Existence, uniqueness and continuity of solutions of integral equations. An addendum.

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Summary. - *This note corrects an error in the continuous dependence theorem of an earlier paper on the subject stated in the title. Here it is shows that the topology of one of the spaces of functions can be modified in order to obtain the desired continuity results.*

The proof of Theorem 3 in [1] contains a technical flaw. The difficulty occurs on page 147, lines 15-19, where we use the inequality

$$(1) \qquad \int_0^{\xi} |g_n(x_n(s), s)|^p \, ds \leq \sup_{x \in K} \int_0^{\xi} |g_n(x, s)|^p \, ds,$$

which is not valid under our assumptions. Specifically if one chooses $g_n(x, t)$ to be continuous and satisfying

i) $g_n(nt, t) = n$, for all t,

ii) $0 \leq g_n(x, t) \leq n$, for all x and t, and

iii) $g_n(x, t) = 0$, whenever $|x - nt| \geq n^{-(p+1)}$,

then Ineq. (1) fails for $x_n(t) = nt$. This same example, which was suggested by Prof. LUCIEN NEUSTADT, also shows that Theorem 3, as stated in [1], is incorrect.

We propose to show here that with a slightly modified definition of the topology T_c, Theorem 3 is then correct. We shall use the notation of [1] in this note.

New definition of T_c: We define a topology on $\mathcal{G}_p$ by defining the closed

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sets in $\mathcal{G}_p$. Specifically we shall give the conditions under which a generalized sequence $\{g_n\}$ converges to a limit g in $\mathcal{G}_p$. Then a set A is closed if and only if every convergent generalized sequence in A has its limits in A.

Let $\{g_n\}$ be a generalized sequence in $\mathcal{G}_p$. We say that $g_n \rightarrow g$ if for every compact interval $I \subset R^+$ and every compact set $\mathcal{K} \supset C(I, R^n)$ and every compact set $W \subset R^n$ there is a real number Γ and a generalized sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ and such that

$$\sup_{x(\cdot)\in\mathcal{K}} \int_I |g_n(x(s), s) - g(x(s), r)|^p ds \leq \varepsilon_n^p , \tag{2}$$

and for all $J \subset I$ and $x(\cdot) \in C(J, W)$ one has

$$\left\{\int_J |g_n(x(s)s,) - g(x(s), s)|^p ds\right\}^{1/p} \leq \Gamma m(J) + \varepsilon_n \tag{3}$$

were $m(J)$ is the Lebesgue measure of J.

Let F_c denote the collection of all closed sets in $\mathcal{G}_p$ and the T_c denote all open sets, i. e. complements of closed sets. In order to show that T_c is a topology one must show that F_c is closed under finite unions and arbitrary intersections. Both of these points are easily verified and we shall omit the details.

Let $g \in \mathcal{G}_p$. One can then view g as a nonlinear mapping of $C(I, R^n)$ into $L_p(I, R^n)$. The topology T_c is the nonlinear analogue of the weak topology. Condition (2) merely assures us of uniform convergence on compact sets and condition (3) is an appropriate generalization of the uniform boundedness principle.

Modification of the proof of Theorem 3 in [1]: We now consider Theorem 3 with the topology T_c as defined above. In this proof we shall make use of the following lemma which can easily be proved by using the method of argumentation normally used in the proof of the ARZELA-ASCOLI Theorem: « Let $\{y_n\}$ be a sequence of continuous functions defined for $0 \leq t \leq \beta$ such that

$$\sup_n \sup_t |y_n(t)| < \infty , \tag{4}$$

and such that for every t, $0 \leq t \leq \beta$, and every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|h| < \delta$ then

$$|y_n(t+h) - y_n(t)| \leq \varepsilon + \varepsilon_n \tag{5}$$

for all n, where $\varepsilon_n \to 0$. Then $\{y_n\}$ contains a subsequence that converges uniformly for $0 \leq t \leq \beta$.»

The new proof of Theorem 3 uses the notation introduced on page 145 and up through line 15 on page 146.

Since $g_n \to g$ in T_c this means that there is a real number Γ and a sequence of positive numbers $\{\varepsilon_n\}$ such that $\varepsilon_n \leq 1$, $\varepsilon_n \to 0$ and for every interval $I \subset [0, \beta]$ and every continuous function $y(\cdot)$ in $C([0, \beta], K)$ one has

$$\left\{ \int_I |g_n(y(s), s) - g(y(s), s)|^p ds \right\}^{1/p} \leq \Gamma m(I) + \varepsilon_n .$$

By using the last inequality with the MINKOWSKI inequality we get

$$\left\{ \int_I |g_n(y(s), s)|^p ds \right\}^{1/p} \leq \Gamma m(I) + \varepsilon_n + \left\{ \int_I m(s)^p ds \right\}^{1/p} \tag{6}$$

whenever $y(\cdot)$ is in $C([0, \beta], K)$.

Let B be the bound for $\{a_n\}$ defined by [1, Eqn. (8)]. Define $y_n(t)$ by

$$\begin{aligned} y_n(t) &= x_n(t),\ 0 \leq t \leq T_n , \\ &= x_n(T_n),\ T_n \leq t \leq \beta , \end{aligned}$$

where T_n satisfies the following conditions:

1) $y_n(\cdot)$ is continuous, $y_n(t) \in K$ for $0 \leq t \leq \beta$, and

2) T_n is maximal with respect to 1).

Since $f_n \to f$ we can find an N_1 so that $T_n > 0$ for $n \geq N_1$. Furthermore, for $n \geq N_1$ one has

$$\sup_{n \geq N_1} \sup_{0 \leq t \leq \beta} |y_n(t)| < \infty ,$$

so that (4) is satisfied.

Consider now the difference $|y_n(t+h) - y_n(t)|$ where $0 \leq t \leq t + h \leq \beta$. Clearly it suffices to look at this difference for $0 \leq t \leq t + h \leq T_n$. We then have

$$\begin{aligned} |y_n(t+h) - y_n(t)| \leq |f_n(t+h) - f_n(t)| &+ \left| \int_t^{t+h} a_n(t+h, s) g_n(x_n(s), s) ds \right| \\ &+ \left| \int_0^t [a_n(t+h, s) - a_n(t, s)] g_n(x_n(s), s) ds \right| \end{aligned} \tag{7}$$

$$\leq |f_n(t+h) - f_n(t)| + B\left[\Gamma h + \varepsilon_n + \left\{\int_t^{t+h} m(s)^p ds\right\}^{1/p}\right]$$

$$+\left\{\int_0^\beta |a_n(t+h, s) - a_n(t, s)|^q ds\right\}^{1/q}\left\{\Gamma\beta + \varepsilon_n + \left\{\int_0^\beta m(s)^p ds\right\}^{1/p}\right\},$$

where Ineq. (6) is used in the last step. By using the argument starting at line 4 on page 148, one can associate with every $\varepsilon > 0$ a $\delta (0 < \delta \leq \varepsilon)$ so that if $|h| < \delta$ one then has

$$|f_n(t+h) - f_n(t)| < \varepsilon$$

$$\left\{\int_0^\beta |a_n(t+h, s) - a_n(t, s)|^q ds\right\}^{1/q} < \varepsilon$$

$$\left\{\int_t^{t+h} m(s)^p ds\right\}^{1/p} < \varepsilon .$$

Thus for $|h| < \delta$ and $n \geq N_1$ we get

$$|y_n(t+h) - y_n(t)| \leq \varepsilon\left[2 + B(\Gamma + 1) + \beta + \left\{\int_0^\beta m(s)^p ds\right\}^{1/p}\right] + B\varepsilon_n ,$$

since $\varepsilon_n \leq 1$ and $\delta \leq \varepsilon$. We can now apply the lemma stated above and conclude that there is a subsequence of $\{y_n\}$ that converges, say that

$$y = \lim y_n \qquad \text{(uniformly for } 0 \leq t \leq \beta).$$

This implies that T_n is bounded away from zero. Hence $y_n(t) = x_n(t)$ on some nontrivial interval $[0, \sigma]$. The rest of the argument of Theorem 3 in [1] can be used to show that for $0 \leq t \leq \sigma$, $y(t)$ is a solution of the limiting equation. In a similar way one can also show that the interval $[0, \sigma]$ can be extended to $[0, \beta]$ by a finite number of repetitions of the above argument.

REMARK. - One can weaken the topology T_c somewhat if on is interested in only solutions of differential equations, that is where the matrix $a(t, s)$ becomes the identity matrix. In this case one can weaken (1) (2) and (3) by

(1) Recall that if I is compact, then $L_p(I) \subset L_1(I)$ and if $h_n \to h$ in $L_p(I)$, then h_n h in $L_1(I)$.

bringing the absolute value signs outside the integral. That is, by replacing (2) and (3) with

$$\sup_{x(\cdot)\in\mathcal{K}} \left| \int_I [g_n(x(s),\ s) - g(x(s),\ s)]\, ds \right| \leq \varepsilon, \tag{8}$$

and for all $J \subset I$ and $x(\cdot) \in C(J,\ W)$ one has

$$\left| \int_J [g_n(x(s),\ s) - g(x(s),\ s]\, ds \right| \leq \Gamma m(J) + \varepsilon_n . \tag{9}$$

Now Inequality (6) becomes

$$\left| \int_I g_n(y(s)\ ,s) ds \right| \leq \Gamma m(I) + \varepsilon_n + \int_I m(s) ds,$$

and Inequality (7) becomes

$$|y_n(t+h) - y_n(t)| \leq |f_n(t+h) - f_n(t)| + \Gamma h + \varepsilon_n + \int_t^{t+h} m(s) ds.$$

The remainder of the argument is now unchanged.

REMARK. - One can define the topology T_c in other ways in order to obtain Theorem 3 of [1]. For example consider the sets of the form

$$U(\varepsilon,\ I,\ \mathcal{K}) = \left\{ (g,\ h):\ \sup_{x(\cdot)\in\mathcal{K}} \left\{ \int_I |g(x(s),\ s) - h(x(s),\ s)|^p ds \right\}^{1/p} < \varepsilon \right\}$$

and of the form

$$V(\varepsilon,\ I,\ W) = \left\{ (g,\ h): \left| \left\{ \int_I (\sup_{x\in W} \left| g(x,\ s) \right|)^p ds \right\}^{1/p} - \right.\right.$$

$$\left.\left. \left\{ \int_I (\sup_{x\in W} \left| g(x,\ s) \right|)^p ds \right\}^{1/p} \right| < \varepsilon \right\}$$

where $\varepsilon > 0$, I is any bounded subinterval of R^+, $W \subset R^n$ is any compact set and $\mathcal{K} \subset C(I,\ R^n)$ is any compact family of functions. This family of sets is a subbasis for a uniformity on $\mathcal{G}_p$. Let $T_c(U)$ be the corresponding uniform topology. If $\{g_n\}$ is a generalized sequence such that $g_n \to g$ in $T_c(U)$, then there exists a sequence $\{\varepsilon_n\}$ such that

$$\left| \left\{ \int_0^\beta \left(\sup_{x \in W} \left| g_n(x, s) \right| \right)^p ds \right\}^{1/p} - \left\{ \int_0^\beta (\sup_{x \in W} | g(x, s) |)^p ds \right\}^{1/p} \right| < \varepsilon_n .$$

Therefore if $x(\cdot) \in C([0, \beta], W)$ and $0 \leq t \leq \beta$ one has

$$\left\{ \int_0^t | g_n(x(s), s) |^p ds \right\}^{1/p} \leqq \left\{ \int_0^\beta (\sup_{x \in W} | g_n(x, s) |)^p ds \right\}^{1/p}$$

$$\leqq \varepsilon_n + \left\{ \int_0^\beta (\sup_x | g(x, s) |)^p ds \right\}^{1/p}$$

$$\leqq \varepsilon_n + \left\{ \int_0^\beta m(s)^p ds \right\}^{1/p}$$

where $m(s)$ is any majorizing function for $g(x, t)$. This shows that the estimates in [1, p. 147] lines 18-19 are now true. The rest of the proof Theorem 3 can then proced word for word as in [1].

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