Global existence without uniqueness (*).

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Summary. - *In this paper we investigate the extendability of solutions of ordinary differential equations on $[to, \infty)$ with the use of Liapunov functions. The results extend to non-unique systems the work done by Conti and Strauss, as well as establishing an equivalence between two properties of a Liapunov function.*

1. - Introduction.

Liapunov functions have been used to determine many properties of solutions of ordinary differential equations including stability, boundedness and invariance (see Yoshizawa [8]). More recently Liapunov functions have been used to investigate the extendability of solutions on $[t_0, \infty)$ (existence in the future) and on $(-\infty, \infty)$ (existence forever) under the assumption that solutions are unique. In this paper we use Liapunov functions to determine the existence in the future of every solution $x(t, t_0, x_0)$ of $\dot{x} = f(t, x)$ where $f: R^1 \times R^d \to R^d$ is merely continuous. A theorem (Theorem 1) is given which extends to non-unique systems the work done by Conti [1] and Strauss [6]. Kneser's theorem as well as other theorems related to the properties of solution funnels are used in the proofs. A second theorem (Theorem 2) is presented which establishes an equivalence between two properties of a Liapunov function, assuming that all solutions exist in the past.

2. - Main Results.

Let R^d denote Euclidean d-space and let $|\cdot|$ denote any norm in R^d. For $x, y \in R^d$ define $d(x, y) = |x - y|$. Consider the system

$$\text{(E)} \qquad \dot{x} = f(t, x)$$

where $f: R^1 \times R^d \to R^d$ is continuous. Denote a solution of (E) through the point (t_0, x_0) by $x(\cdot, t_0, x_0)$.

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Let I be an interval and consider a function $V : I \times R^d \to R^1$. V is said to be locally Lipschitz if it is continuous on $I \times R^d$ and if for each point (t, x) there exists a neighborhood N of (t, x) and a constant $K > 0$ such that

$$|V(s, y) - V(s, z)| \leq K|y - z|$$

for all (s, y) and (s, z) in $N \cap (I \times R^d)$. Define

$$\dot{V}(t, x) = \limsup_{h \to 0^+} h^{-1}((V(t + h, x + hf(t, x)) - V(t, x)).$$

If V is locally Lipschitz, YOSHIZAWA [8 page 3] has proved that

$$\dot{V}(t, x) = \limsup_{h \to 0^+} h^{-1}(V(t + h, x(t + h, t, x)) - V(t, x)).$$

We now state our main results.

THEOREM 1. - Assume $V : R^1 \times R^d \to R^1$ is locally Lipschitz, satisfying

$$V(t, x) \to \infty \quad \text{as} \quad |x| \to \infty \tag{2.1}$$

for each fixed $t \in R^1$

and

$$\dot{V}(t, x) \leq \varphi(t, V(t, x)) \tag{2.2}$$

where we assume $\varphi : R^1 \times R^1 \to R^1$ is continuous; and for every real t_0 and r_0 the maximal solution $r(t, t_0, r_0)$ of the comparison equation $\dot{r} = \varphi(t, r)$ exists in the future. Then all solutions of (E) exist in the future.

Using the same reasoning as that used in Theorem 1 we have the following corollary.

COROLLARY 1. - Let $V : R^1 \times R^d \to R^1$ be locally Lipschitz and satisfy (2.1) and

$$\dot{V}(t, x) \geq \Psi(t, V(t, x)),$$

where we suppose $\Psi : R^1 \times R^1 \to R^1$ is continuous; and for every real t_0 and r_0 the maximal solution $\rho(t, t_0, r_0)$ of $\dot{r} = \Psi(t, r)$ exists in the past. Then every solution of (E) exists in the past.

THEOREM 2. - Let $V : R^1 \times R^d \to R^1$ be locally Lipschitz and satisfy (2.1) and (2.2). If solutions exist in the past then we have

$$V(t, x) \to \infty \quad \text{as} \quad |x| \to \infty \tag{2.3}$$

uniformly for t in compact sets.

REMARK. - Theorem 1 was first stated by CONTI [1]. His proof, however, needed the stronger hypothesis (2.3); see for example [4 page 108]. Later, STRAUSS [6] gave a proof of Theorem 1. These results were for systems (E) with uniqueness, and indeed STRAUSS' proof relied heavily on this fact. In proving Theorem 1, as a result of the non-uniqueness of solutions, we make great use of the properties of solution funnels; and we then provide an example which demonstrates how complicated the behavior of solutions can be in the case of non-uniqueness.

As we mentioned before, originally Theorem 1 was proved using (2.3) instead of (2.1) by CONTI. Later KATO and STRAUSS [3] showed that if solutions of $\dot{x} = f(t, x)$, where f is locally Lipschitz, exist in the future then there exists a function $V(t, x)$ such that $V : R^1 \times R^d \to R^1$ is locally Lipschitz satisfying (2.2) and (2.3). In another paper [6], STRAUSS provided an example of a particular V satisfying (2.1) and (2.2) but not (2.3), so that KATO and STRAUSS' earlier result shows some other V must satisfy (2.2) and (2.3). In STRAUSS' example some solutions did not exist in the past. A natural question then is whether it is precisely the existence in the past which is needed to prove (2.1) and (2.3) are equivalent. Theorem 2 affirmatively answers the question. Once again the assumption of uniqueness is not needed.

3. - Proofs.

We will need to carefully analyze the properties of solution funnels. For $(t_0, x_0) \in R^1 \times R^d$ define the positive and negative solution funnels respectively as

$$F^+_{t_0, x_0} = \{(t, x(t)) : t \geq t_0, \ x(t_0) = x_0\} \subset R^{d+1}$$

and

$$F^-_{t_0, x_0} = \{(t, x(t)) : t \leq t_0, \ x(t_0) = x_0\} \subset R^{d+1}$$

where $\dot{x}(t) = f(t, x(t))$. The solution funnel through (t_0, x_0), denoted by F_{t_0, x_0}, is defined as

$$F_{t_0, x_0} = F^+_{t_0, x_0} \cup F^-_{t_0, x_0}.$$

When the initial point is understood we shall only write F^+, F^-, and F.

The τ-cross-section, denoted by $F(\tau)$, is a subset of R^d formed by the intersection of F and the hyperplane $t = \tau$, that is

$$F(\tau) = F \cap (\{\tau\} \times R^d).$$

We shall now state without proof the following known results which will prove useful in this section.

LEMMA 1 [2, page 14]. - Let $x_n(t)$ be a solution of (E), $x_n(t_n) = x_n$. We suppose $t_n \to t_0$ and $x_n \to x_0$. Then there exists a subsequence of solutions $x_{n(1)}(t)$, $x_{n(2)}(t)$ ... such that $x_{n(k)}(t)$ approaches $x_0(t)$ as $k \to \infty$ uniformly on compact subsets of the domain of $x_0(t)$ where $x_0(t)$ is a solution of (E) satisfying $x_0(t_0) = x_0$.

The following result is due to KNESER [2, page 15].

LEMMA 2. - The τ-cross-section of the solution funnel through (t_0, x_0) is compact and connected if all solutions exist on $[t_0, \tau]$.

The following lemma is a generalized convergence result proved by STRAUSS and YORKE [7].

LEMMA 3. - Let f be a continuous mapping on an open set $D \subset R^1 \times R^d$. Let $(\tau_0, \xi_0) \in D$ and suppose all solutions of (E) through (τ_0, ξ_0) exist on $[a, b]$, $\tau_0 \in [a, b]$. Then for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, such that if $d((\tau, \xi), (t_0, \xi_0)) < \delta$, then for each solution $x(\cdot, \tau, \xi)$ of (E), there exists a solution of (E) through (τ_0, ξ_0), $x(\cdot, \tau_0, \xi_0)$, such that $|x(t, \tau, \xi) - x(t, \tau_0, \xi_0)| < \varepsilon$ for all $t \in [a, b]$.

We shall define a metric topology on the class $\mathcal{K}$ of non empty compact sets of R^d. Let $A \in \mathcal{K}$ and $\varepsilon > 0$ be given and define

$$R(A, \varepsilon) = \{S \in \mathcal{K} : S \subset B(A, \varepsilon)\},$$

$$\rho^*(S, A) = \inf\{\varepsilon : S \subset R(A, \varepsilon)\}, \text{ and}$$

$$\rho(S, A) = \max(\rho^*(S, A), \rho^*(A, S)).$$

It can be shown that ρ is a metric on $\mathcal{K}$. This topology is referred to as the Hausdorff metric topology.

LEMMA 4 ([5]). - Consider the funnel of solutions of (E) through the point $(t_0, x_0) = p$. Suppose all solutions exist in the future. Then the t-cross-section through p, $F_p(t)$, has the property that $F_p : R^1 \to \mathcal{K}$ is continuous in the Hausdorff metric topology.

With these preliminaries we now state and prove a lemma concerning funnel cross-sections when some solutions do not exist in the future.

LEMMA 5. - Assume all solutions through (t_0, x_0) exist up to but not necessarily at w^*. Suppose there exists a solution $y(t, t_0, x_0)$ such that $|y(t, t_0, x_0)| \to \infty$ as $t \to w^*$, and there exists a solution $x(t, t_0, x_0)$ which is defined at w^*. Then the w^*-cross-section, $F(w^*)$, is unbounded.

PROOF OF LEMMA 5. - We define $M = |x(w^*, t_0, x_0)|$, where $M < \infty$ by assumption. By an application of Lemma 4, we may conclude that there exists a $\delta > 0$, such that if $t \in [w^* - \delta, w^*]$, we have

$$(3.1) \qquad |x(t, t_0, x_0)| \leq 2M.$$

Let $\{t_n\}$ be any sequence of points in the closed interval $[w^* - \delta, w^*]$ such that $t_n \to w^*$ as $n \to \infty$. By hypothesis we have

$$|y(t_n, t_0, x_0)| \quad \infty \text{ as } t_n \to w^*.$$

Define

$$(3.2) \qquad k_n = |y(t_n, t_0, x_0)|;$$

hence $k_n \to \infty$ as $n \to \infty$. Assume that $k_n > 2M$ for all $n > 0$.

For each n we consider the t_n-cross-section of the funnel of solutions through (t_0, x_0), By Lemma 2, these sets are connected and using (3.1) and (3.2) we conclude there exists, for each i, a sequence of solutions $\{\varphi_i^n(\cdot, t_0, x_0)\}$, in which the domain of $\varphi_i^n(\cdot)$ includes $[t_0, t_n]$ and such that

$$|\varphi_i^n(t_n, t_0, x_0)| = j_i, \qquad \text{for } i \text{ fixed, for all } n,$$

where $2M < j_i < k_i$ and $j_i \to \infty$ as $i \to \infty$.

By an application of Lemma 1 we have, for fixed i, a solution $\psi_i(\cdot, t_0, x_0)$ such that (relabeling the subsequence)

$$(3.3) \qquad \varphi_i^n(\cdot) \to \psi_i(\cdot) \quad \text{as} \quad n \to \infty$$

uniformly on compact subsets of the domain of $\psi_i(\cdot)$. By the definition of w^*, $\psi_i(\cdot)$ is defined on $[t_0, w^*)$. Since $|\varphi_i^n(t_n)| = j_i$ for all n, we have without loss of generality,

$$\varphi_i^n(t_n) \to y_0 \quad \text{as} \quad n \to \infty, \quad \text{and} \quad |y_0| = j_i.$$

We now construct a rectangle about the point (w^*, y_0). Consider, first, any number ρ, with $0 < \rho < w^* - t_0$. For all points (t, x) in the rectangle

$$R = \{(t, x) : |t - w^*| \leq \rho, \quad |x - y_0| \leq 1\},$$

there exists a $T > 0$ such that $|f(t, x)| \leq T$. Define

$$\beta = \min (1/2T, \rho/2)$$

and construct the rectangles

$$W = \{(t, x) : |t - w^*| \leq \beta/2, \ |x - y_0| \leq 1/2\}$$

and

$$L = \{(t, x) : t \in [w^* - \beta/2, w^*], \ |x - y_0| \leq 1/4\}.$$

Consider n so large that $(t_n, \varphi_i^n(t_n)) \in L$. From the construction, the graph of $\varphi_i^n(\cdot)$ is defined in W for $t \in [w^* - \beta/2, w^*]$ and satisfies the following inequality

$$j_i - 1/2 \leq |\varphi_i^n(t)| \leq j_i + 1/2.$$

Using the convergence theorem we have for $t \in [w^* - \beta/2, w^*)$,

$$j_i + 1/2 \geq |\psi_i(t)| \geq j_i - 1/2;$$

hence, $\psi_i(w^*)$ is defined and

$$|\psi_i(w^*)| \geq j_i - 1/2.$$

Now we repeat this same procedure for each i and obtain the existence of a sequence of solutions $\{\psi_i(\cdot, t_0, x_0)\}$ such that

$$|\psi_i(w^*, t_0, x_0)| \geq j_i - 1/2$$

$$\Rightarrow |\psi_i(w^*, t_0, x_0)| \to \infty \quad \text{as} \quad i \to \infty.$$

Hence the w^*-cross-section is unbounded, thereby proving the lemma. We are now in a position to prove the main result.

PROOF OF THEOREM 1. - Suppose the conclusion is false. Then there exist a point $(\tau, \widehat{x}_0)$, a solution $x(\cdot, \tau, \widehat{x}_0)$ and a point $w > \tau$ such that

$$x(\cdot) \in F^+_{\tau, \widehat{x}_0},$$

$$|x(t, \tau, \widehat{x}_0)| \to \infty \quad \text{at} \quad t \to w^-,$$

and $x(\cdot, \tau, \widehat{x}_0)$ is defined on $[\tau, w)$. There exists a point (t_0, y_0) such that $\tau < t_0 < w$ and all solution through (t_0, y_0) exist on $[t_0, w]$. Define

$$x(t_0, \tau, \widehat{x}_0) = x_0,$$

and we see that $x_0 \neq y_0$.

Define

$$L = \{z_\lambda = \lambda x_0 + (1 - \lambda) y_0 : 0 \leq \lambda \leq 1\},$$

and let

$$\lambda^* = \sup \{\lambda : x(w, t_0, z_\lambda) \text{ is finite}\}.$$

Since $z_0 = y_0$, we have by an application of Lemma 3 that $0 < \lambda^* \leq 1$.

We claim that not all solutions through (t_0, z_{λ^*}) exist up to $t = w$ for if they do, then by a consequence of Lemma 3, there would exist a neighborhood of z_{λ^*} such that all solutions through that neighborhood exist at $t = w$, contradicting the definition of λ^*.

This establishes the claim.
Define

$$w^* = \sup \{M\colon \text{all solutions through } (t_0, z_{\lambda^*}) \text{ exist on } [t_0, M)\}.$$

Hence, there exists a solution $y(\cdot)$ such that

$$|y(t, t_0, z_{\lambda^*})| \to \infty \quad \text{as} \quad t \to w^{*-},$$

where $t_0 < w^* \leq w$. Define the set

$$B = \{|x(w^*, t_0, z_\lambda)| : 0 \leq \lambda < \lambda^*, \qquad x(\cdot) \subset F^+_{t_0, z_\lambda}\}.$$

We shall consider the two cases:

1. B is unbounded,

2. B is bounded,

and arrive at a contradiction for each case, thus proving the theorem.

Assume B is unbounded. Then we can choose a sequence of solutions $\{x(w^*, t_0, z_{\lambda_i})\}$, where $0 \leq \lambda_i < \lambda^*$, such that $\lambda_i \to \lambda^*$ and

$$|x(w^*, t_0, z_{\lambda_i})| \to \infty \quad \text{as} \quad i \to \infty.$$

By the continuity of V, we know there exists an $r_0 > 0$ such that

$$V(t_0, z_\lambda) \leq r_0 \quad \text{for} \quad 0 \leq \lambda \leq 1.$$

Using the differential inequality (2.2) we conclude that

$$V(t, x(t, t_0, z_{\lambda_i})) \leq r(t, t_0, r_0)$$

for $t_0 \leq t \leq w^*$, where $r(t, t_0, r_0)$, the maximal solution of $\dot{r} = \varphi(t, r)$, exists in the future. In particular,

$$V(w^*, x(w^*, t_0, z_{\lambda_i})) \leq r(w^*, t_0, r_0)),$$

a contradiction to (2.1).

Assume now that B is bounded. Then there exists an $M > 0$ such that

$$|x(w^*, t_0, z_\lambda)| \leq M$$

for $0 \leq \lambda < \lambda^*$ and for all $x(\cdot) \in F^+_{t_0, z_\lambda}$. Consider any sequence $\{\lambda_i\}$ such that $z_{\lambda_i} \to z_{\lambda^*}$, and for each i, select a solution $x_i(\cdot, t_0, z_{\lambda_i})$ from $F^+_{t_0, z_{\lambda_i}}$. Hence by an application of Lemma 1, there exists a solution $x(\cdot, t_0, z_{\lambda^*})$ and a subsequence of solutions $\{x_{i_k}(\cdot, t_0, z_{\lambda_{i_k}})\}$ of $\{x_i(\cdot, t_0, z_{\lambda_i})\}$ such that

$$x_{i_k}(\cdot, t_0, z_{\lambda_{i_k}}) \to x(\cdot, t_0, z_{\lambda^*}) \quad \text{as} \quad i_k \to \infty$$

uniformly on compact subsets of the domain of $x(\cdot, t_0, z_{\lambda^*})$. As long as $x(\cdot, t_0, z_{\lambda^*})$ exists, we have

$$|x_{i_k}(t, t_0, z_{\lambda_{i_k}}) - x(t, t_0, z_{\lambda^*})| \leq M$$

for i_k sufficiently large and $t \in [t_0, T]$ where $[t_0, T]$ is in the domain of $x(\cdot, t_0, z_{\lambda^*})$. Therefore,

$$|x(t, t_0, z_{\lambda^*})| \leq |x_{i_k}(t, t_0, z_{\lambda_{i_k}})| + M. \tag{3.4}$$

Since $|x_{i_k}(w^*, t_0, z_{\lambda_{i_k}})| \leq M$, we have the existence of a $\delta > 0$ such that

$$|x_{i_k}(t, t_0, z_{\lambda_{i_k}})| \leq 2M \quad \text{for} \quad t \in [w^* - \delta, w^*]. \tag{3.5}$$

Since the domain of $x(\cdot, t_0, z_{\lambda^*})$ is at least as large as $[t_0, w^*)$, there exists a $K > 0$ such that

$$|x(t, t_0, z_{\lambda^*})| \leq K \quad \text{for} \quad t \in [t_0, w^* - \delta];$$

and from (3.4) and (3.5) we also can conclude that

$$|x(t, t_0, z_{\lambda^*})| \leq 3M \quad \text{for} \quad t \in [w^* - \delta, w^*).$$

If we define $K_1 = \max(3M, K)$ then

$$|x(t, t_0, z_{\lambda^*})| \leq K_1 \quad \text{for} \quad t \in [t_0, w^*);$$

hence $|x(w^*, t_0, z_{\lambda^*})| \leq K_1$. Thus we have precisely the conditions needed for Lemma 5. Hence there exists a sequence of solutions $\{\psi_i(\cdot, t_0, z_{\lambda^*})\}$, such that

$$|(\psi_i(w^*, t_0, z_{\lambda^*})| \to \infty \quad \text{as} \quad i \to \infty.$$

Therefore we have reduced the problem to the point where we can use the techniques used in Case 1, thus completing the proof.

PROOF OF THEOREM 2. - Assume V does not satisfy (2.3). Then there exist sequences $\{x_i\}$, $\{t_i\}$, and an $M > 0$ such that $|x_i| \to \infty$, $t_i \to t_0$, and $V(t_i, x_i) \leq M$. We can assume without loss of generality that $t_i \to t_0$ monotonically.

CASE I. - Assume $t_i \nearrow t_0$. For each i pick any $y_i \in F_{t_i, x_i}(t_0)$, which exists since solutions exists in the future. We claim

$$|y_i| \to \infty \quad \text{as} \quad i \to \infty .$$

Suppose the claim is false, that is there exists a subsequence $\{y_{i_k}\}$ such that $y_{i_k} \to y_0$ as $i_k \to \infty$. The solution funnel through (t_0, y_0) is compact on $[t_0 - \delta, t_0]$ for $\delta > 0$. By an application of Lemma 3, we have the existence of points $(t_i, z_i) \in F^-_{t_0, y_0}$ such that $|z_i - x_{i_k}| < 1/2$. Thus we have $|z_i| \to \infty$ as $i \to \infty$, a contradiction to the fact that the funnel through (t_0, y_0) $F^-_{t_0, y_0}$, is compact, a consequence of Lemma 2.

Since $\{y_i\}$ are unbounded and $y_i = x_i(t_0, t_i, x_i)$, we have as a result of (2.1) and (2.2) that

$$V(t_0, y_i) \leq r(t_0, t_i, V(t_i, x_i)) \quad \text{and}$$

$$V(t_0, y_i) \to \infty \quad \text{as} \quad i \to \infty .$$

Since $V(t_i, x_i) \leq M$, we have as a result of the differential inequality,

$$V(t_0, y_i) \leq r(t_0, t_i, M).$$

But $r(t_0, t_i, M)$ is bounded for all i since solutions of $\dot{r} = \varphi(t, r)$ exist in the future, thus leading to a contradiction.

Now assume $t_i \searrow t_0$. Let $\hat{t}$ be any point greater than t_0. Assume $t_i \in [t_0, \hat{t}]$ and choose $y_i \in F_{t_i, x_i}(\hat{t})$. We claim $\{y_i\}$ are unbounded which would then reduce to the previous case.

Assume $\{y_i\}$ are bounded; there exists $\bar{y}$ such that $y_{i_k} \to \bar{y}$ as $i_k \to \infty$. On $[t_0, \hat{t}]$, $F^-_{\hat{t}, \bar{y}}$ is compact, since solutions exist in the past. There exists a $\delta > 0$ such that if $|y_{i_k} - \bar{y}| < \delta$, then there exists a sequence of points $(t_i, z_i) \in F^-_{\hat{t}, \bar{y}}$ such that $|x_{i_k} - z_i| < 1/2$. Since $\{z_i\}$ are bounded, we arrive at a contradiction, thus proving the theorem.

4. - An Example.

The set B used in the proof of Theorem 1 is always unbounded when solutions are assumed to be unique. To show that B may be bounded when we do not require uniqueness, we consider the following example.

Consider the scalar equation

$$\text{(S)} \qquad \begin{aligned} &\dot{x} = 0 && x < 0, \\ &\dot{x} = 2x^{1/2} && 0 \leq x < 1 \\ &\dot{x} = 2x^2 && x \geq 1. \end{aligned}$$

Letting

$$t_0 = 0, \qquad y_0 = -1, \qquad x_0 = 0,$$

we have

$$z_1 = 0, \qquad z_0 = -1, \qquad \lambda^* = 1, \qquad z_{\lambda^*} = 0.$$

There exists a solution $y(\cdot, 0, 0)$ of (S) such that

$$|y(t, 0, 0)| \to +\infty \quad \text{as} \quad t \to 3/2.$$

Therefore, letting $w^* = 3/2$ we have that

$$B = (0, 1].$$

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