The Wazewski topological method for contingent equations (*).

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Summary. - *A Wazewski - type theorem for contingent equations is obtained using the fundamental theory of contingent equations.*

1. - Introduction.

The topological method of WAZEWSKI [12], which was originally stated for ordinary differential equations whose solutions are unique with respect to initial conditions, has been subsequently extended to equations without uniqueness (see for example, BIELECKI [4], JACKSON and KLASSEN [9], or BEBERNES and SCHUUR [3]) and to contingent equations (see for example, BIELECKI and KLUCZNY [5], or KLUCZNY [10]).

KLUCZNY discusses several of these extensions. And in his paper he shows that if a family of curves satisfies certain postulates, then a WAZEWSKI theorem may be proved for this family. By proving that the family of solutions of a contingent equation satisfies these postulates, it the follows that a WAZEWSKI theorem holds.

Here we develop the fundamental theory of contingent equations along lines suggested by YORKE [14], [15]. We then use this theory to study the properties of a set-valued consequent mapping and to obtain a WAZEWSKI theorem for contingent equations.

Let $c(R^n)$ $(cc(R^n))$ denote the family of all nonempty compact (compact and convex) subsets of R^n. For $x \in R^n$ and $A, B \in c(R^n)$, let $|x|$ be the Euclidean norm, $r(x, B) = \inf \{|x - y| : y \in B\}$, $r(A, B) = \sup \{r(x, B) : x \in A\}$, and $d(A, B) = \max \{r(A, B), r(B, A)\}$. Let $N_a(A) = \{x : r(x, A) < a\}$.

Let $R^1 \times R^n = W$ and denote points of W by $P = (t_p, x_p)$, or just (t, x), or by $P_n = (t_n, x_n)$. For $P, Q \in W$, $|P - Q|$ is the Euclidean norm. For $A \subset W$ $Fr(A)$ is the frontier of A and $\bar{A}$ is the closure of A.

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For $E \subset W$ and P an accumulation point of E, the *positive contingent of E at P* is

$$D^+(E : P) = \{ y \in R^n : \text{ there exists a sequence } \{P^n\} \subset E \text{ such that}$$

$$\text{as } n \to \infty,\ P_n \to P,\ t_n > t_p, \text{ and } |(x_n - x_p)(t_n - t_p)^{-1} - y| \to 0 \}.$$

The negative contingent and contingent of E at P are defined in a similar manner. If E is the graph of a function $\varphi : R^1 \to R^n$ and $P = (t_0, x_0)$ we write $D^+\varphi(t_0)$ instead of $D^+(E : P)$

A mapping $F : W \to c(R^n)$ is *usc* (upper semi-continuous) at $P \in W$ if, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|Q - P| < \delta$ implies $r(F(Q), F(P)) < \varepsilon$. The mapping F is *usc on a set* $E \subset W$ if it is *usc* at each point of the set. If, in this definition, we replace $r(F(Q), F(P))$ by $d(F(Q), F(P))$, then F is *continuous.*

We shall consider the contingent equation

(1) $\qquad x' \in F(t, x)$ where $F : W \to cc(R^n)$ is usc on W.

By a *solution of* (1) we mean a continuous function $\varphi : I \to R^n$, where I is some interval in R^1, such that $D\varphi(t) \subset F(t, \varphi(t))$ for $t \in I$. (If I contains end-points we use one sided contingents where necessary). It is known [13] that if $\varphi(t)$ is a solution of (1) on I, then $\varphi(t)$ is absolutely continuous and $\varphi'(t) \in F(t, \varphi(t))$ a.e. on I.

The *trajectory of a solution* $\varphi(t)$, defined on I, is the set $\{(t, \varphi(t)) : t \in I\}$. Later we discuss the maximal interval of existence of a solution $\varphi(t)$ which we shall denote by D_φ.

We note that our results hold equally well for W an open subset of $R^1 \times R^n$.

2. - Basic Theory.

ZAREMBA [16] proved that given $P \in W$ there exists a solution $\varphi(t)$ of (1) which is defined on some open interval containing t_p and which satisfies $\varphi(t_p) = x_p$. (An account of Zaremba's work may also be found in [6]).

This theorem is the counterpart of the PEANO existence theorem for ordinary differential equations. In developing the WAŻEWSKI method in [3] we found it convenient to use not the PEANO theorem, but an existence theorem of NAGUMO [11]. The NAGUMO theorem translates to contingent equations. We include a proof for the sake of completenness. (The NAGUMO theorem for differential inequalities, and a remark on contingent equations, may be found in [7]).

DEFINITION. - A set $A \subset W$ is *right admissible* (with respect to (1)) if, for each $P \in A$ there exists an $a > 0$ and a solution $\varphi(t)$ of (1) such that $\varphi(t)$ is defined and $(t, \varphi(t)) \in A$ for $t_p \leq t \leq t_p + a$ and $\varphi(t_p) = x_p$.

THEOREM 1. - A closed set $E \subset W$ is right admissible if and only if $D^+(E : P) \cap F(P) \neq \emptyset$ for all $P \in E$.

PROOF. - That this condition is necessary for right admissibility is immediate.

We assume that in a neighborhood of P, $F(P)$ is continuous. If this is not the case, then in the manner of ZAREMBA ([16, Theorem II.8] or [6, Theorem 1.3]), we may approximate F by a family of such functions. We shall have the theorem for each member of the family and by a limiting argument the theorem holds for F.

We next observe that the following conditions are equivalent: *i)* $D^+(E : P) \cap F(P) \neq \emptyset$ for all $P \in E$; *ii)* for $P \in E$ there exists a $v \in F(P)$ such that given $\varepsilon > 0$ there exists a $Q \in E$ with $t_p < t_Q < t_p + \varepsilon$ and $|(x_Q - x_p)(t_p - t_Q)^{-1} - v| < \varepsilon$; *iii)* for $P \in E$ and $\varepsilon > 0$ there exists a $Q \in E$ with $t_p < t_Q < t_p + \varepsilon$ and $r((x_Q - x_p)(t_Q - t_p)^{-1}, F(P)) < \varepsilon$.

For $P_0 \in E$ there exist a, $L > 0$ such that $r(F(P), 0) \leq L$ on

$$T = \{P : t_0 \leq t_p \leq t_0 + a, |x_p - x_0| \leq (L + 1) |t_p - t_0|\}.$$

For ε, $0 < \varepsilon < 1$, let $\{P_n\}$ be a sequence of points *in* E satisfying

$$t_{n-1} < t_n < t_{n-1} + \varepsilon, \; r((x_n - x_{n-1})(t_n - t_{n-1})^{-1}, F(P_{n-1})) < \varepsilon$$

$(n = 1, 2, \ldots)$ and P_0 given above. Let M be the collection of all such points (for fixed ε) and let $b = \sup\{t : (t, x) \in M\}$.

If $b \leq t_0 + a$, then M has a limit point $(b, y) \in E \cap T$.

Hence, there exists a $(c, z) \in E$ such that

$$b < c < b + \varepsilon, \qquad r((z - y)(c - b)^{-1}, F(b, y)) < \varepsilon.$$

Now using the continity in (b, y) of this last expression we can find a $P_n \in M$ such that

$$t_n < c < t_n + \varepsilon, \; r((z - x_n)(c - t_n)^{-1}, F(t_n, x_n)) < \varepsilon$$

which contradicts the definition of b. So $b > t_0 + a$.

For each integer $n > 1$ there is a sequence $P_0, \ldots, P_m$ in E such that $t_m > t_0 + a$ and $t_{i-1} < t_i < t_{i-1} + \left(\frac{1}{n}\right)$,

$$r((x_i - x_{i-1})(t_i - t_{i-1})^{-1}, F(P_{i-1})) < \left(\frac{1}{n}\right) \qquad (i = 1, \ldots, m).$$

Let $\varphi_n(t)$ be the polygonal function which joins $P_0, \ldots, P_m$.

Then $|\varphi_n(t_2) - \varphi_n(t_1)| \leq (L+1)\,|t_2 - t_1|$ so by the ASCOLI theorem there exists a sub-sequence $\{\varphi_k(t)\}$ of $\{\varphi_n(t)\}$ which converges uniformly to $\varphi(t)$ on $[t_0,\ t_0 + a]$. Then $\varphi(t_0) = x_0$, $\varphi(t)$ is continuous (in fact absolutely continuous), and $(t,\ \varphi(t)) \in T \cap E$ for $t_0 \leq t \leq t_0 + a$. The latter is true since for $t \in [t_0,\ t_0 + a]$, *i)* each $(t,\ \varphi_n(t)) \in T$ and *ii)* there is a sequence of points in E which are vertices of the approximating polygonal lines which approach $(t,\ \varphi(t))$.

The proof can be completed by showing that given $t \in [t_0,\ t_0 + a]$ and $\nu > 0$ there exists $\eta > 0$, $N > 0$ such that

$$r((\varphi_k(s) - \varphi_k(t))(s-t)^{-1},\ F(t,\ \varphi(t))) < 2\nu \quad \text{for all} \quad k \geq N,$$

$s \in [t_0,\ t_0 + a]$ with $0 < |s - t| < \eta$. For then we can first let $k \to \infty$ and then $\nu \to 0$ and conclude that $D\varphi(t) \subset F(t,\ \varphi(t))$.

Assume $t \in (t_0,\ t_0 + a)$; the proof at the endpoints is similar. Since $F(P)$ is usc there exists an η, $0 < 2\eta < \min(t - t_0,\ t_0 + a - t)$ such that $r(F(u,\ x),\ F(t,\ \varphi(t))) < \nu$ on $R = \{(u,\ x) : |u - t| \leq 2\eta,\ |x - \varphi(u)| \leq \eta\}$. Choose N such that $k \geq N$ implies *i)* $(1/k) < \nu$, *ii)* $|\varphi_k(u) - \varphi(u)| \leq \eta$ for $|u - t| \leq 2\eta$, and *iii)* $(1/k) < \eta$ (so that if $P_0, \ldots, P_m$ are the vertices of $\varphi_k(t)$, then $\max\{(t_i - t_{i-1}) : i = 1, \ldots, m\} < \eta$). Then $k \geq N$, $|u - t| \leq \eta$ implies $r(F(u,\ \varphi_k(u)),\ F(t,\ \varphi(t))) < \nu$. Now $D\varphi_k(u) = (x_j - x_{j-1})(t_j - t_{j-1})^{-1}$ where $P_{j-1} \in R$ if $|u - t| \leq \eta$. So $r(D\varphi_k(u),\ F(t,\ \varphi(t))) \leq r(D\varphi_k(u),\ F(P_{j-1})) + r(F(P_{j-1}),\ F(t,\ \varphi(t))) < 2\nu$ for $|u - t| \leq \eta$, $k \geq N$.

If $\psi(t)$ is a continuous function defined on $[d,\ e]$ and if $D\psi(t) \subset A$, a non-empty, convex set, for $d \leq t \leq e$, then $(\psi(d) - \psi(e))(d - e)^{-1} \in A$. (See [16, Theorem II.5] or [1]).

Since $S = \{x : r(x,\ F(t,\ \varphi(t))) \leq 2\nu\}$ is compact and convex and since $D\varphi_k(u) \subset S$ for $k \geq N$, $0 < |u - t| < |s - t|$, we can conclude that $(\varphi_k(s) - \varphi_k(t))(s - t)^{-1} \in S$ for $k \geq N$, $0 < |s - t| < \eta$.

REMARKS. - 1) As in the theory of ordinary differential equations, any solution $\varphi(t)$ of (1) can be extended to a maximal interval of existence which we shall denote by D_φ.

2) If $E \subset W$ is locally compact and $F(P)$ is defined and usc only on E, the theorem still holds. For then E is relatively closed in some open $U \subset W$, F has a usc extension to all of U, and the preceding applies.

3) If F is defined and usc on a locally compact subset $E \subset W$, if E is right admissible, if $\varphi(t,\ P)$ is a solution of (1) passing through $P \in E$, and if $(t,\ \varphi(t,\ P)) \to Q \in Fr(E)$ as $t \to Fr(D_\varphi)$, then $Q \notin Fr(E) \cap E$. (For remarks on the continuation of solutions see [2]).

The next definition and two theorems follow YORKE [14], [15].

DEFINITION. - A set $A \subset W$ is *positively weakly invariant* (with respect to (1)) if and only if for each $P \in A$ there exists a solution $\varphi(t)$ of (1) such that $\varphi(t_p) = x_p$ and $(t, \varphi(t)) \in A$ for $t \in D_\varphi \cap [t_p, \infty)$. Negatively weakly invariant and weakly invariant are defined in a similar manner.

THEOREM 2. - a) A set $A \subset W$ is weakly invariant if and only if it is a union of a family of trajectories of solutions of (1). b) A closed set $E \subset W$ is positively weakly invariant if and only if it is right admissible and hence if and only if $D^+(E : P) \cap F(P) \neq \emptyset$ for all $P \in E$.

PROOF. - The proof is straightforward.

THEOREM 3. - If G_1, $G_2 \subset W$ are closed and positively weakly invariant and if $W = G_1 \cup G_2$, then $H = G_1 \cap G_2$ is positively weakly invariant.

PROOF. - Assume H is not positively weakly invariant and let $P_0 \in H$. Then there exists two trajectories $\varphi_i(t)$ of (1) with $\varphi_i(t_0) = x_0$ $(i = 1, 2)$ and an $a > 0$ such that $(t, \varphi_i(t)) \in G_i$ for $t_0 \leq t < t_0 + a$, $i = 1, 2$. Let $L(t)$ be the segment joining $(t, \varphi_1(t))$ to $(t, \varphi_2(t))$ and let $(t, x(t)) \in L(t) \cap (G_1 \cap G_2)$. Here we use the restrictions placed on G_1, G_2.

Then $x(t) = \alpha(t)\varphi_1(t) + (1 - \alpha(t))\varphi_2(t)$, $0 \leq \alpha(t) \leq 1$.
Choose a sequence $\{t_n : t_n \to t_0 +$, and $\alpha(t_n) \to \alpha_0\}$. Then

$$(x(t_n) - x_0)(t_n - t_0)^{-1} = (\alpha(t_n)(\varphi_1(t_n) - x_0) + (1 - \alpha(t_n))$$
$$(\varphi_2(t_n) - x_0))(t_n - t_0)^{-1} .$$

We can now choose a subsequence $\{t_k\}$ of $\{t_n\}$ such that in the limit the left side of this expression belongs to $D^+(H : P_0)$ and the right side equals $\alpha_0 v_1 + (1 - \alpha_0)v_2$ where $v_i \in D^+\varphi_i(t_0)\,(i = 1, 2)$.

By convexity this vector is in $F(P_0)$ hence $D^+(H : P_0) \cap F(P_0) \neq \emptyset$ and H is positively weakly invariant.

In the sequel we shall use the following KAMKE-type convergence theorem. (See for example [8, theorem 3.2]).

THEOREM 4. - Let $\{P_n \in W : P_n \to P_0$ as $n \to \infty\}$ and let $\varphi_n(t)$ be a solution of (1) with $\varphi_n(t_n) = x_n$. Then there exists a solution $\varphi(t)$ of (1) with $\varphi(t_0) = x_0$ and with maximal interval of existence D_φ and a subsequence $\{\varphi_k(t)\}$ of $\{\varphi_n(t)\}$ such that for any compact subinterval $I \subset D_\varphi$ and all k sufficiently large, $\varphi_k(t)$ is defined on I and $\varphi_k(t) \to \varphi(t)$ as $k \to \infty$ uniformly on I.

PROOF. - Let $\varphi_n(t)$ be as in the hypotheses. There exists a, $N > 0$ such that if $n \geq N$, then $\varphi_n(t)$ is defined and uniformly bounded for $|t - t_0| \leq a$. Just let $R = \{(t, x) : |t - t_0| \leq 2b, |x - x_0| \leq 2b\}$ for some $b > 0$, $M = \max \{r(F(P), 0) : P \in R\}$, and choose $a = \min((b/2), (b/2M))$ and N such that $n \geq N$ implies $|P_n - P| < b$.

If $\{\varphi_n(t)\}$ is a sequence of solutions of (1) which are defined, uniformly bounded, and equicontinuous on an interval I, then exists a subsequence $\{\varphi_k(t)\}$ such that $\varphi_k(t) \to \psi(t)$ uniformly on I as $k \to \infty$ (by the ARZELA-ASCOLI theorem) and $\psi(t)$ is a solution of (1) (by ZAREMBA, [16, theorem II.6]).

Let $\{\varphi_n(t)\}$ be as in the hypotheses and choose a as in the first paragraph of the proof. Let $\{\varphi_k(t)\}$ be a subsequence of $\{\varphi_n(t)\}$ such that $\varphi_k(t) \to \psi(t)$ uniformly on $|t - t_0| \leq a$ where $\psi(t)$ is a solution of (1) with $\psi(t_0) = x_0$.

Let (b, c) be the maximal interval such that there exists a solution $\varphi(t)$ defined on (b, c); $\varphi(t) = \psi(t)$ on $|t - t_0| < a$; and for each oompact $I \subset (b, c)$, some subsequence of $\{\varphi_k(t)\}$ is defined and converges to $\varphi(t)$ on I.

We claim that $\varphi(t)$ is not defined at $t = b$, c and hence $(b, c) = D_\varphi$ and φ satisfies the conclusion of the theorem. For suppose $\varphi(b)$ is defined. We can find a subsequence of $\{\varphi_k(t)\}$ and a sequence $\{t_m : t_m \to b -\}$ such that $|\varphi_m(t) - \varphi(t)| < \left(\dfrac{1}{m}\right)$ on $[t_0, t_m]$. Now using the first two paragraphs of the proof we can extend $\varphi(t)$ beyond b.

3. - The consequent mapping and Wazewski's theorem.

DEFINITIONS. - Let V be an open subset of W. A point $Q \in Fr(V)$ is a *consequent of a point* $P \in V$ (relative to (1)) if there exists a solution $\varphi(t, P)$ of (1) and a t_1, $t_p < t_1 \leq t_Q$, such that $\varphi(t, P)$ is defined on $[t_p, t_Q]$, $(t, \varphi(t, P)) \in V$ for $t_p \leq t < t_1$, $(t, \varphi(t, P)) \in Fr(V)$ for $t_1 \leq t \leq t_Q$, and $(t_Q, \varphi(t_Q, P)) = Q$. The point $Q \in Fr(V)$ is a *consequent of a point* $P \in Fr(V)$ (and P may equal Q) if there exists a solution $\varphi(t, P)$ of (1) such that $(t, \varphi(t, P)) \in Fr(V)$ for $[t_p, t_Q]$. Consequents will also be called *points of egress* and the set of all egress points will be denoted by S.

A point $Q \in S$ is a *strict egress point* if for every solution $\varphi(t, Q)$, $c_\varphi = \sup\{t : (s, \varphi(s, Q)) \in Fr(V),\ t_Q \leq s \leq t\} < \infty$ and there exists a sequence $\{t_n : t_n \to c_{\varphi +}\}$ with $(t_n, \varphi(t_n, Q)) \in W - \bar{V}$. The mapping $C : (V \cup S) \to Fr(V)$ defined by $C(P) = \{Q \in Fr(V) : Q$ is a consequent of $P\}$ is the *consequent mapping.*

A solution $\varphi(t, P)$, $P \in V$, *leaves* V if there exists some $t_1 \in D_\varphi \cap [t_p, \infty)$ such that $(t_1, \varphi(t_1, P)) \in W - V$.

THEOREM 5. - Let $C : (V \cup S) \to Fr(V)$ be the consequent mapping and let $P \in V \cup S$. If all solutions through P leave V and if all points of egress are strict, then $C(P)$ is compact.

PROOF. - We shall show that any sequence $\{Q_n\} \subset C(P)$ contains a subsequence which converges to a point of $C(P)$.

Let $\varphi_n(t)$ be a solution of (1) through P such that $(t_n, \varphi_n(t_n)) = Q_n$. By Theorem 4 there is a subsequence $\{\varphi_k(t)\}$ of $\{\varphi_n(t)\}$ which converges to a solution $\varphi(t)$ of (1) with $\varphi(t_p) = x_p$. By assumption, $\varphi(t)$ leaves V and every point of egress is strict.

Hence there exists a $t_a > t_p$ such that $(t_a, \varphi(t_a)) \in W - \overline{V}$. For all k sufficiently large, $(t_a, \varphi_k(t_a)) \in W - \overline{V}$ hence $t_p < t_k < t_a$. Choose a subsequence $\{t_m\}$ of $\{t_k\}$ such that $t_m \to t_b < t_a$. Them $Q_m = (t_m, \varphi_n(t_m)) \to (t_b, \varphi(t_b))$.

Now $(t_b, \varphi(t_b)) \notin C(P)$ implies that for m sufficiently large, Q_m is pot an egress point. The possibilities (for example, $(t_c, \varphi(t_c)) \in W - \overline{V}$ for some $t_c \in (t_p, t_b)$) are easily checked. Thus $C(P)$ is compact.

THEOREM 6. - Let $C : (V \cup S) \to Fr(V)$ be the consequent mapping and let $P \in V \cup S$. If all solutions through P leave V and all points of egress are strict, then $C(P)$ is connected.

PROOF. - If $P \in S$, the proof is immediate. So assume $P \in V$.

If $C(P)$ is not connected, then $C(P) = C_1 \cup C_2$ where C_1 and C_2 are disjoint compact subsets of W.

For $\varphi(t, P)$ a solution of (1) with $P \in V$ let $D_\varphi = (\alpha, \omega)$ be the maximal interval of existence of $\varphi(t, P)$ relative to W and let $E_\varphi = (\beta, \gamma)$ be the maximal interval of existence of $\varphi(t, P)$ relative to V, i.e. E_φ is the largest open interval about t_p such that $(t, \varphi(t, P)) \in V$ for $t \in E_\varphi$. Let $\Phi(P)$ denote the trajectory of $\varphi(t, P)$ relative to V, i.e. $\Phi(P) = \{(t, \varphi(t, P)) : t \in E_\varphi\}$. Let $\sigma(A, B) = \inf \{|a - b| : a \in A, b \in B\}$.

Let $R = \{\Phi(Q) : Q \in V$ and $\sigma(\Phi(Q), C_1) \leq \sigma(\Phi(Q), C_2)\}$ and let S be a set similarly defined but with the inequality reversed. Then R, S are positively weakly invariant and closed, relative to V, $V = R \cup S$, and $P \in R \cap S$. We may conclude by Theorem 8 that there exists a $\varphi(t, P)$ such that $(t, \varphi(t, P)) \in R \cap S$ for $t_p \leq t < \gamma$.

Since $\varphi(t, P)$ leaves V, $\gamma < \omega$ and $(\gamma, \varphi(\gamma, P)) \in Fr(V)$, say $(\gamma, \varphi(\gamma, P)) \in C_1$. Then $0 = \sigma(\Phi(P), C_1) = \sigma(\Phi(P), C_2) > 0$ and from this contradiction we conclude that $C(P)$ is connected.

THEOREM 7. - Let $C : (V \cup S) \to Fr(V)$ be the consequent mapping and let A be a nonempty subset of $V \cup S$. If all solutions through P leave V, for each $P \in A$, and if all points of egress are strict, then C is usc on A.

PROOF. - By Theorem 5, $C(P)$ is compact for each $P \in A$.

If C is not usc at some $P_0 \in A$, then there exists an $\varepsilon > 0$ and a sequence $\{P_n \in A : P_n \to P_0$ and $r(C(P_n), C(P_0)) \geq \varepsilon$ as $n \to \infty\}$.

Hence, for each n, there is a $Q_n \in C(P_n)$ such that $r(Q_n, C(P_0)) \geq \varepsilon$. Let $\varphi_n(t)$ be a solution of (1) through P_n and Q_n. By Theorem 4 there is a subsequence $\{\varphi_k(t)\}$ of $\{\varphi_n(t)\}$ which converges to a solution $\varphi(t)$ of (1) with $\varphi(t_0) = x_0$. By assumption, $\varphi(t)$ leaves V and every point of egress is strict. Now just as in the proof of Theorem 5 we can find a subsequence $\{t_m\}$ of $\{t_k\}$ such that $(t_m, \varphi_m(t_m)) = Q_m \to (t_b, \varphi(t_b)) \in C(P_0)$ which is a contradiction.

REMARK. - If A is a compact and connected subset of $V \cup S$, then under the hypotheses of Theorem 7, $C(A)$ is compact and connected. This is a consequence of the properties of usc functions. (See for example [3], Theorem 1).

DEFINITION. - Let A, B be subsets of R^{n+1} with $B \subset A$. If there exists a usc mapping $G : A \to c(R^{n+1})$ such that $G(x) \subset B$ and $G(x)$ is connected for all $x \in A$ and $x \in G(x)$ for all $x \in B$, then B is a *set-valued retract* of A and G is *a set-valued retraction* from A into B.

THEOREM 8. - If there exists a set $Z \subset V \cup S$ such that $Z \cap S$ is a set-valued retract of S but not of Z and if all points of S are points of strict egress, then there exists a $P \in Z$ and a solution $\varphi(t, P)$ of (1) such that $(t, \varphi(t, P)) \in V$ for $t \in D_\varphi \cap [t_p, \infty)$.

PROOF. - If, for all $P \in Z$, every solution $\varphi(t, P)$ of (1) leaves V, then the consequent mapping $C : Z \to S$ is usc.

Let $H : S \to Z \cap S$ be the set-valued retraction which is given in the hypotheses. Then $HC: Z \to Z \cap S$ is usc and for each $P \in Z \cap S$, $P \in H(P) \subset HC(P)$. Thus $Z \cap S$ is a set-valued retract of Z which is a contradiction.

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