, $\sigma(B)=\sigma(A^{\#})=\sigma(A)^{\#}$. Since $A^*=B+S_2$, we have $A=B^*+S_2^*$ and so $S_2^*=A-B^*$.

Applying the preceding results to the operator B, we deduce that $B^*=B^{\#}+S$, where $S(\partial/\partial x)^\beta$ is bounded in L^p norm for all $1<p<\infty$ and all $|\beta|=m$, and where

$$\sigma(B^{\#})=\sigma(B)^{\#}=[\sigma(A)^{\#}]^{\#}=\sigma(A)$$

on account of the Claim. Accordingly, $A-B^{\#}$ belongs to $\mathfrak{J}_m$, since $\sigma(A-B^{\#})=$ $=\sigma(A)-\sigma(B^{\#})=0$, thus, in particular, $(A-B^{\#})(\partial/\partial x)^\beta$ is bounded in L^p norm for $1<p<\infty$ and $|\beta|=m$. Therefore, the operator

$$S_2^*(\partial/\partial x)^\beta=(A-B^{\#})(\partial/\partial x)^\beta-S(\partial/\partial x)^\beta$$

also satisfies this property.

Finally, we verify the Claim by induction. We note that if $m=1$, or if $\rho\geq m-1$, the conclusion is obvious since $\sigma(A)^{\#}=\overline{\sigma(A)}$ in these cases. Thus, with $m\geq2$, we denote by $\sigma(A)^b$ the «adjoint» in Σ_{m-1} and we may assume that $[\sigma(A)^b]^b=\sigma(A)$.

Supposing for simplicity that $\rho=0$, we observe that

$$\sigma(A)^{\#}=\sigma(A)^b+(2\pi i)^{1-m}\sum_{|\alpha|=m-1}1/\alpha!\ \partial_\xi^\alpha\partial_x^\alpha\overline{a}$$

and

$$[\sigma(A)^b]^{\#}=[\sigma(A)^b]^b+(2\pi i)^{1-m}\sum_{|\alpha|=m-1}1/\alpha!\ \partial_\xi^\alpha\partial_x^\alpha a.$$

Consequently,

$$[\sigma(A)^{\#}]^{\#}=[\sigma(A)^b]^{\#}-(2\pi i)^{1-m}\sum_{|\alpha|=m-1}1/\alpha!\ \partial_\xi^\alpha\partial_x^\alpha a=[\sigma(A)^b]^b=\sigma(A)$$

and the Claim is established. Q.E.D.

Turning to the composition of operators in $\mathcal{S}_m$, we shall see how it can be handled in a similar fashion.

LEMMA 2.2 - If A and B are in $\mathcal{S}_m$, then $AB - A \circ B$ belongs to $\mathfrak{I}_m$.

PROOF. - It is an immediate consequence of Lemma 1.5 and Remarks 1.6 that composition of operators in $\mathcal{S}_m$ with operators in $\mathfrak{I}_m$ yields operators in $\mathfrak{I}_m$. Consequently, with $A = A_j$, $p = p_j$, $0 \leq \rho = \rho_j < m$, and with $B = B_k$, $q = q_k$ and $0 \leq \rho' = \rho_k < m$, it suffices to consider products of the form AB where, for all f in $\mathcal{S}$,

$$[Af](x) = \int e^{2\pi i<x,\zeta>} p(x, \zeta) \hat{f}(\zeta)\, d\zeta = \int h(x, x-y) f(y)\, dy$$

and

$$[Bf](x) = \int e^{2\pi i<x,\zeta>} q(x, \zeta) \hat{f}(\zeta) d\zeta = \int k(x, x-y) f(y) dy.$$

Then,

$$[A(Bf)](x) = \int h(x, x-y) \left\{ \int k(y, y-z) f(z)\, dz \right\} dy \tag{2.9}$$

where the approppriate integrals are taken in the principal value sense if ρ or ρ' equals zero.

If $[\rho + \rho'] \geq m$, then AB belongs to $\mathfrak{I}_m$ since, on account of Remark 1.6 (a), $A \in \mathfrak{I}_{[\rho]}$ and $B \in \mathfrak{I}_{[\rho']}$. Otherwise, setting $r' = m - [\rho + \rho']$, we consider the following Taylor expansion in y at the point $y = x$:

$$k(y, \xi) = \sum_{|\alpha| < r'} 1/\alpha!\, k_\alpha(x, \xi)(y-x)^\alpha + R(x; y, \xi)$$

where $k_\alpha(x, \xi) = \partial_x^\alpha k(x, \xi)$. Letting $\xi = y - z$ and substituting back into (2.9), we obtain

$$A(Bf) = \sum_{|\alpha| < r'} 1/\alpha! \int h(x, x-y)(y-x)^\alpha \left\{ \int k_\alpha(x, y-z) f(z) dz \right\} dy + A(Sf) \tag{2.10}$$

where

$$[Sf](y) = \int R(y, y-z) f(z)\, dz \tag{2.11}$$

with $R(y, \xi) = R(x; y, \xi)$, x being kept fixed.

According to the definition of the kernel k,

$$k_\alpha(x, y-z) = \mathfrak{F}_\zeta^{-1}[\partial_x^\alpha q(x,\zeta)](y-z)$$

whence, since $f \in \mathcal{S}$,

$$\int k_\alpha(x, y-z) f(z)\, dz = \int \mathfrak{F}_\zeta^{-1}[\partial_x^\alpha q(x,\zeta)](y-z)\, \mathfrak{F}_\zeta^{-1}[\widehat{f}(\zeta)](z) dz =$$

$$= \mathfrak{F}_\zeta^{-1}\{[\partial_x^\alpha q(x,\zeta)]\widehat{f}(\zeta)\}(y).$$

Again, by definition of the kernel h,

$$h(x, x-y)(x-y)^\alpha = \mathfrak{F}_\zeta^{-1}\left[\left(\frac{i}{2\pi}\right)^{|\alpha|} \partial_\xi^\alpha p(x,\zeta)\right](x-y)$$

whence

$$h(x, x-y)(y-x)^\alpha = (2\pi i)^{-|\alpha|}\, \mathfrak{F}_\zeta^{-1}[\partial_\xi^\alpha p(x,\zeta)](x-y).$$

Therefore, the summation on the right-side of (2.10) becomes

$$\sum_{|\alpha|<r'} 1/\alpha!\,(2\pi i)^{-|\alpha|} \int \mathfrak{F}_\zeta^{-1}[\partial_\xi^\alpha p(x,\zeta)](x-y)\, \mathfrak{F}_\zeta^{-1}\{[\partial_x^\alpha q(x,\zeta)]\widehat{f}(\zeta)\}(y) dy =$$

$$= \sum_{|\alpha|<r'} 1/\alpha!(2\pi i)^{-|\alpha|} \int e^{2\pi i<x,\zeta>}[\partial_\xi^\alpha p(x,\zeta)][\partial_x^\alpha q(x,\zeta)]\widehat{f}(\zeta) d\zeta = (A \circ B) f$$

as is readily seen by the definition of r'. In other words, we have

$$AB = A \circ B + AS.$$

Consequently, since $A \in \mathcal{S}_m$ has the form prescribed above, in order to prove that $AS \in \mathfrak{I}_m$ it suffices to show that the operator S given by (2.11) belongs to $\mathfrak{I}_m$. Since AB and $A \circ B$ are bounded in L^p norm for all $1 < p < \infty$, the same holds for AS. Hence S must also have this property, on account of Lemma 1.5. Finally, estimating the derivatives $\partial_\xi^\beta R(y, \xi)$ of our remainder, the desired conclusion follows as in the proof of Lemma 2.1. Q.E.D.

The two preceding Lemmas exhaust the proof of Theorem 1.8.

REMARK 2.3. - The algebra $\mathcal{S}_1$ contains, as a subalgebra, the algebra generated by all singular integral operators of class C_2^∞ (see [3]) and their adjoints.

III.

Henceforth, the principal symbol (cf. Definition 1.7) of an operator A in $\mathcal{S}_m$ will be denoted by $\sigma_p(A)$. If $A \in \mathfrak{I}_m$, we shall say that A is smoothing of degree m.

LEMMA 3.0. - If A and B belong to $\mathcal{S}_m$ and are smoothing of degree ρ and ρ' respectively, $0 \leq \rho,\ \rho' < m$, then $[A, B] = AB - BA$ is smoothing of degree $\geq \min\{\rho + \rho' + 1, m\}$. If $[A, B]$ is smoothing of degree $\rho + \rho' + 1 < m$, then its principal symbol is given by

$$\sigma_p([A, B]) = (2\pi i)^{-1} \sum_{k=1}^{n} \left\{ \frac{\partial \sigma_p(A)}{\partial \zeta_k} \frac{\partial \sigma_p(B)}{\partial x_k} - \frac{\partial \sigma_p(B)}{\partial \zeta_k} \frac{\partial \sigma_p(A)}{\partial x_k} \right\} \tag{3.0}$$

This result follows directly from Definition 1.7 and Theorem 1.8.

For any positive real s, the fractional integral operator I^s belongs to $\mathcal{S}_m$ for all integers $m \geq 1$. If $s < m$, we have $\sigma(I^s) = \sigma_p(I^s) = |\zeta|^{-s}$, whereas $I^s \in \mathfrak{I}_m$ if $s \geq m$. We shall set $\Lambda = I^{-1}$.

THEOREM 3.1. - If A in $\mathcal{S}_m$ is smoothing of degree ρ, $0 \leq \rho < m$, then $A\Lambda^\rho$ and $\Lambda^\rho A$ belong to $\mathcal{S}_\mu$, where

$$\mu = [m - [\rho]/2], \tag{3.1}$$

and are smoothing operators of degree zero with principal symbol $\sigma_p(A\Lambda^\rho) = \sigma_p(\Lambda^\rho A) = \sigma_p(A)|\zeta|^\rho$.

PROOF. - Using formula (1.3) we express the operator A in the form

$$A = A_0 I^\rho + A_1 I^{\rho j+1} + \dots + S = [A_0 + A_1 I^{\rho j+1-\rho} + \dots + SI^{-\rho}] I^\rho \tag{3.2}$$

where $\sigma_p(A_0) \in B_r^\infty(E^n \times S^{n-1})$, with $r = 2m - [\rho]$, is homogeneous of degree zero in ζ. Consequently, $\sigma_p(A_0)$ is a B_μ^∞ homogeneous symbol of degree zero where $\mu \geq 1$ is the integer given by (3.1). Likewise, we see that the operator P, given by the expression in brackets in (3.2), belongs to $\mathcal{S}_\mu$ and is smooting of degree zero. Thus, $A\Lambda^\rho = P$ and $\sigma_p(A\Lambda^\rho) = \sigma_p(A_0) = \sigma_p(A)|\zeta|^\rho$.

Moreover, by Lemma 3.0, $\Lambda^\rho A = \Lambda^\rho P I^\rho = P + \Lambda^\rho[P, I^\rho] \equiv Q$ is also an operator of degree zero in $\mathcal{S}_\mu$, with $\sigma_p(\Lambda^\rho A) = \sigma_p(P) = \sigma_p(A)|\zeta|^\rho$. Q. E. D.

COROLLALY 3.2. - Let A in $\mathcal{S}_m$ be smoothing of degree $\rho < m$ and let μ be as in (3.1). Then there exist operators P and Q in $\mathcal{S}_\mu$ of degree zero, with $\sigma_p(P) = \sigma_p(Q) = \sigma_p(A)|\zeta|^\rho$, such that

$$A = PI^\rho = I^\rho Q. \tag{3.3}$$

Let Ω be an open set in E^n.

DEFINITION 3.3. - An operator A in $\mathcal{S}_m$ is said to be *elliptic on* Ω if

(i) A is of degree zero,

(*ii*) $\sigma_p(A)(x, \zeta) \neq 0$ for all (x, ζ) in $\Omega \times S^{n-1}$.

The operator A is said to be *elliptic* if it is elliptic on all of E^n.

EXAMPLE 3.4. - Let $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a differential operator with coefficients $a_\alpha(x)$ in $B_r(E^n)$, $r = m + |\alpha|$, and with characteristic polynomial $P_m(x, \zeta) = \sum_{|\alpha|=m} a_\alpha(x) \zeta^\alpha$. Since

$$D^\alpha = \mathfrak{F}^{-1} \zeta^\alpha \mathfrak{F} = \{\mathfrak{F}^{-1} [\zeta^\alpha d(\zeta)^{-m}] \mathfrak{F}\} \cdot \{\mathfrak{F}^{-1} d(\zeta)^m \mathfrak{F}\} = H_\alpha \Lambda^m$$

we obtain

$$P(x, D) = \{\sum_{|\alpha| \leq m} a_\alpha(x) H_\alpha\} \Lambda^m = A \Lambda^m \tag{3.4}$$

where $\Lambda^m = I^{-m}$ is an isomorphism on $\mathcal{S}$ (and $\mathcal{S}^*$), and the operator A belongs to $\mathcal{S}_m$, with principal symbol

$$\sigma_p(A) = \{\sum_{|\alpha|=m} a_\alpha(x) \zeta^\alpha\} |\zeta|^{-m} = P_m(x, \zeta) |\zeta|^{-m}$$

homogeneous of degree zero in ζ. Therefore, the differential operator $P(x, D)$ is elliptic on Ω if and only if the corresponding integral operator A in $\mathcal{S}_m$ is elliptic on Ω.

Let us recall that a formal power series $S = \sum_{\nu=0}^{\infty} a_\nu X^\nu$, over the complex field (for example), is said to be invertible if there exists another series $T = \sum_{\nu=0}^{\infty} b_\nu X^\nu$ such that

$$S \cdot T = \sum_{k=0}^{\infty} \{\sum_{\mu+\nu=k} a_\mu b_\nu\} X^k = 1.$$

Then, as easily seen, such a series is invertible if and only if its constant term $a_0 \neq 0$.

As before, we denote by M_φ the operator $[M_\varphi f](x) = \varphi(x) f(x)$

LEMMA 3.5. - Let A in $\mathcal{S}_m$ be elliptic on Ω. Then, for all $\varphi \in C_0^\infty(\Omega)$, there exists a B in $\mathcal{S}_m$ such that BA-M_φ and AB-M_φ are smoothing of degree 1. Moreover, there exist operators B and C in $\mathcal{S}_m$ such that BA-M_φ and AC-M_φ belong to $\mathfrak{I}_m$.

PROOF. - Let B in $\mathcal{S}_m$ be an operator with principal symbol $\sigma_p(B) = \varphi(x)[\sigma_p(A)]^{-1}$. Then the first conclusion follows from Theorem 1.8, since $\sigma_p(AB) = \sigma_p(A)\,\sigma_p(B) = \varphi(x) = \sigma_p(BA)$.

Again from Theorem 1.8, we have that BA-M_φ belongs to $\mathfrak{J}_m$ if and only if $\sigma(B) \circ \sigma(A) = \varphi(x)$. Choosing $b_0 = \sigma_p(B)$ as before, we can solve successively for the remaining b_j by an argument analogous to be one used for formal power serier. The existence of C is determined in the same way. Q.E.D.

The following result is a direct consequence of Lemma 3.5.

LEMMA 3.6. - Let A in $\mathcal{S}_m$ be elliptic on Ω and let Ω_1 be a relatively compact subset of Ω. Then there exists an operator B in $\mathcal{S}_m$, with $\sigma(BA) = 1$ on Ω_1, such that for all $f \in L^p_{-m}$, $1 < p < \infty$,

$$BAf = f + Sf$$

where $\sigma(S) = 0$ on Ω_1.

Another immediate consequence of Theorem 1.8 is the following «pseudo-locality» property.

LEMMA 3.7. - Let A belong to $\mathcal{S}_m$. Then, for any φ_1 and φ_2 in $B^\infty(E^n)$ with disjoint supports, the operator $M_{\varphi_1} A M_{\varphi_2}$ belongs to $\mathfrak{J}_m$.

PROOF. - Since $\sigma(M_{\varphi_1}) = \sigma_p(M_{\varphi_1}) = \varphi_1(x)$, it follows that

$$\sigma(M_{\varphi_1} A M_{\varphi_2}) = \varphi_1(x)\,\sigma(AM_{\varphi_2}) =$$

$$= \varphi_1(x)\,\Sigma\,(1/\alpha!)(2\pi i)^{-|\alpha|}[\partial_\xi^\alpha \sigma(A)][\partial_x^\alpha \varphi_2] = 0$$

since φ_1 and φ_2 have disjoint supports. Q.E.D.

Let us consider the local spaces

$$L^{p,s}_{\text{loc}}(\Omega) = \{f \in \mathcal{S}^* : \varphi f \in L^p_s \text{ for all } \varphi \in C_0^\infty(\Omega)\}.$$

LEMMA 3.8. - Let A in $\mathcal{S}_m$ be smoothing of degree ρ, $0 \leq \rho \leq m$. Then, for all $1 < p < \infty$ and $0 \leq |s| \leq |s + \rho| \leq m$, if $f \in L^{p,s}_{\text{loc}}(\Omega)$ then $(Af) \in L^{p,s+\rho}_{\text{loc}}(\Omega)$.

PROOF. - Let $\varphi \in C_0^\infty(\Omega)$ and choose a corresponding $\psi \in C_0^\infty(\Omega)$ such that $\psi(x) = 1$ on a neighborood of the support of φ. Then

$$\varphi Af = \varphi A \psi f + \varphi A(1 - \psi)f.$$

Now, since ψf is in L^p_s, $\varphi A(\psi f)$ belongs to $L^p_{s+\rho}$ for $0 \leq |s| \leq |s + \rho| \leq m$. Moreover, by Lemma 3.7, since φ and $(1-\psi)$ have disjoint supports, $M_\varphi A M_{(1-\psi)}$

is in $\mathfrak{J}_m$ and hence also $\psi A(1-\psi)f$ belongs to $L^p_{s+\rho}$ for $|s+\rho|\leq m$. Therefore, φAf is in $L^p_{s+\rho}$. Q.E.D.

We are now ready to obtain the following (local) regularity property of elliptic operators.

THEOREM 3.9. - Let A in $\mathcal{S}_m$ be elliptic on Ω and let Ω_1 be a relatively compact subset of Ω. Then for all $1<p<\infty$ and all f in L^p_{s-m}, $0\leq s\leq m$, if $(Af)\in L^{p,s}_{\mathrm{loc}}(\Omega_1)$ then $f\in L^{p,s}_{\mathrm{loc}}(\Omega_1)$.

PROOF. - By Lemma 3.6 there exists a B in $\mathcal{S}_m$ such that $BAf=f+Sf$ where $\sigma(S)=0$ on Ω_1. By Lemma 3.8, since B has degree zero, Af in $L^{p,s}_{\mathrm{loc}}(\Omega_1)$ implies that BAf belongs to $L^{p,s}_{\mathrm{loc}}(\Omega_1)$.

Now let $\varphi\in C_0^\infty(\Omega_1)$. Then $M_\varphi S$ belongs to $\mathfrak{J}_m$ since $\sigma(S)=0$ on Ω_1. Thus, φSf is in $L^p_{s-m+m}=L^p_s$ and so $(Sf)\in L^{p,s}_{\mathrm{loc}}(\Omega_1)$. Consequently, $f=BAf-Sf$ belongs to $L^{p,s}_{\mathrm{loc}}(\Omega_1)$. Q.E.D.

We note that combining Theorem 3.9 with Example 3.4, one obtains another proof of the interior regularity for solutions of certain elliptic partial differential equations on Ω. Finally, we remark that the algebra $\mathcal{S}_m$ is invariant under smooth local diffeomorphisms and that, for such a map $\varphi:\Omega\to E^n$, the principal symbol of an operator in $\mathcal{S}_m$ behaves as a function defined on the (trivial) cotangent bundle over Ω. This result, together with its proof, is the analogue of Theorem 7 in [7]. The behavior of the (full) symbol under local diffeomorphism is more complicated and requires a discussion of certain (jet) equivalence-classes of maps. Thus, we shall defer its description to another occasion.

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