On Weyl's identity.

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Summary. - *The Weyl Identity (as orignally stated by H. Weyl and used by A. Wintner) is not totally correct. Presented here is the correct version of the Identity, a concise proof of it and some applications.*

Introduction.

HERMAN WEYL published the paper [10] in which the WEYL problem was introduced and partially solved. One of the crucial steps in the solution is to obtain an a priori bound for the mean curvature in terms of the metric coefficients. WEYL deduces such a bound from his identity (33). Aurel WINTNER published the paper [12] in which he named equation (33) the WEYL identity and showed that this identity has other interesting applications.

WEYL does not include the details of the computations which lead to the WEYL identity. He describes them as a "langweilige Rechnung„ and states "Es ist wahrscheinlich, dass ein geschickterer RECHNER die FORMEL (33) auf viel leichterem WEGE wird ermitteln konnen, als hier angeduetet wurde.„ As far as we know except for some related results derived by CHERN in [1] WEYL's request has thus far been unfulfilled.

The WEYL identity is not correct. A counterexample is given in section 2. A corrected identity (3.6) was found in [3] by essentially following the tedious algebraic procedure suggested by WEYL in [10]. NIRENBERG in [6] uses the WEYL inequality which he proves directly by using an identity (10.9) in [6] which is esentially equivalent to the corrected WEYL Identity (3.6) except that NIRENBERG's (10.9) is not in invariant form. As WEYL predicted, a more direct proof was discovered and is given in section 3. We generalize the identity in section 4 by eliminating the restriction that the GAUSSIAN curvature does not vanish. The corrected identity (3.6) leaves some applications unaffected but does show that some of WINTNER's formulas in [12] are not correct. The related results derived by CHERN in [1] are shown in [2] to imply the

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corrected identity (3.6) and not WEYL's identity as was claimed. In later sections we correct other results in WINTNER's paper which are not consequences of his working with the wrong identity.

1. - Formulas from the theory of surfaces.

Let M be a surface in EUCLIDEAN 3-space of class $C^r(r \geq 4)$ with first fundamental form given by

$$ds^2 = g_{ij} du^i du^j$$

and second fundamental form by

$$l_{ij} du^i du^j$$

The functions g_{ij} and l_{ij} are of class C^{r-1} and C^{r-2} respectively. When $K \geq 0$ we pick the unit normal to the surface N so that $l_{ii} \geq 0$ $i = 1, 2$.

We will be concerned with a neighborhood of a point P on a surface of class C^r, $r \geq 5$ which is neither an umbilic nor a flat point. In this neighborhood we will be using lines of curvature coordinates. There is a loss in differentiability involved in changing to lines of curvature coordinates. (See [11, p. 860] for a discussion of this point).

The parametric curves are the lines of curvature if and only if $g_{12} = e_{12} = 0$, and in this case $l_{ii} = k_i g_{ii}$, $i = 1, 2$ where k_1 and k_2 are the principle curvatures.

We denote the GAUSSIAN curvature by K, the mean curvature by H and define the quantity

$$J = (1/2)(k_1 - k_2)$$

and note that $J^2 = H^2 - K$.

The MAINARDI-CODAZZI equations in lines of curvature coordinates become as in [5, p. 66]

$$(1.1) \qquad (l_{11})_v = (g_{11})_v H$$

$$(l_{22})_u = (g_{22})_u H$$

and Theorem Egregium takes the form (in any orthogonal coordinates):

$$(1.2) \qquad K = -(1/(2(g_{11}g_{22})^{\frac{1}{2}}))(a + b)$$

$$a = ((g_{22})_u/(g_{11}g_{22})^{\frac{1}{2}})_u$$

$$b = ((g_{11})_v/(g_{11}g_{22})^{\frac{1}{2}})_v$$

As defined in [4, pp. 230–1] we will denote the BELTRAMI parameters of the first and second kind with respect to the first fundamental form by $\nabla'(\varphi, \theta)$ and $\Delta'\varphi$ respectively and with respect to the second fundamental form by $\nabla''(\varphi, \theta)$ and $\Delta''\varphi$ respectively.

Let the metric on the surface M be represented by the matrix A. For each point $x \in M$, the metric induces an isomorphism between the tangent and cotangent spaces and hence leads us to define the operator g_A, between Γ, the vector fields on M, to Λ^1, the space of one forms on M, as follows:

For w' and $w \in \Gamma$

$$(g_A(w'))(w) = (w', w)_A$$

where $(w', w)_A$ is the inner product.

Let $\omega_A = (\det A)^{\frac{1}{2}} du \wedge dv$ be the volume element. We use cf. [7] the operator

$$m_A : \Lambda^1 \to \Gamma$$

defined as follows:

For $\alpha \in \Lambda^1$

$$m_A(\alpha) : \Lambda^1 \to C^0(M, R^1)$$

where for $\beta \in \Lambda^1$

$$m_A(\alpha)(\beta) = s \text{ where } \alpha \wedge \beta = s\omega_A .$$

For the divergence of a vector field X, we use the formula cf. [7].

$$\operatorname{div}_A(X)\omega_A = -\, dm_A^{-1}(X). \tag{1.3}$$

2. - A counterexample to Weyl's identity.

WEYL's identity is the following relation between K, H and their differential parameters:

$$\Delta''H - \frac{1}{2}\Delta'K - 2KJ^2 = (2/J)\nabla''(H, J) - (1/J)\nabla'(K, J)$$

Consider the one parameter family of surfaces $X(u, v) = \lambda X(u, v)$ for $\lambda \in R^1$.

The WEYL identity applied to the surface $X_\lambda(u, v)$ yields:

$$(1/\lambda)^2(\Delta''H - (2/J)\nabla''(H, J)) = (1/\lambda)^4\left(\frac{1}{2}\Delta'K + 2KJ^2 - (1/J)\nabla'(K, J)\right). \tag{2.1}$$

The coefficients of $(1/\lambda)^2$ and $(1/\lambda)^4$ in (2.1) must be zero since (2.1) must hold for all $\lambda \neq 0$. The surface defined by $X(u, v) = (u, v, u^2 + 2v^2)$ is a counterexample to WEYL's identity since the origin is a non-umbilic point at wich $\Delta''H \neq 0$, $\nabla''(H, J) = 0$ and $K > 0$.

3. - Concise proof of the corrected Weyl identity.

Preliminares

LEMMA 1: *Given a Riemannian n-dimensional manifold M with two Riemannian metrics corresponding to the diagonal matrices* I *and* II. *Then for any vector field X*

$$\operatorname{div}_I(X)\omega_I = \operatorname{div}_{II}((\det \mathrm{I}/\det \mathrm{II})^{\frac{1}{2}} X)\omega_{II} \tag{3.1}$$

PROOF - It follows directily from the definitions in section 1 that

$$m_I^{-1} = (\det \mathrm{I}/\det \mathrm{II})^{\frac{1}{2}} m_{II}^{-1} \qquad \text{and}$$

$$\omega_I = (\det \mathrm{I}/\det \mathrm{II})^{\frac{1}{2}} \omega_{II}.$$

By (1.3)

$$\operatorname{div}_I(X)\omega_I = -\,\mathrm{d}m_I^{-1}(X) \qquad \text{which by the above}$$

$$= -\,d((\det \mathrm{I}/\det \mathrm{II})^{\frac{1}{2}} m_{II}^{-1}(X))$$

which from the linearity of m_{II}^{-1} and the definition of div_{II} give the desired result.

LEMMA 2: *Let M be a two-dimensional Riemannian manifold Let e_1, e_2 be an orthonormal basis in the tangent space. Then the Gaussian curvature K satisfies the following equation*:

$$K\omega = -\operatorname{div}((\operatorname{div} e_1)e_1 + (\operatorname{div} e_2)e_2)\omega \tag{3.2}$$

where the volume element ω is chosen in accordance with the metric.

We note here that Weatherbnrn in [9] first proved this result for surfaces. Below we outline a more direct proof.

We observe from the form of Theorem Egregium in orthogonal coordinates (1.2) that K can be expressed as the divergence of a vector field with respect to the metric $E = (\delta_{ij})$. Using Lemma 1 we express K as the divergence of a vector field with respect to the metric induced on M by its embedding in E^3. Further applying this technique to the components of the vector field we arrive at equation (3. 2).

We now consider a two dimensional surface imbedded in 3-space with $K > 0$, and which is free from umbilics. Let I and II represent the matrices associated with the first and second fundamental forms respectively in lines of curvature coordinates. Let (e_1, e_2) be an orthonormal basis in the directions of the u and v parametric curves respectively and let (e_1^*, e_2^*) be the dual basic in the cotangent space. Then we have according to our convention:

(3.3) $$g_I^{-1}(e_i^*) = e$$

$$g_{II}^{-1}(e_i^*) = e_i/k_i \quad \text{for} \quad i = 1,2.$$

Since $\text{grad}_A f = g_A^{-1}df$ we may sometimes write $\text{grad}_I f \cdot \tau$ for any of the expressions:

$$(\text{grad}_I f,\ \tau)_I = (\text{grad}_{II} f,\ \tau)_{II} = df(\tau).$$

We use the relations

(3.4) $$k_1 = H + J$$

$$k_2 = H - J$$

to rewrite the MAINARDI - CODAZZI equations (1.1) by replacing the coefficients of the second fundamental form by expressions involving H and J:

$$((H + J)g_{11})_v = (g_{11})_v H$$

$$\text{and} \qquad ((H - J)g_{22})_u = (g_{22})_u H.$$

By carrying out the differentiation in the above equations and rearranging terms we get

(3.5) $$(g_{11})_v/g_{11} = (H_v + J_v)/(-J) \qquad \text{and}$$

$$(g_{22})_u/g_{22} = (H_u - J_u)/J.$$

We now prove the following form of the WEYL identity (3.6) in the neighborhood of a point P which is not an umbillic and at which $K \neq 0$.

(3.6) $$2KJ^2 = K\Delta''H - K\nabla''(J^2,\ H)/J^2 + \nabla''(K,\ H)/2 - \Delta'K/2 + \nabla'(J^2,\ K)/(2J^2).$$

CASE 1. - $K > 0$.

By Lemma 2, $K\omega_I = -\,\text{div}_I((\text{div}_I\, e_1)e_1 + (\text{div}_I\, e_2)e_2)\omega_I$.
We proceed to write the vector $(\text{div}_I\, e_1)e_1 + (\text{div}_I\, e_2)e_2$ in terms of H, J, and K. It is easy to see that

$$\text{div}_I\, e_1 = (1/2(g_{11})^{1/2})((g_{22})_u/g_{22})$$

which becomes by the form (3.5) of the MAINARDI - CODAZZI equations

(3.7) $$\text{div}_I\, e_1 = (1/2(g_{11})^{1/2})(H_u - J_u)/J$$

$$\text{div}_I\, e_2 = (1/2(g_{22})^{1/2})(-\,H_v - J_v)/J$$

where the latter formula in (3.7) is similarly obtained. From $J^2 = H^2 - K$ we have

(3.8) $$J_u = (2HH_u - K_u)/2J$$

and an analagous equation with u replaced by v. By substituting (3.8) in (3.7) and using (3.4) we write $\text{div}_I e_1$ and $\text{div}_I e_2$ in terms of J, H, K, k_1 and k_2 as follows:

$$\text{div}_I e_1 = (1/(2(g_{11})^{\frac{1}{2}}))(-k_2 H_u/J^2 + K_u/2J^2) \tag{3.9}$$

$$\text{div}_I e_2 = (1/(2(g_{22})^{\frac{1}{2}})(-k_1 H_v/J^2 + K_v/2J^2)$$

By writing $k_i = K/k_j$ for $i = 1, 2, i \neq j$ and rearranging terms we have

$$(\text{div}_I e_1)e_1 + (\text{div}_I e_2)e_2 = A + B \qquad \text{where} \tag{3.10}$$

$$A = (1/4J^2)((K_u/(g_{11}{}^{\frac{1}{2}})e_1 + (K_v/(g_{22})^{\frac{1}{2}})e_2)$$

and

$$B = -(K/2J^2)(H_u/(g_{11})^{\frac{1}{2}})(e_1/k_1) + (H_v/((g_{22})^{\frac{1}{2}})(e_2/k_2))$$

It is easily seen that A and B can be expressed by

$$A = (1/4J^2)(dK(e_1)e_1 + dK(e_2)e_2)$$
$$B = -(K/2J^2)(dH(e_1)(e_1/k_1) + dH(e_2)(e_2/k_2)).$$

We note here that the discussion up to this point holds for $K > 0$ and $K < 0$. We now assume that $K > 0$.

Using equations (3.3) and the linearity of g_I and g_{II} we get

$$A = (1/4J^2)(g_I{}^{-1}(dK(e_1)e_1^* + dK(e_2)e_2^*))$$
$$B = -(K/2J^2)(g_{II}{}^{-1}(dH(e_1)e_1^* + dH(e_2)e_2^*))$$

which is simply

$$A = (1/4J^2)g_I{}^{-1}(dK)$$
$$= (1/4J^2)\,\text{grad}_I K \qquad \text{and}$$
$$B = -(K/2J^2)g_{II}{}^{-1}(dH)$$
$$= -(K/2J^2)\,\text{grad}_{II} H.$$

Thus from (3.10) and the above we write in terms of H, J, K the vector

$$(\text{div}_I e_1)e_1 + (\text{div}_I e_2)e_2 = (1/4J^2)\,\text{grad}_I K - (K/2J^2)\,\text{grad}_{II} H.$$

So by Lemma 2 we have

$$K\omega_I = \text{div}_I((K/2J^2)\,\text{grad}_{II} H - (1/4J^2)\,\text{grad}_I K)\omega_I. \tag{3.11}$$

Using Lemma 1 on the first factor of the right hand side of (3.11) and noting that $K=\det \mathrm{II}/\det \mathrm{I}$ we have

$$K\omega_I = \operatorname{div}_{II}(((K)^{\frac{1}{2}}/2J^2)\operatorname{grad}_{II} H)\omega_{II} - \operatorname{div}_I((1/4J^2)\operatorname{grad}_I K)\omega_I. \tag{3.12}$$

Observing $\omega_{II}=(K)^{\frac{1}{2}}\omega_I$, equating the coefficients of ω_I in (3.12), expanding, substituting $\operatorname{grad}_A(1/J^2) = -(\operatorname{grad}_A(J^2))/J^4$ for $A=\mathrm{I}$ and II and finally multiplying by J^2 we get the WEYL identity (3.6).

CASE 2. - $K<0$.

The use of a metric corresponding to a negative definite matrix does not affect the formal definition of g_A and thus the proof is valid through (3.11). The proof appears to break down at equation (3.12), since the volume element ω_{II} is equal to $(\det \mathrm{II})^{1/2}\, du \wedge dv$ where det II is negative and also we see a factor of $(K)^{1/2}$. We overcome this difficulty by considering the complex tangent space Γ_C as generated by the vectors $\partial/\partial u$ and $\partial/\partial v$ over the complex field C. Replacing R^1, Λ^i, Γ by C, Λ^i_C, Γ_C, we define m, d, and div exactly as in section 1. Lemma 1 remains valid for real vector fields X, since $\omega_A = (\det)^{\frac{1}{2}} du \wedge dv$, $m_A^{-1}(\partial/\partial u) = -(\det A)^{\frac{1}{2}} dv$ and $m_A^{-1}(\partial/\partial v) = (\det A)^{\frac{1}{2}} du$, for $A=\mathrm{I}$ and II. The entire proof for $K<0$ follows verbatim from the proof which was done in the real GRASSMAN manifold. If $K<0$ the identity (3.6) is still a real valued identity since div_{II} of a real vector field is real.

Although the preceding proof is valid for C^k surfaces where $k \geq 5$, we will show that the identity holds for $k \geq 4$. Given any C^4 surface, corresponding to each point p, there exists a C^∞ surface $X(u, v)$ each of whose coordinates is the first five terms of the two dimensional TAYLOR series expansion at P, of the corresponding coordinates of the original C^4 surface. This C^∞ surface and the original C^4 surface have the same values for all the quantities appearing in (3.6) at P.

We remark that by integrating the identity (3.6) we can get the following identity which is valid on surfaces of class C^3:

$$\oint (\sqrt{K}/J^2)(\operatorname{grad}_{II} H,\ m_{II})_{II}\, ds_{II} =$$

$$(1/2)\oint (1/J^2)(\operatorname{grad}_I K,\ m_I)_I\, ds_I + 2\int K\, d\sigma_I$$

where m_I, ds_I and $d\sigma_I$ are the outward normal, the arc length and the area element with respect to the metric I and similarly for II.

4. - The generalized Weyl identity.

When $K=0$ the corrected WEYL identity contains the terms $K\triangle''H$ and $K\,\mathrm{grad}_{II}\,H$ which take the form 0/0. Below we define an operator S_{II} which permits us to generalize the WEYL identity to:

$$2KJ^2 = L_2H - L_1K \tag{4.1}$$

where the $L_i i=1, 2$ are differential operators for which L_1 is elliptic and L_2 is hyperbolic, parabolic or elliptic at a point according as K at that point is less than zero, equal to zero or greater than zero respectively. (By parabolic we also include flat points at which all the second order derivatives of K vanish.)

Let $\mathrm{II} = (e_{ij})$ represent the second fundamental form. We define the operator

$$S_{II}: C^1(M, R^1) \to \Gamma \qquad \text{as}$$

$$S_{II}(f) = \sum_{i,k} (\partial f/\partial u^i) l_{ki}^*(\partial/\partial u^k)$$

where l_{pq}^* is the cofactor of l_{pq}. We remark that $S_{II}(f)$ is well define forp all values of det (l_{ij}). Furthermore if det $(l_{ij}) \neq 0$ we define

$$\mathrm{grad}_{II} f = (1/(\det(l_{pq}))) \sum_{i,k} (\partial f/\partial u^i) l_{ki}^*(\partial/\partial u^k).$$

We note that if det $(l_{ij}) > 0$ then this definition of $\mathrm{grad}_{\mathrm{II}} f$ agrees with the one in section 1 and if $\det(e_{ij}) \neq 0$ then the relation

$$S_{\mathrm{II}}(f) = \det(l_{ij})\, \mathrm{grad}_{\mathrm{II}} f$$

holds.

The differential operators which appear in the generalized form of the WEYL identity, (4.1) can be written as:

$$L_2H = \mathrm{div}_{\mathrm{I}}(S_{\mathrm{II}}(H)/\det \mathrm{I}) - (1/J^2)(S_{\mathrm{II}}(H)/\det \mathrm{I})\cdot \mathrm{grad}_{\mathrm{I}} J^2 \tag{4.1a}$$

$$L_1K = (1/2)\triangle''K - (1/2J^2)\nabla'(K, J^2) \tag{4.1b}$$

where the inner product in L_2H is given as in section 3.

We remark that for $\det \mathrm{II} \neq 0$, $S_{\mathrm{II}}(H)/\det \mathrm{I} = K\,\mathrm{grad}_{\mathrm{II}}\,H$ and by expanding (4.1) we obtain (3.6).

To prove (4.1) we observe that when $K=0$ the proof of the corrected WEYL identity in lines of curvature coordinates is valid up to equation (3.9). Using (3.9) we obtain

$$(\mathrm{div}_{\mathrm{I}}\, e_1)e_1 + (\mathrm{div}_{\mathrm{I}}\, e_2)e_2 = (1/(4J^2))\,\mathrm{grad}_{\mathrm{I}}\, K +$$

$$- (1/(2J^2))(k_2H_u/(g_{11})^{\frac{1}{2}})e_1 + (k_1H\,/(g_{22})^{\frac{1}{2}})e_2).$$

We note that the second vector on the right hand side of the above is precisely $S_{II}(H)/\det \mathrm{I}$. Thus from the above and Lemma 2 we have

$$K = \mathrm{div}_{\mathrm{I}}\,((1/2J^2)(S_{II}(H)/\det \mathrm{I}) - (1/4J^2)\,\mathrm{grad}_{\mathrm{I}}\, K$$

Again expanding, letting $(\mathrm{grad}_{\mathrm{I}}\, J^2)/J^4$ replace $-\mathrm{grad}_{\mathrm{I}}\,(1/J^2)$ and multiplying by $2J^2$ we have equation (4.1) which is valid at nonumbillics for all values of K.

To see that L_1 is an elliptic operator on K and that L_2 is an operator on H whose type is determined by the sign of K we compute the coefficients of the higher derivatives and we obtain

$$L_1K = (g_{22}K_{uu} - 2g_{12}K_{uv} + g_{11}K_{vv})/(2(g_{11}g_{22} - g_{12}^2) + K_u(\ldots) + H_v(\ldots). \qquad \text{and}$$

$$L_2H = (l_{22}H_{uu} - 2l_{12}H_{uv} + l_{11}H_{vv})/(g_{11}g_{22} - g_{12}^2) + H_u(\ldots) + H_v(\ldots).$$

We see that L_1 is elliptic since the matrix $(g_{ij}/(2(g_{11}g_{22} - g_{12}^2)))$ has determinant equal to $1/(4(g_{11}g_{22} - g_{12}^2))$ which is strictly greater than zero. The matrix $(l_{ij}/(g_{11}g_{22} - g_{12}^2))$ corresponding to L_2 has determinant $(l_{11}l_{22} - l_{12}^2)/(g_{11}g_{22} - g_{12}^2)^2$ which is equal to $K/(g_{11}g_{22} - g_{12}^2)$ and hence L_2 is a hyperbolic, parabolic or elliptic operator at a point P according to whether K at P is negative, zero or positive respectively.

We remove the restriction that our surface contains no unbilics by applyng the method WINTNER used in section 4 of [12] to formula (4.1) obtaining:

$$4KJ^4 = J^2(2\,\mathrm{div}_{\mathrm{I}}\,(S_{II}(H)/\det \mathrm{I}) - \triangle' K) + $$
$$+ (-2)(S_{II}(H)/\det \mathrm{I}) \cdot \mathrm{grad}_{\mathrm{I}}\, J^2 + \nabla'(KJ^2).$$

5. - Some applications of the corrected Weyl identity.

At the end of his paper, WEYL concludes from the invalid « WEYL identity » the inequality:

$$H^2(u,\ v) \leq \max_{S}\, (K - (4K)^{-1}\triangle' K)$$

for surfaces S which are compact and on which the GAUSSIAN curvature is positive. The validity of this conclusion, in spite of WEYL's using an incorrect identity to start with, is due to the fact that WEYL's argument uses the identity at extremum points for H. At such points the corrected WEYL identity and WEYL's original identity agree. In [6] NIREMBERG proves the above inequality directly from his identity (10.9) which is basically equivalent to (3.6). In [12] WINTNER extended the inequality to surfaces containing umbilics, however his proof is not correct, A corrected version follows:

Let P be the point on S at which H attains its maximum. We only treat the case for P an umbilic point since otherwise WINTNER's proof is correct. If WEYL's inequality does not hold we have:

$$H^2(u_0, v_0) > \max_S (K - \Delta' K/4K) \geq K(u_0, v_0) - (\nabla' K)(u_0, v_0)/4K(u_0, v_0).$$

Since $X(u_0, v_0) = P_0$ is an umbilic, $H^2(u_0, v_0) = K(u_0, v_0)$ and we have $0 < \Delta'(K)/4K$ at P_0. But this contradicts the generally true fact that if $H(u, v)$ assumes its maximal value at an umbilic, K also assumes its maximal value there, for:

$$K(u, v) \leq H^2(u_0, v_0) \leq H^2(u_0, v_0) = K(u_0, v_0).$$

In section 6 of his paper WINTNER lets the third fundamental form replace the first fundamental form in WEYL's suggested proof of the WEYL identity to deduce a second invalid identity. We let the third fundamental form play the first in the proof of (3.6) to get the identity:

$$((H^*)^2 - K^*)(\Delta'' H^* - \Delta^* K^*/2K^* - 2((H^*)^2 - K^*)/K^*)$$
$$= \nabla''(H^*, H^{*2} - K^*) - \nabla^*(K^*, (H^*)^2 - K^*)/2K^* - ((H^*)^2 - K^*)\nabla''(K^*, H^*)/2K^*$$

where H^*, K^*, Δ^* and ∇^* are defined as in WINTNER's paper. This is an invoriant form of (16.8) in [6] which NIRENBERG proved directly. For reasons similar to those given at the beginning of this section pertaining to the WEYL inequality, the method WINTNER uses in section 6 of [12] does prove MIRANDA's inequality from the above identity.

We now use the corrected WEYL identity to derive a necessary condition for the existence of surfaces with constant GAUSSIAN curvature. Let $K = c$ on S. Then (3.6) reduces to:

$$(1/2)(H^2 - c)\Delta'' H - H \nabla'' H - (H^2 - c)^2 = 0, \quad \text{or}$$
$$\operatorname{div}_{II} (\operatorname{grad}_{II} H/(H^2 - c)) = 2$$

This expression is not equivalent to WINTNER's formula (25) in [12] because WINTNER employed WEYL's identity rather than its corrected form (3.6).

6. - Smoothness of J.

In section 2 of [12] WINTNER claims that $J(u, v)$ is of class C^{n-2} whenever $X(u, v)$ is C^n for $n > 1$. In section 9, WINTNER attempts to prove this statement, but the proof is not valid because the functions he refers to as «a» and «b» cannot be defined in general to be continuous. The following example shows that there exist C^∞ surfaces with $K > 0$, on which J is not everywhere C^1.

$$M^*(u, v) = (1/(u^2 + v^2 + (1 + uv^2)^2))(u, v, 1 + uv^2).$$

This surface is the inversion in the unit sphere of the surface $M(u, v) = (u, v, 1 + uv^2)$. A direct computation shows that on $M(u, v)$,

$$J = (1/(\det \mathrm{I})^{3/2})(u^2 + 4v^2 + (9u^2v^8 + 4v^6 + 10u^2v^4))^{1/2}$$

which is not C^1 at $M(0, 0)$. However at $M(0, 0)$, $K = 0$. In [8] WEATHERBURN computes the relation of J^* to J and K^* to K for an inversion in the unit sphere to be:

$$(a) \qquad J^* = -|M(u, v)|^2 J$$

$$(b) \qquad K^* = |M|^4 K + 2|M|^2 \rho J + 4\rho^2 \text{ where } \rho = N(u, v) \cdot M(u, v)$$

From (a) it is clear that if J is not C^1 neither is J^*. From (b) we see that $K^*(0, 0) = 4 > 0$. Thus at $M^*(0, 0)$, J^* is not C^1 and $K^* > 0$.

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