On Tangent Bundles with Sasakian Metrics of Finslerian and Riemannian Manifolds.

KENTARO YANO (Japan) and (*) TANJIRO OKUBO (Canada) (**)

Abstract. - *On the basis of the so-called phase completion the notion of vertical, horizontal and complete objects is defined in the tangent bundles over Finslerian and Riemannian manifold. Such a tangent bundle is made into a manifold of almost Kaehlerian structure by endowing it with Sasakian metric. The components of curvature tensors with respect to the adapted frame are presented. This having been done it is shown possible to study the differential geometry of Finslerian spaces by dealing with that of their own tangent bundles.*

Introduction.

There exist several essentially different points of view with regard to Finslerian spaces. C. CARATHÉODORY [2] dealt with the spaces by means of variational calculus, while since 1925 J. L. SYNGE [17] and L. BERWALD [1] developed tensor calculus by understanding that Finslerian space is a manifold with a metric tensor whose components are the second derivatives of $\frac{1}{2}F^2(x, dx)$ where F is a function satisfying the properties that it is positive, homogeneous of degree one in the differentials and convex in the latter. Regarding the differentials involved in F as the components of the element of support and providing with the socalled base connection, which we call δ_i*-derivation.* E. CARTAN [3] endowed Finslerian spaces with Euclidean connection. The metric tensor being functions of position and element of support, a trend arose to observe that Finslerian space can be derived from a Riemannian space by the application of homogeneous contact transformation [6], [8] which by K. YANO and E. T. DAVIES [19] was elevated to the contact tensor calculus. There the special frame of reference called the first and second contact frames introduced by M. S. KNEBELMAN [9] and Y. MUTO [13], respectively, served to define the sub-distributions complementary and non-holonomic in general.

Recently the present authors [24] developed the differential geometry of tangent bundles over affinely connected spaces of Finslerian type, which are called *generalized spaces of paths* [6], [11] and showed that it is always possible to regard the geometry of such a space as that of its tangent bundle

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by means of the so-called *phase completion.* There the definition of horizontal distribution was given by the use of E. CARTAN's base connection which helped to define the first contact frame in the contact tensor calculus. Thus the geometry of spaces of Finslerian type lies in this category needless to say.

However, the problem we face consists in giving the metrics in tangent bundles. For though it would be very natural for which to use the so-called *vertical, horizontal and complete tensor fields* of type (0.2) that arise from the Finsler metric g derived from the fundamental function F of the base manifold, either of them renders the tangent bundle unable to be a proper Riemannian manifold. On the other hand S. SASAKI [16] introduced a special metric g^S so that the tangent bundle over a Riemannian manifold may be a proper Riemannian manifold, and later on K. YANO and E. T. DAVIES adopted this scheme to study the differential geometry of the tangent bundles over Finslerian and Riemannain manifolds [21]. The purpose of the present paper is to deepen K. YANO and E. T. DAVIES' theory by applying what we obtained in our theory of tangent bundles over generalized spaces of paths. The Sasakian metric g^S makes them into almost Kaehlerian manifolds and we prove that they can not be Einstein manifolds. Also we derive the curvature properties of base Finslerian manifolds uniformly by computing the curvature tensor of their tangent bundles, which may by all means be one of the contributions to the theory of tangent bundles to Finsler geometry.

§ 1. - Tangent bundle over a Finslerian manifold.

Let M be an n-dimensional manifold whose class of differentiability is assumed to be as high as required. Then its *tangent bundle* $T(M)$ is by definition

$$T(M) = \bigcup_{P \in M} T_P(M)$$

where P is a point of M and $T_P(M)$ is the tangent plane of M at P. A point $\tilde{P}$ of $T(M)$ is an ordered pair (P, y_P) of a point P and a vector $y_P \in T_P(M)$. π is the *projection* $T(M) \to M$ defined by $\tilde{P} = (P, y_P) \to P$. The set $\pi^{-1}(P)$ is called the *fibre* over P, and M is the *base manifold.*

Suppose that the manifold M is covered by a system of coordinate neighbourhoods $\{U, x^h\}$ (1), where (x^h) is the local coordinate systems in the neighbourhood U. Let (y^h) be the system of Cartesian coordinates in each tangent space $T_P(M)$ of M with respect to the natural base (∂_h), where $\partial_h = \partial/\partial x^h$. Then in the open set $\pi^{-1}(U)$ of $T(M)$ we can introduce local coordinates (x^h, y^h) for $\tilde{P}$, which we call coordinates in $\pi^{-1}(U)$ induced from (x^h), or simply *induced coordinates in* $\pi^{-1}(U)$.

(1) The indices $a, b, c, d, \ldots, h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, n\}$.

If U' is another coordinate neighbourhood of P in M, then $\pi^{-1}(U')$ contains $\tilde{P}$ and the induced coordinates of $\tilde{P}$ relative to $\pi^{-1}(U')$ are $(x^{h'}, y^{h'})$, where

$$(1.2)\qquad \begin{aligned} x^{h'} &= x^{h'}(x) \\ y^{h'} &= \partial_h y^{h'} \cdot y^h , \end{aligned}$$

On writing (1.2) shortly as

$$(1.3)\qquad x^{A'} = x^{A'}(x)\ (^2)$$

where we understand that $x^{\bar{h}} = y^h$ and $x^{\bar{h}'} = y^{h'}$, the Jacobian of the transformation (1.3) is given by

$$(1.4)\qquad (\partial_B x^{A'}) = \begin{bmatrix} \partial_i x^{h'} & 0 \\ y^a \partial_a \partial_i x^{h'} & \partial_i x^{h'} \end{bmatrix} .$$

Thus the tangent bundle $T(M)$ is orientable [21].

We suppose that there is given in $T(M)$ a function $F(x, y)$ satisfying the properties

(a) $F(x, y) > 0$ for $y \neq 0$,

(b) $F(x, \alpha y) = \alpha F(x, y)$ for real α,

(c) $F(x, y + z) = F(x, y) + F(x, z)$,

(d) For $x^h = x_0^h$, the surface $F(x_0, y) = 1$ is a convex surface.

Then it is clear that the n^2 symmetric function $g_{ji}(x, y)$ defined in $\pi^{-1}(U)$ by

$$(1.5)\qquad g_{ji}(x, y) = \partial^2 \frac{1}{2} F^2 / \partial y^j \partial y^i$$

undergo the law of transformation

$$(1.6)\qquad g_{j'i'} = \partial_{j'} x^j \cdot \partial_{i'} x^i \cdot g_{ji}$$

subject to (1.3), and when y^h is replaced by the direction $\dot{x}^h = \dfrac{dx^h}{dt}$ of a point $P(x^h) \in U$, g_{ji} can make M into a Finslerian space. Then $T(M)$ is *the tangent bundle over a Finslerian manifold M.*

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(2) The indices $A, B, C, D, E, \ldots$ run over the range $\{1, 2, \ldots, n; \bar{1}, \bar{2}, \ldots, \bar{n}\}$.

We suppose moreover that in each $\pi^{-1}(U)$ there are given n^2 functions $\Gamma_i^h(x, y)$ satisfying the properties

(a) $\Gamma_i^h(x, \alpha y) = \alpha\Gamma_i^h(x, y)$

(b) $\Gamma_i^h(x, y)$ undergo the law of transformation

$$\Gamma_{i'}^{h'}(x', y') = \partial_h x^{h'} \cdot (\partial_i x^{i'} \cdot \Gamma_i^h + y^a \partial_{a'} \partial_{j'} x^{h'}), \tag{1.7}$$

subject to (1.3).

By the use of Γ_i^h we introduce the operator δ_i defined by

$$\delta_i = \partial_i - \Gamma_i^a \partial_a, \tag{1.8}$$

then for any function $f(x, y)$ of class C^r, $r \geq 1$, $\delta_i f$ can be defined in $T(M)$ globally in virtue of (1.2), (1.7) and

$$\partial_{i'} f = \partial_{i'} x^i \cdot \partial_i f + \partial_{i'} y^a \cdot \partial_a f.$$

Since each of δ_i and ∂_i serves to be the base of $T(M)$, and denoting them respectively by B_i^A and C_i^A, we introduce in $T(M)$ a special frame of reference $A_\alpha^A = (B_i^A, C_i^A)$, which we call *adapted frame* [21].

$$B_i^A = (\delta_i^h - \Gamma_i^h), \ C_i^A = (0, \delta_i^h). \tag{1.9}$$

The dual frame $B_B^\beta = (B_B^h, C_B^{\bar{h}})$ has the components inverse to A_α^A and they are given by

$$B_B^h = (\delta_i^h, 0), \ C_B^{\bar{h}} = (\Gamma_i^h, \delta_i^h). \tag{1.10}$$

Obviously the equations of a fibre are given by

$$dx^h = B_B^h dx^B = 0, \tag{1.11}$$

while its complementary sub-distribution is given by n equations

$$C_B^{\bar{h}} dx^B = dy^h + \Gamma_i^h(x, y) dx^i = 0 \tag{1.12}$$

and it is called the *horizontal distribution of* $T(M)$.

The non-holonomic object with respect to A_α^A is given by

$$\Omega_{\gamma\beta}^\alpha = -\Omega_{\beta\gamma}^\alpha = A_A^\alpha(\delta_\alpha A_\beta^A - \delta_\beta A_\gamma^A) \tag{1.13}$$

where

$$\delta_\alpha = \delta_i \text{ for } \alpha = i, \text{ and } \delta_\alpha = \delta_{\bar{i}} \text{ for } \alpha = \bar{i},$$

and the non-vanishing components of $\Omega_{\gamma\beta}^{\alpha}$ will be

$$\Omega_{ji}^{h} = -\Omega_{ij}^{h} = -(\delta_j\Gamma_i^h - \delta_i\Gamma_j^h),$$
(1.14)
$$\Omega_{\bar{j}i}^{h} = -\Omega_{i\bar{j}}^{h} = -\delta_{\bar{j}}\Gamma_i^h.$$

Let $C: x^h = x^h(t)$ be a curve of class C^r, $r \geq 1$, in U of M and suppose that its natural lift $(x^h, \dot{x}^h)$ in $\pi^{-1}(U)$ lies in horizontal distribution always, that is to say, $x^h(t)$ satisfies the equations

$$\frac{d^2x^h}{dt^2} + \Gamma_{ji}^h(x, \dot{x})\frac{dx^j}{dt}\frac{dx^i}{dt} = 0 \tag{1.15}$$

where,

$$\Gamma_{ji}^h\dot{x}^i = \Gamma_j^h,$$

then we call C the generalized geodesic of M, or simply the *geodesic*.

In M we take a vec or field X along such geodesic C and consider the infinitesimal transforma on

$$\bar{x}^h(t) = x^h(t) + X^h(x(t))\delta u, \tag{1.17}$$

where $X^h(x(t))$ are the components of X with respect to (∂_h). Then the direction $\dot{x}(t)$ of C is transformed into

$$\dot{\bar{x}}^h(t) = \dot{x}^h(t) + X^h{}_{,j}\dot{x}^j(t)\delta u. \tag{1.18}$$

We say that if the transformation (1.14) carries C to a path of M within the preservation of the parameter t, the vector field X defines an *affine collineation*. The necessary and sufficient condition for X to define an affine collineation is the vanishing of the lie derivatives $\mathfrak{L}_X\Gamma_{ji}^h$ [11], [14], [20]:

$$\begin{aligned}\mathfrak{L}_X\Gamma_{ji}^h(x, \dot{x}) &= \partial_j\partial_i x^h - \Gamma_{ji}^a\partial_a V^h + \Gamma_{ja}^h\partial_i V^a + V^a\partial_a\Gamma_{ji}^h \\ &+ \Gamma_{ji}^h{}_{\dot{b}}\,\dot{x}^b\partial_b V^a \\ &= \nabla_j\nabla_i X^h + K_{kji}^h X^k + \Gamma_{jia}^h\dot{x}^b\nabla_b X^a = 0,\end{aligned} \tag{1.19}$$

where we have put

$$\nabla_j X^h = \delta X^h + \Gamma_{ja}^h X^a \tag{1.20}$$

$$K_{kji}{}^h(x, \dot{x}) = \delta_k \Gamma^h_{ji} - \delta_j \Gamma^h_{ka} + \Gamma^h_{ka}\Gamma^a_{ji} - \Gamma^h_{aj}\Gamma^a_{ki}, \tag{1.21}$$

and we notice that $\delta_j X^h(x(t)) = \partial_j X^h(x(t))$ for the case.

Let $X^h(x, y)$ and $\omega_i(x, y)$ be two set of functions defined in $\pi^{-1}(U)$ such that they obey the law of transformation

$$X^{h'} = \partial_h x^{h'} \cdot \omega_{i'} = \partial_{i'} x^i \cdot \omega_i$$

subject to (1.31). Since such functions can always be defined in U of M by replacing the y' s by x' s as the standard work of defining vectors and 1-forms in Finsler geometry, our concern is to define any vector or 1-form in $T(M)$ by the use of these n functions given a priori. Such an action may be called *phase completion* of the tangent bundle over a Finslerian manifold. There would be many ways for phase completion, but we shall deal with the following three kinds. We define the *vertical, horizontal and complete vector fields* denoted respectively by X^V, X^H and X^C which have in $\pi^{-1}(U)$ the components

$$X^V = \begin{pmatrix} 0 \\ X^h \end{pmatrix}, \; X^H = \begin{pmatrix} X^h \\ -\Gamma^h_i X^i \end{pmatrix}, \; X^C = \begin{pmatrix} X^h \\ y^a \delta_a X^h \end{pmatrix}, \tag{1.22}$$

and the *vertical, horizontal and complete 1-form's* denoted respectively by ω^V, ω^H and ω^C which have in $\pi^{-1}(U)$ the components

$$\omega^V = (\omega_i, 0), \; \omega^H = (\omega_i, \Gamma^h_i \omega_h), \; \omega^c = (y^a \delta_a \omega_i, \omega_i)$$

where each components are expressed in terms of the natural base (∂_A).

Let P and Q be the two sets composed respectively of n^{r+s} functions $P_{i_r i_{r-1} \cdots i_1}{}^{h_s h_{s-1} \cdots h_1}$ and n^{t+u} functions $Q_{j_t j_{t-1} \cdots j_1}{}^{k_u k_{u-1} \cdots k_1}$ such that they obey the law of transformation

$$P_{i'_r \cdots i'_1}{}^{h'_s \cdots h'_1} = \partial_{i'_r} x^{i_r} \cdots \partial_{i'_1} x^{i_1} \cdot \partial_{h_s} x^{h'_s} \cdots \partial_{h_1} x^{h'_1} \cdot P_{i_r \cdots i_1}{}^{h_s \cdots h_1}$$

$$Q_{j'_t \cdots j'_1}{}^{k'_u \cdots k'_1} = \partial_{j'_t} x^{j_t} \cdots \partial_{j'_1} x_{j_1} \cdot \partial_{k_u} x^{k'_u} \cdots \partial_{k_1} x^{k'_1} Q_{j_t \cdots j_1}{}^{k_u \cdots k'_1},$$

and denote by $P \times Q$ the formal product of these two sets of functions.

We define the vertical, horizontal and complete tensor fields by

$$(P \times Q)^V = P^V \otimes Q^V,$$

$$(P \times Q)^H = P^H \otimes Q^V + P^V \otimes Q^H,$$

$$(P \times Q)^C = P^C \otimes Q^V + P^V \otimes Q^C,$$

respectively [24].

From these definitions the vertical, horizontal and complete tensor fields G of type (0.2) have in $\pi^{-1}(U)$ the components

$$G^V = \begin{pmatrix} G_{ji} & 0 \\ 0 & 0 \end{pmatrix}, \quad G^H = \begin{pmatrix} \Gamma_j^t G_{ti} + \Gamma_i^t G_{jt} & G_{ji} \\ G_{ji} & 0 \end{pmatrix}, \quad G^C = \begin{pmatrix} y^a \partial_a G_{ji} & G_{ji} \\ G_{ji} & 0 \end{pmatrix}$$

respectively. Thus, on taking g_{ji} given by (1.5), we can metrize $T(M)$ by g^c, However, such g^c makes $T(M)$ into a pseudo Riemannian manifold [23], and we now introduce in $T(M)$ a special metric g^S which we call *Sasakian lift* of the metric g of M and was adapted by S. SASAKI [16] in order to make the tangent bundle over a Riemannian manifold with the metric $g_{ji}(x)$ into property Riemannian. On putting $g_{js}\Gamma^s_{ja} y^a = \Gamma_{ij}$, it is given by

$$g^S = \begin{pmatrix} g_{ji} + g_{cb}\Gamma_j^c \Gamma^b_{\ i} & \Gamma_{ij} \\ \Gamma_{ji} & g_{ji} \end{pmatrix} \tag{1.23}$$

and with respect to the adapted frame it has the components

$$g^S = \begin{bmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{bmatrix} \tag{1.24}$$

and consequently the element of arc length in $T(M)$ is given by

$$dS^2 = g_{ji} dx^j dx^i + g_{ji} \delta y^j \delta y^i. \tag{1.25}$$

Thus we have proved.

THEOREM 1. - *The tangent bundle $T(M)$ over a Finsler space M is a Riemannian manifold with respect to Sasakian lift g^S.*

Denoting by G^S the inverse to g^S, we find G^S has the components

$$G^S = \begin{pmatrix} g^{ih} & -\Gamma^{hi} \\ -\Gamma^{ih} & g^{ih} + g^{cb}\Gamma^h_c \Gamma^i_b \end{pmatrix} \tag{1.26}$$

with respect to the natural base (∂A) and

$$G^S = \begin{bmatrix} g^{ih} & 0 \\ 0 & g_{ih} \end{bmatrix} \tag{1.27}$$

with respect to the adapted frame, where $g_{ji} g^{ih} = \delta^h_i$.

§ 2, - **Euclidean connection in** $T(M)$.

$T(M)$ having been made into a proper Riemannian manifold, there exists a unique connection ∇^S that keeps g^S covariantly constant and is torsion free. It is given by [12]

$$2g^S(\overset{S}{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) = \tilde{X}g^S(\tilde{Y}, \tilde{Z}) + \tilde{Y}g^S(\tilde{Z}, \tilde{X}) - \tilde{Z}g^S(\tilde{X}, \tilde{Y}) + g^S([\tilde{X}, \tilde{Y}], \tilde{Z}) + g^S([\tilde{Z}, \tilde{X}], \tilde{Y}) + g^S(\tilde{X}, [\tilde{Z}, \tilde{Y}]), \tag{2.1}$$

where $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ are any vector fields of $T(M)$, and in terms of the local coordinates (x^h, y^h) of $\pi^{-1}(U)$ the parameter Γ^A_{CB} of ∇^S is the Cristoffel symbol. Its corresponding coefficients with respect to the adapted frame are given by

$$\Gamma^\alpha_{\gamma\beta} = A^\alpha_A(\delta_\gamma A^A_\beta + \Gamma^A_{CB}A^C_\gamma A^B_\beta) \tag{2.2}$$

whence

$$\Gamma^\alpha_{\gamma\beta} - \Gamma^\alpha_{\beta\gamma} = \Omega^\alpha_{\gamma\beta}. \tag{2.3}$$

The covariant derivative of $\overset{S}{g}_{\gamma\beta}$ is given by

$$\nabla_\delta g_{\gamma\beta} = \delta_\delta g_{\gamma\beta} - \Gamma^\varepsilon_{\delta\gamma}g_{\varepsilon\beta} - \Gamma^\varepsilon_{\delta\beta}g_{\gamma\varepsilon} = 0, \tag{2.4}$$

from which we obtain

$$\Gamma^\alpha_{\gamma\beta} = \frac{1}{2} g^{\alpha\varepsilon}(\delta_\gamma g_{\varepsilon\beta} + \delta_\beta g_{\gamma\varepsilon} - \delta_\varepsilon g_{\gamma\beta}) + \frac{1}{2}(\Omega_{\gamma\beta}{}^\alpha + \Omega^{\ \alpha}_{\cdot\gamma\beta} + \Omega^\alpha_{\cdot\beta\gamma}), \tag{2.5}$$

where we have written

$$\Omega^\alpha{}_{\cdot\gamma\beta} = g^{\alpha\delta}g_{\delta\beta}\Omega_{\varepsilon\gamma}{}^\delta. \tag{2.6}$$

The particular values of Γ for different indices, on taking account of (1.13) and (1.24), are found to be

(a) $2\Gamma^h_{ji} = g^{ha}(\delta_j g_{ia} + \delta_i g_{ja} - \delta_a g_{ji})$

(b) $2\Gamma^h_{\bar{j}i} = g^{ha}\partial_{\bar{j}} g_{ia} + \Omega^h{}_{i\bar{j}}$,

(c) $2\Gamma^h{}_{j\bar{i}} = g^{ha}\partial_{\bar{i}} g_{ja} + \Omega^h{}_{j\bar{i}}$

(d) $2\Gamma^h_{\bar{j}\bar{i}} = - g^{ha}\delta_a g_{ji} + \Omega^h{}_{\bar{j}\bar{i}} + \Omega^h{}_{\bar{i}\bar{j}}$.

(2.7)

(e) $2\Gamma^{\bar{h}}_{j\,i} = -g^{ha}\partial_{\bar{a}}g_{ji} + \Omega_{ji}{}^{\bar{h}}$

(f) $2\Gamma^{\bar{h}}_{\bar{j}\;\;} = g^{ha}\delta_i g_{ja} + \Omega_{\bar{j}}{}^{\;\;\bar{h}} + \Omega^{\bar{h}}{}_{i\bar{j}}$,

(g) $2\Gamma^{\bar{h}}_{j\,\bar{i}} = g^{ha}\delta_j g_{ia} + \Omega_{j\bar{i}}{}^{\bar{h}} + \Omega^{\bar{h}}{}_{j\bar{i}}$,

(h) $2\Gamma^{\bar{h}}_{\bar{j}\,\bar{i}} = g^{ha}\partial_{\bar{j}}g_{ia}$.

If we use the notations

$$2\left\{ {h \atop ji} \right\} = g^{ha}(\partial_j g_{ai} + \partial_i g_{ja} - \partial_a g_{ji}), \tag{2.8}$$

$$2\, C_{jih} = \partial_{\bar{j}} g_{jh}, \tag{2.9}$$

the (2.7a) becomes

$$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} - g^{ha}(\Gamma^b_j C_{bia} + \Gamma^b_i C_{bja} - \Gamma^b_a C_{bji}). \tag{2.10}$$

Taking account of homogeneity properties of C's, we have immediately

$$\Gamma^h_{ji} y^j y^i = \left\{ {h \atop ji} \right\} y^j y^i \equiv 2G^h, \tag{2.11}$$

and if we now take

$$\Gamma^h_i = \partial_{\bar{i}} G^h, \tag{2.12}$$

we have an expression on the right hand side of (2.10) as

$$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} - (\partial_{\bar{j}} G^a) C_{ia}{}^h - (\partial_{\bar{i}} G^a) C_{ja}{}^h + g^{ha}(\partial_{\bar{a}} G^b) C_{bji}, \tag{2.13}$$

which coincides with the expression for Γ^* in Cartian's theory [3].

The fact that

$$\Gamma^h_i = \Gamma^h_{ji} y^j, \tag{1.16$'$}$$

$$\delta_k F = \partial_k F - \Gamma^a_k \partial_{\bar{a}} F = 0, \tag{2.14}$$

$$y^a \partial_{\bar{j}} \Gamma^h_{ai} = y^a \partial_{\bar{i}} \Gamma^h_{ja} = y^a \nabla_a C_{ji}{}^h \tag{2.15}$$

are easily verified.

With (2.16)′ we can determine the non-vanishing components of the non-holonomic objects that were given in (1.14). For Ω_{ji}^{h} we have

$$\Omega_{ji}{}^{h} = -K_{jia}{}^{h}y^{a} \tag{2.16}$$

where $K_{kji}{}^{h}$ are those given in (1.21) whose variables $\dot{x}'s$ are replaced by the $y's$.

Similarly

$$\Omega_{\bar{j}\bar{i}}{}^{h} = -\partial_{\bar{j}}\partial_{\bar{i}}G^{h} \equiv G_{ji}{}^{h}.$$

From this definition we have

$$g_{ha}G_{ji}{}^{a} = g_{ha}\Gamma_{ji}^{a} + (\partial_{\bar{j}}g_{ha}\Gamma_{ib}^{a})y^{b} - 2C_{jha}\Gamma_{ib}^{a}y^{b}$$

from which, substituting from (2.10), we have

$$g_{ha}G_{ji}{}^{a} = g_{ha}\Gamma_{ji}^{a} + y^{a}\nabla_{a}C_{jih}$$

or

$$\Omega_{\bar{j}\bar{i}}{}^{h} = G_{ji}{}^{h} = \Gamma_{ji}^{h} + \nabla_{0}A_{ji}{}^{h}, \tag{2.17}$$

where

$$A_{ji}{}^{h} = F\,C_{ji}{}^{h}, \quad \nabla_{0} = \frac{1}{F}y^{a}\nabla_{a}. \tag{2.18}$$

We can now rewrite (2.7) in the form (³)

(2.19)

(a) $\Gamma_{ji}^{h} = \Gamma_{ji}^{*h}$ of Cartan,

(b) $2\Gamma_{\bar{j}i}^{h} = 2C_{ji}{}^{h} - K^{h}{}_{iaj}y^{a}$,

(c) $2\Gamma_{j\bar{i}}^{h} = 2C_{ji}{}^{h} - K^{h}{}_{jai}y^{a}$,

(d) $\Gamma_{\bar{j}\bar{i}}^{h} = \nabla_{0}A_{ji}{}^{h}$,

(e) $2\Gamma_{ji}^{\bar{h}} = -2C_{ji}{}^{h} - K_{jia}{}^{h}y^{a}$,

(f) $\Gamma_{\bar{j}i}^{\bar{h}} = -\nabla_{0}A_{ji}{}^{h}$,

(g) $\Gamma_{j\bar{i}}^{\bar{h}} = \Gamma_{ji}^{h}$,

(h) $\Gamma_{\bar{j}\bar{i}}^{\bar{h}} = C_{ji}{}^{h}$,

(³) In the formulas of $\Gamma's$ enlisted as (3.21) by K. YANO and E. T. DAVIES' paper [21], the plus sign printed in (*b*) and (*c*) should be read to be the minus sign as seen above correspondingly.

If we construct the vector field A_i by contracting $A_{ji}{}^h$ given in (2.18) with respect to j and h, we have

$$A_i = A_{ji}{}^j = \frac{1}{2} F \partial_i \log g, \quad g = \det.(g_{ji}), \tag{2.20}$$

and with regard to which A. DEIKE [5] proved.

LEMMA. - *If in a Finslerian space the n functions* A_i *which are equal to* $\frac{1}{2} F \partial_i \log g$ *vanish identically, then the space is Riemannian.*

For the case, the function $F(x, y)$ defined in § 1 becomes

$$F^2 = g_{ji}(x) y^j y^i$$

and Γ' s in (2.19) reduce to

$$\begin{aligned}
&(a) \quad \Gamma_{ji}^h = \left\{ {h \atop ji} \right\}, \\
&(b) \quad 2\Gamma_{\bar{j} i}^h = -K_{jai}{}^h y^a, \\
&(c) \quad 2\Gamma_{j \bar{i}}^h = -K_{iaj}{}^h y^a, \\
&(d) \quad \Gamma_{\bar{j} \bar{i}}^h = 0, \\
&(e) \quad 2\Gamma_{ji}^{\bar{h}} = -K_{jia}{}^h y^a, \\
&(f) \quad \Gamma_{\bar{j} i}^{\bar{h}} = 0, \\
&(g) \quad \Gamma_{j \bar{i}}^{\bar{h}} = \Gamma_{ji}^h, \\
&(h) \quad \Gamma_{\bar{j} \bar{i}}^{\bar{h}} = 0.
\end{aligned} \tag{2.21}$$

If we express X^V, X^H and X^C given by (1.22) in terms of the adapted frame, they have the components

$$X^V = \begin{bmatrix} 0 \\ X^h \end{bmatrix}, \quad X^H = \begin{bmatrix} X^h \\ 0 \end{bmatrix}, \quad X^C = \begin{bmatrix} X^h \\ y^a \nabla_a X^h \end{bmatrix}. \tag{2.22}$$

respectively. When M is Riemannian, the X' s appearing in (2.2) are the components of a vector field in U, and X^V, X^H and X^C are called the *vertical, horizontal and complete lift of* X [16], [23].

If we compute the covariant derivatives of those vector fields, we have by using (2.19) for the vertical vector X^V

$$(2.23)\quad \begin{aligned} &\text{(a)}\quad \overset{S}{\nabla}_j \tilde{X}^\alpha = \left[(C_{ji}{}^h - \frac{1}{2} K^h{}_{jbi} y^b) X^i,\ \nabla_j X^h\right], \\ &\text{(b)}\quad \overset{S}{\nabla}_{\bar{j}} \tilde{X}^\alpha = [(\nabla_0 A_{ji}{}^h) X^i,\ \partial_{\bar{j}} X^h + C_{ji}{}^h X^i], \end{aligned}$$

and for the horizontal vector X^H

$$(2.24)\quad \begin{aligned} &\text{(a)}\quad \overset{S}{\nabla}_j \tilde{X}^\alpha = [\nabla_j X^h,\ -K_{jia}{}^h y^a x^i], \\ &\text{(b)}\quad \overset{S}{\nabla}_{\bar{j}} \tilde{X}^\alpha = \left[(C_{ji}{}^h - \frac{1}{2} K^h{}_{jai} y^a) X^i,\ -(\nabla_0 A_{ji}^{*h}) X^i\right], \end{aligned}$$

and. for the complete vector X^C

$$(2.25)\quad \begin{aligned} &\text{(a)}\quad \overset{S}{\nabla}_j \tilde{X}^\alpha = \Big[\nabla_j X^h + \Big(C_{ji}{}^h - \frac{1}{2} K^h{}_{jai} y^a\Big) y^b \nabla_b X^i,\ \frac{1}{2} y^a \nabla_j{}_a V^h + \\ &\qquad\qquad + \frac{1}{2}(\nabla_j \nabla_a X^h + K_{bja}{}^h X^b) - C_{ji}{}^h X^i\Big]. \\ &\text{(b)}\quad \overset{S}{\nabla}_{\bar{j}} \tilde{X}^\alpha = \Big[\partial_j X^h + \Big(C_{ji}{}^h - \frac{1}{2} K^h{}_{iaj} y^a\Big) X^i - C_{ji}{}^h y^b \nabla_b V^i, \\ &\qquad\qquad \partial_j (y^a \nabla_a X^h) + C_{ji}{}^h (y^a \nabla_a X^i) - X^a \nabla_0 A_{ja}{}^h\Big]. \end{aligned}$$

Especially if M is Riemannian we have for the vertical lift X^V

$$(2.23)'\quad \begin{aligned} &\text{(a)}\quad \overset{S}{\nabla}_j \tilde{X}^\alpha = \left[\frac{1}{2} y_a \nabla_a \nabla_j X^h - \frac{1}{2} y^b \mathfrak{L}_X \left\{ {h \atop ji} \right\},\ \nabla_j X^h\right] \\ &\text{(b)}\quad \overset{S}{\nabla}_{\bar{j}} \tilde{X}^\alpha = [0,\ 0], \end{aligned}$$

and for the horizontal lift X^H

$$(2.24)'\quad \begin{aligned} &\text{(a)}\quad \overset{S}{\nabla}_j \tilde{X}^\alpha = \left[\nabla_j X^h,\ y^a \mathfrak{L}_X \left\{ {h \atop aj} \right\} - y^a \nabla_j \nabla_a X^h\right], \\ &\text{(b)}\quad \overset{S}{\nabla}_{\bar{j}} \tilde{X}^\alpha = \left[\frac{1}{2} y^a (\nabla_a \nabla_j - \nabla_j \nabla_a) X^h,\ 0\right], \end{aligned}$$

and for the complete lift X^C.

$$\text{(a)}\quad \overset{S}{\nabla}_j \tilde{X}^\alpha = \left[\nabla_j X^h - \frac{1}{2} K_{iaj}{}^h y^a y^b \nabla_b X^i,\right.$$

$$(2.25)'\qquad \left. \frac{1}{2} y^a \nabla_j \nabla_a X^h + \frac{1}{2} y^a \mathfrak{L}_X \begin{Bmatrix} h \\ ja \end{Bmatrix} \right],$$

$$\text{(b)}\quad \overset{S}{\nabla}_{\bar{j}} \tilde{X}^\alpha = \left[-\frac{1}{2} y^a (\nabla_j \nabla_a - \nabla_a \nabla_j) X^h,\ \nabla_j X^h\right],$$

from which we obtain.

THEOREM 2. - *The vertical, horizontal and complete lifts of a vector field X are parallel in $T(M)$ if and only if X is a parallel vector field and defines an affine collineation.*

§ 3. - Almost complex structure of $T(M)$.

Let $T(M)$ be the tangent bundle over a Finslerian manifold and have Sasakian lift g^S as its metric.

We consider in $T(M)$ a tensor field $\tilde{F}$ of type (1.1) such that it acts on a vertical and horizontal vectors under the following rule:

$$(3.1)\qquad \tilde{F} X^V = X^H \quad \text{and} \quad \tilde{F} X^H = - X^V.$$

Then we find that $\tilde{F}$ satisfies

$$(3.2)\qquad \tilde{F}^2 = - \tilde{E},$$

where $\tilde{E}$ is the unit tensor field in $T(M)$. Taking account of (2.2) they are found to have the components

$$(3.3)\qquad \tilde{F} = \begin{bmatrix} 0 & \delta_i^h \\ -\delta_i^h & 0 \end{bmatrix},$$

and

$$(3.4)\qquad \tilde{E} = \begin{bmatrix} \delta_i^h & 0 \\ 0 & \delta_i^h \end{bmatrix},$$

with respect to the adapted frame.

(3.2) or (3.3) implies that $\tilde{F}$ is an almost complex structure [22] and has the components,

$$\tilde{F} = \begin{bmatrix} \Gamma_j^h & \delta_j^h \\ -\delta_j^h - \Gamma_j^a \Gamma_a^h & -\Gamma_j^h \end{bmatrix} \tag{3.5}$$

with respect to the natural frame (∂_A).

An using (1.23) and (1.26) if we compute the components F_{BA} of the tensor field $\tilde{F}\, g^S$ of type (1.1), they are found to be

$$F_{BA} = \begin{pmatrix} \Gamma_{ji} - \Gamma_{ij} & g_{ji} \\ -g_{ji} & 0 \end{pmatrix}, \tag{3.6}$$

where we have put

$$\Gamma_{ji} = g_{it} \Gamma_{bj}^t y^b, \tag{3.7}$$

and taking account of (2.10), (3.6) can be writen as

$$F_{BA} = \begin{pmatrix} y^a(\partial_j g_{ia} - \partial_i g_{ja}) & g_{ji} \\ -g_{ji} & 0 \end{pmatrix}. \tag{3.8}$$

This implies that the exterior differential of the 1-form $\theta = g_{ji} y^j dx^i$ globally defined in $T(M)$ is

$$d\theta = \frac{1}{2} F_{CB} dx^C \wedge dx^B$$

and consequently F_{AB} is closed. Thus we have

THEOREM 3. - *The tangent bundle $T(M)$ over a Finslerian or a Riemannian manifold M whose metric is Sasakian lift g^S is an almost Kaehlerian manifold.*

The Nijenhuis tensor $\tilde{N}$ of the almost complex structure $\tilde{F}$ is, by definition, given by [12]

$$\tilde{N}(\tilde{X}\ \tilde{Y}) = [\tilde{F}\tilde{X},\ \tilde{F}\tilde{Y}] - \tilde{F}[\tilde{F}\tilde{X},\ \tilde{Y}] - \tilde{F}[\tilde{X},\ \tilde{F}\tilde{Y}] - [\tilde{X},\ \tilde{Y}]$$

where $\tilde{X}$ and $\tilde{Y}$ are arbitrary vector fields in $T(M)$, and $\tilde{N}$ has in $\pi^{-1}(U)$ the components of the form

$$N_{CB}{}^A = F_C{}^D(\partial_D F_B{}^A - \partial_B F_D{}^A) - F_B{}^D(\partial_D F_C{}^A - \partial_C F_D{}^A)$$

with respect to the natural frame. In terms of the adapted frame they are [21].

$$\begin{aligned} [N]_{\gamma\beta}{}^\alpha = {} & F_\gamma{}^\delta(\delta_\delta F_\beta{}^\alpha - \delta_\beta F_\delta{}^\alpha) - F_\beta{}^\delta(\delta_\delta F_\gamma{}^\alpha - \delta_\gamma F_\delta{}^\alpha) \\ & + F_\varepsilon{}^\alpha(F_\gamma{}^\delta \Omega_{\delta\beta}{}^\varepsilon - F_\beta{}^\delta \Omega_{\delta\gamma}{}^\varepsilon) + (\delta_\gamma^\varepsilon \delta_\beta^\delta + F_\gamma{}^\varepsilon F_\beta{}^\delta)\Omega_{\varepsilon\delta}{}^\alpha. \end{aligned} \tag{3.9}$$

On taking account of (2.16), (2.17) and (3.3) if we compute the particular values of $N_{\gamma\beta}{}^{\alpha}$ for the different indices, we find that

$$[N]_{ji}{}^{h} = [N]_{ji}{}^{h} = -K_{jia}{}^{h}y^{a} \tag{3.10}$$

all others being zero. Hence we have [18], [21].

THEOREM 4. - *The tangent bundle $T(M)$ over a Riemannian manifold M having Sasakian lift g^S as metric is a Kaehlerian manifold if and only if*

$$K_{jia}{}^{h}y^{a} = 0. \tag{3.11}$$

THEOREM 5. - *The tangent bundle $T(M)$ over a Riemannian manifold M having Sasakian lift g^S as metric is a Kaehlerian manifold if and only if M is flat* [21].

A vector $\tilde{X}$ with respect to which the tensor $\tilde{F}$ has the vanishing Lie derivative is said to be almost analytic. Denoting ay $[\mathfrak{L}_X F]_{\beta}{}^{\alpha}$ the components of $\mathfrak{L}_X F^A{}_B$ with respect to the adapted frame, we have [21]

$$[\mathfrak{L}_X F]_{\beta}{}^{\alpha} = X^{\varepsilon}\delta_{\varepsilon}F_{\beta}{}^{\alpha} - F_{\beta}{}^{\varepsilon}\delta_{\varepsilon}X^{\alpha} + F_{\varepsilon}{}^{\alpha}\delta_{\beta}X^{\varepsilon} + X^{\varepsilon}(\Omega_{\varepsilon\delta}{}^{\alpha}F_{\beta}{}^{\delta} - \Omega_{\varepsilon\beta}{}^{\delta}F_{\delta}{}^{\alpha}).$$

Then we obtain for the vertical vector X^V

$$\begin{aligned} &[\mathfrak{L}_{X^V}F]_j{}^h = \nabla_j X^h + X^a\nabla_0 A_{aj}{}^h, \quad [\mathfrak{L}_{X^V}F]_{\bar{j}}{}^h = \partial_{\bar{j}}X^h, \\ &[\mathfrak{L}_{X^V}X]_j{}^{\bar{h}} = \partial_{\bar{j}}X^h, \quad [\mathfrak{L}_{X^V}F]_{\bar{j}}{}^{\bar{h}} = -\nabla_j X^h - X^a\nabla_0 A_{aj}{}^h. \end{aligned} \tag{3.12}$$

and for the horizontal vector X^H

$$\begin{aligned} &[\mathfrak{L}_{X^H}F]_j{}^h = \partial_{\bar{j}}X^h, \quad [\mathfrak{L}_{X^H}F]_{\bar{j}}{}^h = -\nabla_j X^h - X^a\nabla_0 A_{aj}{}^h, \\ &[\mathfrak{L}_{X^H}F]_j{}^{\bar{h}} = -\nabla_j X^h - X^a\nabla_0 A_{aj}{}^h, \quad [\mathfrak{L}_{X^H}F]_{\bar{j}}{}^{\bar{h}} = -\partial_{\bar{j}}X^h, \end{aligned} \tag{3.13}$$

from which we have

THEOREM 6. - *The necessary and sufficient condition for the vertical vector X^V or the horizontal vector field X^H to be almost analytic is that X^h satisfies in the base Finslerian manifold that*

$$\partial_{\bar{j}}X^h = 0, \qquad \nabla_j X^h + X^a\nabla_0 A_{aj}{}^h = 0.$$

THEOREM 7. - *The necessary and sufficient condition for the vertical or horizontal lift of a vector field X in the base Riemannian manifold to be almost analytic is that X is a parallel vector field.*

As to the complete vector field X^C we have

$$
\begin{aligned}
&[\mathfrak{L}_{X^C}F]_j{}^h = \partial_{\bar{j}}X^h + y^a\{\nabla_j\nabla_a X^h + K_{kja}{}^h X^k + (\partial_{\bar{C}}\Gamma^h_{aj})y^b\nabla_b X^C\},\\
(3.14)\qquad &[\mathfrak{L}_{X^C}F]_{\bar{j}}{}^h = [\mathfrak{L}_{X^C}F]_j{}^{\bar{h}} = y^a\nabla_a\partial_{\bar{j}}X^h,\\
&[\mathfrak{L}_{X^C}F]_{\bar{j}}{}^{\bar{h}} = y^a\{\nabla_j\nabla_a X^h + K_{kja}{}^h X^k + (\partial_{\bar{C}}\Gamma^h_{aj})y^b\nabla_b X^C\}.
\end{aligned}
$$

Therefore if X^h are the components of a vector field X defined along a geodesic C of the base Finslerian manifold, then by taking account of (1.19), we have

THEOREM 8. - *The necessary and sufficient condition for the complete vector field X^C lifted from a vector field X defined along a geodesic C in the base Finslerian manifold to be almost analytic is that X defines an affine collineation.*

We note that the equations of a geodesic given in (1.15) is, by taking the arc tengths S as parameter, written as [11], [15]

$$\frac{d^2x^h}{dS^2} + \left\{ {h \atop ji} \right\} \frac{dx^i}{dS}\frac{dx^i}{dS} = 0$$

because of (2.10), and also that the statement similar to theorem 6 holds for the Riemannian case [21].

Let us consider the Lie derivative of the tensor F_{CB}. Denoting by $[\mathfrak{L}_X F]_{CB}$ the components of ${}_XF$ with respect to the adapted frame, we have

$$[\mathfrak{L}_X F]_{\gamma\beta} = X^\delta\delta_\delta F_{\gamma\beta} + F_{\delta\beta}\delta_\gamma X^\delta + F_{\gamma\delta}\delta_\beta X^\delta + X^\varepsilon(\Omega_{\varepsilon\gamma}{}^\delta F_{\delta\beta} + \Omega_{\varepsilon\beta}{}^\delta F_{\gamma\delta}),$$

so that we have for the vertical vector field X^V

$$
\begin{aligned}
&[\mathfrak{L}_{X^V}F]_{ji} = \nabla_j X_i - \nabla_i X_j,\quad [\mathfrak{L}_{X^V}F]_{\bar{j}i} = -\nabla_j X_i,\\
(3.15)\qquad &[\mathfrak{L}_{X^V}F]_{\bar{j}\bar{i}} = 0,
\end{aligned}
$$

where we have put

$$X_i = g_{hi}X^h,\qquad \nabla_{\bar{j}}X^i = \partial_j X_i - C_{ji}{}^a X^a,$$

and for the horizontal vector field X^H

$$
\begin{aligned}
&[\mathfrak{L}_{X^H}F]_{ji} = 0,\qquad [\mathfrak{L}_{X^H}F_{ji}] = \nabla_j X_i - X^a\nabla_0 A_{jai}\\
(3.16)\qquad &[\mathfrak{L}_{X^H}F]_{ji} = \partial_{\bar{i}}X_j - \partial_{\bar{j}}X_i,
\end{aligned}
$$

from which we have for $T(M)$ over a Finslerian manifold M

THEOREM 9. - *The necessary and sufficient condition for* (a) $\mathfrak{L}_{X^V}F_{CB}$

(b) $\mathfrak{L}_{X^H}F_{CB}$ to vanish is that a) $\nabla_j X_i - \nabla_i X_j = 0$ and

$\nabla_{\bar{j}} X_i = 0$ (b) $\nabla_j X_i = X^a \nabla_0 A_{jai}$ *and* $\partial_{\bar{j}} X_i = \partial_{\bar{i}} X_j$

and for $T(M)$ over a Riemannian manifold M [21].

THEOREM 10. - *The necessary and sufficient condition for* (a) $\mathfrak{L}_{X^V}F_{CB}$

(b) $\mathfrak{L}_{X^H}F_{CB}$ *to vanish is that* a) *the vector field X in M is closed*

(b) *X in M is parallel.*

As to the complete vector field X^C

$$[\mathfrak{L}_{X^C}F]_{ji} = g_{bj}\{\nabla_i \nabla_a X^b + K_{cia}{}^b X^C + (\partial_{\bar{d}}\Gamma_{ia}^b) y^c \nabla_d\} y^a$$

$$- g_{bi}\{\nabla_j \nabla_a X^b + K_{cja}{}^b X^C + (\partial_{\bar{d}}\Gamma_{ja}^b) y^c \nabla_c X^d\} y^a,$$

$$[\mathfrak{L}_{X^C}F]_{\bar{j}i} = \nabla_j X_i + \nabla_i X_j + C_{ji}{}^h y^a \nabla_a X_h,$$

$$[\mathfrak{L}_{X^C}F]_{\bar{j}\bar{i}} = \partial_{\bar{i}} X_j - \partial_{\bar{j}} X_i.$$

Let X be a vector field defined along a curve $D : x^h = x^h(S)$ of class C^r, $r \geq 1$, and consider the infinitesimal transformation of the form given by (1.17). The necessary and sufficient condition for the transformation to preserve the arc lengths of the curve D is given by the vanishing of the Lie derivative $\mathfrak{L}_X\, g_{ji}$ [11], [20]:

$$\mathfrak{L}_X\, g_{ji}(x, \dot{x}) = \nabla_j X_i + \nabla_i X_j + C_{ji}{}^h y^a \nabla_a X_h = 0. \tag{3.18}$$

The vector field $X(x(t))$ of a Finslerian manifold which satisfies this condition is called *Killing vector*. If furthermore the Killing vector defines an affine collineation, we say that $X(x(S))$ generates *motion* [M. S. KNEBELMAN [11], p. 561].

Hence if in the above formula (3.17) the vector field X^C is supposed to be constructed with the components of a vector field X defined along a geodesic C of M, we have

THEOREM 11. - *The Lie derivative of F_{CB} with respect to X^C vanishes if X generates a motion in the base Finslerian manifold. The similar statement holds for the Riemannian case too* [21].

§ 4. - Curvature tensor of ∇^S.

Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be any vector fields in $T(M)$.

The curvature tensor K^S of the connection ∇^S is, by definition, given by

(4.1) $$K^S(\tilde{X}, \tilde{Y})\tilde{Z} = [\nabla^S_{\tilde{X}}, \nabla^S_{\tilde{Y}}]\tilde{Z} - \nabla^S_{[\tilde{X}, \tilde{Y}]}\tilde{Z}$$

and has in $\pi^{-1}(U)$ the components

$$(K^S)_{DCB}{}^A = \partial_D \Gamma_{CB}{}^A - \partial_C \Gamma_{DB}{}^A + \Gamma_{DE}{}^A \Gamma_{CB}{}^E - \Gamma_{CE}{}^A \Gamma_{DB}{}^E.$$

If we express the Equation (4.1) in terms of the adapted frame, we have

(4.2) $$(\nabla_\delta \nabla_\gamma - \nabla_\gamma \nabla_\delta) Z^\alpha = [K^S]_{\delta\gamma\beta}{}^\alpha Z^\beta$$

where

(4.3) $$[K^S]_{\delta\gamma\beta}{}^\alpha = \delta_\delta \Gamma^\alpha_{\gamma\beta} - \delta_\gamma \Gamma^\alpha_{\delta\beta} + \Gamma^\alpha_{\delta\varepsilon}\Gamma^\varepsilon_{\gamma\beta} - \Gamma^\alpha_{\gamma\varepsilon}\Gamma^\varepsilon_{\delta\beta} - \Omega_{\delta\gamma}{}^\varepsilon \Gamma^\alpha_{\varepsilon\beta}.$$

On using the expression of $\Gamma^\alpha_{\gamma\beta}$ for the various range of indices given in (2.19) and taking account of (2.16) and (2.17), we find that all the particular values of $[K^S]_{\delta\gamma\beta}{}^\alpha$ are

(a) $$[K^S]_{kji}{}^h = K_{kji}{}^h - C_{ka}{}^h C_{ji}{}^a + C_{ja}{}^h C_{ki}{}^a$$
$$- \frac{1}{2}(C_{ka}{}^h K_{jib}{}^a - C_{ja}{}^h K_{kib}{}^a - 2C_{ia}{}^h K_{kjb}{}^a + C_{ki}{}^a K^h{}_{jba} - C_{ji}{}^a K^h{}_{kba}) y^a$$
$$+ \frac{1}{4}(K^h{}_{kcd} K_{jib}{}^d - K^h{}_{icd} K_{kid}{}^d - 2K^h{}_{icd} K_{kjb}{}^d) y^c y^b,$$

(b) $$[K^S]_{\bar{k}ji}{}^h = \nabla_i C_{ki}{}^h - \nabla^h C_{kji} + \frac{1}{2}(\nabla_j K^h{}_{iak}) y^a$$
$$- \frac{1}{2}(K_{jia}{}^b \nabla_0 A_{ka}{}^h + K^h{}_{jab} \nabla_0 A_{ik}{}^b + K^h{}_{iab} \nabla_0 A_{jk}{}^b) y^a$$

(c) $$[K^S]_{k\bar{j}i}{}^h = -\nabla_i C_{kj}{}^h + \nabla^h C_{kji} - \frac{1}{2}(\nabla_k K^h{}_{iaj}) y^a$$
$$+ \frac{1}{2}(K_{kia}{}^b \nabla_0 A_{jb}{}^h + K^h{}_{kab} \nabla_0 A_{jk}{}^b + K^h{}_{iab} \nabla_0 A_{kj}{}^b) y^a,$$

(d) $$[K^S]_{kj\bar{i}}{}^h = \nabla_k C_{ji}{}^h - \nabla_j C_{ki}{}^h - \frac{1}{2}(\nabla_k K^h{}_{jai} - \nabla_j K^h{}_{kai}) y^a$$

(4.4) $$+ K_{kja}{}^b y^a \nabla_0 A_{bi}{}^h,$$

(e) $$[K^S]_{\bar{k}\bar{j}i}{}^h = -\frac{1}{2}(\partial_{\bar{k}} K^h{}_{iaj} - \partial_{\bar{j}} K^h{}_{iak}) y^a - \frac{1}{2}(K^h{}_{ikj} - K^h{}_{ijk})$$
$$+ C_{ka}{}^h C_{ji}{}^a - C_{ja}{}^h C_{ki}{}^a - \nabla_0 A_{ka}{}^h \nabla_0 A_{ja}{}^h + \nabla_0 A_{ja}{}^h \nabla_0 A_{ki}{}^a$$

$$-\frac{1}{2}(C_{ka}{}^{h}K^{a}{}_{ibj}-C_{ja}{}^{h}K^{a}{}_{ibk}+C_{ji}{}^{a}K^{h}{}_{abk}-C_{ki}{}^{a}K^{h}{}_{abj})y^{b}$$

$$+\frac{1}{4}(K^{h}{}_{ack}K^{a}{}_{ibj}-K^{h}{}_{acj}K^{a}{}_{ibk})y^{c}y^{b},$$

(f) $$[K^{S}]_{\bar{k}j\bar{i}}{}^{h}=-\frac{1}{2}(\partial_{\bar{k}}K^{h}{}_{jai})y^{a}-\frac{1}{2}K^{h}{}_{jki}\frac{1}{4}K^{k}{}_{ack}K^{a}{}_{jbi}y^{c}y^{b}$$

$$+\partial_{\bar{k}}C_{ji}{}^{h}+C_{ka}{}^{h}C_{ji}{}^{a}-C_{ja}{}^{h}C_{ki}{}^{a}$$

$$-\frac{1}{2}C_{ka}{}^{h}K^{a}{}_{jbi}y^{b}+\frac{1}{2}C_{ki}{}^{a}K^{h}{}_{jba}y^{b}-\frac{1}{2}C_{ji}{}^{a}K^{h}{}_{kba}y^{b}$$

$$+\nabla_{k}\nabla_{0}A_{ji}{}^{h}-\nabla_{j}\nabla_{0}A_{ki}{}^{h},$$

(g) $$[K^{S}]_{k\bar{j}\bar{i}}{}^{h}=\frac{1}{2}(\partial_{\bar{j}}K^{h}{}_{kai})y^{a}+\frac{1}{2}K^{h}{}_{kji}-\frac{1}{4}K^{h}{}_{acj}K^{a}{}_{kbi}y^{c}y^{b}$$

$$-\partial_{\bar{j}}C_{ki}{}^{h}-C_{ja}{}^{h}C_{ki}{}^{a}-C_{ka}{}^{h}C_{ji}{}^{a}$$

$$+\frac{1}{2}C_{ja}{}^{h}K^{a}{}_{kbi}y^{b}-\frac{1}{2}C_{ji}{}^{a}K^{h}{}_{kba}y^{b}+\frac{1}{2}C_{ki}{}^{a}K^{h}{}_{jba}y^{b}$$

$$-\nabla_{j}\nabla_{0}A_{ki}{}^{h}+\nabla_{k}\nabla_{0}A_{ji}{}^{h},$$

(h) $$[K^{S}]_{\bar{k}\bar{j}\bar{i}}{}^{h}=\partial_{\bar{k}}\nabla_{0}A_{ji}{}^{h}-\partial_{\bar{j}}\nabla_{0}A_{ki}{}^{h}$$

$$+\left(C_{ka}{}^{h}-\frac{1}{2}K^{h}{}_{abk}y^{b}\right)\nabla_{0}A_{ji}{}^{a}-(C_{ja}{}^{h}K^{h}{}_{abj}y^{b})\nabla_{0}A_{ki}{}^{a}$$

$$-C_{ki}{}^{a}\nabla_{0}A_{ja}{}^{h}+C_{ji}{}^{a}\nabla_{0}A_{ka}{}^{h},$$

(i) $$[K^{S}]_{kji}{}^{\bar{h}}=-\nabla_{k}C_{ji}{}^{h}+\nabla_{j}C_{ki}{}^{h}-\frac{1}{2}(\nabla_{j}K_{kib}{}^{h}-\nabla_{k}K_{jib}{}^{h})y^{b}$$

$$-K_{kjb}{}^{a}y^{b}\nabla_{0}A_{ai}{}^{h},$$

(j) $$[K^{S}]_{\bar{k}ji}{}^{\bar{h}}=-\frac{1}{2}\partial_{\bar{k}}(K_{jia}{}^{h}y^{a})-\partial_{\bar{k}}C_{ji}{}^{h}-C_{ka}{}^{h}C_{ji}{}^{a}+C_{ja}{}^{h}C_{ki}{}^{a}$$

$$+\nabla_{j}\nabla_{0}A_{ki}{}^{a}+\frac{1}{2}C_{ki}{}^{a}K_{jab}{}^{h}y^{b}-\frac{1}{2}C_{ji}{}^{a}K_{kab}{}^{h}y^{b}-\frac{1}{2}C_{ja}{}^{h}K^{a}{}_{ibk}y^{b}$$

$$-\frac{1}{4}K_{jac}{}^{h}K^{a}{}_{ibk}y^{c}y^{b}-\nabla_{0}A_{kj}{}^{a}\nabla_{0}A_{ai}{}^{h},$$

(k) $$[K^S]_{k\bar{j}i}{}^{\bar{h}} = \partial_{\bar{j}}(K_{kia}{}^h y^a) + \partial_{\bar{j}} C_{ki}{}^h + C_{ja}{}^h C_{ki}{}^a - C_{ka}{}^h C_{ji}{}^a$$

$$- \nabla_k \nabla_0 A_{ji}{}^a - \frac{1}{2} C_{ji}{}^a K_{kab}{}^h y^b + \frac{1}{2} C_{ki}{}^a K_{jab}{}^h y^b + \frac{1}{2} C_{ka}{}^h K^a{}_{ibj} y^b$$

$$+ \frac{1}{4} K_{kac}{}^h K^a{}_{ibj} y^c y^b + \nabla_0 A_{jk}{}^a \nabla_0 A_{ai}{}^h,$$

(l) $$[K^S]_{kj\bar{i}}{}^{\bar{h}} = K_{kji}{}^h + \frac{1}{4}(K_{kac}{}^h K_{jbi}{}^h - K_{jac}{}^h K_{kbi}{}^a) y^c y^b$$

$$- C_{ka}{}^h C_{ji}{}^a + C_{ji}{}^a C_{ki}{}^a + \frac{1}{2}(C_{ki}{}^a K_{jab}{}^h - C_{ji}{}^a K_{kab}{}^h) y^b$$

$$+ \frac{1}{2}(C_{ka}{}^h K_{jbi}{}^a - C_{ja}{}^h K_{kbi}{}^a + C_{ai}{}^h K_{kjb}{}^a) y^b,$$

(m) $$[K^S]_{\bar{k}\bar{j}i}{}^{\bar{h}} = - \nabla_k C_{ji}{}^h + \nabla_j C_{ki}{}^h + \frac{1}{2} K^a{}_{jbi} y^b \nabla_0 A_{ka}{}^h - \frac{1}{2} K^a{}_{ibj} y^b \nabla_0 A_{ja}{}^h,$$

(n) $$[K^S]_{\bar{k}\ \bar{i}}{}^{\bar{h}} = \nabla_i C_{kj}{}^h + \nabla^h C_{kji} - \frac{1}{2} K^a{}_{jbi} y^b \nabla_0 A_{ka}{}^h - \frac{1}{2} K_{jab}{}^h y^b \nabla_0 A_{ki}{}^a,$$

(o) $$[K^S]_{k\bar{j}\bar{i}}{}^{\bar{h}} = - \nabla_i C_{jk}{}^h + \nabla^h C_{jki} - \frac{1}{2} K^a{}_{kbi} y^b \nabla_0 A_{ja}{}^h - \frac{1}{2} K_{kab}{}^h y^b \nabla_0 A_{ji}{}^a,$$

(p) $$[K^S]_{\bar{k}\bar{j}\bar{i}}{}^{\bar{h}} = \partial_{\bar{k}} C_{ji}{}^h - \partial_{\bar{j}} C_{ki}{}^h + C_{ka}{}^h C_{ji}{}^a - C_{ja}{}^h C_{ki}{}^a$$

$$+ (\nabla_0 A_{ki}{}^a) \nabla_0 A_{ja}{}^h - (\nabla_0 A_{ij}{}^a) \nabla_0 A_{ka}{}^h.$$

From these it is clear that if the base Finslerian manifold is flat, then $C_{ji}{}^h$ and $K_{kji}{}^h$. Conseversely being zero, all the components $[K^S]_{\delta\gamma\beta}{}^\alpha$ vanish and hence $T(M)$ is flat. If $[K^S]_{\delta\gamma\beta}{}^\alpha$ vanish, then by taking account of the homogeneity properties of C' *s* and K' *s*, we find, for instance, from (4.4(j)) that

(4.5) $$\partial_{\bar{k}} C_{ji}{}^h - C_{ka}{}^h C_{ji}{}^a + C_{ja}{}^h C_{ki}{}^a = 0,$$

the left hand side being homogeneous of degree minus two in the y' *s*.

Then transvecting (4.5) by y^k and by using the relations

(4.6) $$y^k \partial_{\bar{k}} C_{ji}{}^h = - C_{ji}{}^h, \quad C_{ka}{}^h y^k = 0,$$

we obtain

$$C_{ji}{}^h = 0,$$

which implies that the base manifold M is Riemannian, and for the case $[K^S]_{\delta\beta\gamma}{}^{\alpha}$ given in (4.4(a) ~ (p)) reduce respectively to

(4.7)

(a) $[K^S]_{kji}{}^h = K_{kji}{}^h + \frac{1}{4}(K_{dck}{}^h K_{jib}{}^d - K_{dcj}{}^h K_{kib}{}^d - 2K_{fci}{}^h K_{kjb})y^b,$

(b) $[K^S]_{\bar{k}ji}{}^h = \frac{1}{2}(\nabla_j K_{kai}{}^h)y^a,$

(c) $[K^S]_{k\bar{j}i}{}^h = -\frac{1}{2}(\nabla_k K_{jai}{}^h)y^a,$

(d) $[K^S]_{kj\bar{i}}{}^h = -\frac{1}{2}(\nabla_k K_{iaj}{}^h - \nabla_j K_{iak}{}^h)y^a,$

(e) $[K^S]_{\bar{k}\bar{j}i}{}^h = K_{kji}{}^h + \frac{1}{4}(K_{kca}{}^h K_{jbi}{}^a - K_{jca}{}^h K_{kbi}{}^a)y^c y \quad ,$

(f) $[K^S]_{k\bar{j}\bar{i}}{}^h = \frac{1}{2}K_{kij}{}^h + \frac{1}{4}K_{kca}{}^h K_{ibj}{}^a y^c y^b,$

(g) $[K^S]_{k\bar{j}\bar{i}}{}^h = +\frac{1}{2}K_{ijk}{}^h - \frac{1}{4}K_{jca}{}^h K_{kbi}{}^a y^c y^b,$

(h) $[K^S]_{\bar{k}\bar{j}\bar{i}}{}^h = 0,$

(i) $[K^S]_{kji}{}^{\bar{h}} = \frac{1}{2}(\nabla_i K_{kja}{}^h)y^a,$

(j) $[K^S]_{\bar{k}ji}{}^{\bar{h}} = -\frac{1}{2}K_{jik}{}^h - \frac{1}{4}K_{jac}{}^h K_{kbi}{}^a y^c y^b,$

(k) $[K^S]_{k\bar{j}i}{}^{\bar{h}} = \frac{1}{2}K_{kji}{}^h + \frac{1}{4}K_{kac}{}^h K_{jbi}{}^a y^c y^b,$

(l) $[K^S]_{kj\bar{i}}{}^{\bar{h}} = K_{kji}{}^h + \frac{1}{4}(K_{kac}{}^h K_{ibj}{}^a - K_{jac}{}^h K_{ibk}{}^a)y^b.$

(m) $[K^S]_{\bar{k}\bar{j}i}{}^{\bar{h}} = 0,$

(n) $[K^S]_{\bar{k}j\bar{i}}{}^{\bar{h}} = 0,$

(o) $[K^S]_{k\bar{j}\bar{i}}{}^{\bar{h}} = 0,$

(p) $[K^S]_{\bar{k}\bar{j}\bar{i}}{}^{\bar{h}} = 0.$

Thus, if $[K^S]_{\delta\beta\gamma}{}^\alpha$ vanish, then we find from (4.7(a)), for example, that $K_{kji}{}^h$ vanishes because it does not depend upon the y' s, while the second term in the right hand side of (4.7(a)) involves them as a linear combination, and hence the base Riemannian manifold is flat. We have proved

THEOREM 12. - *If the tangent bundle $T(M)$ over a Finslerian manifold M whose metric is Sasakian lift g^S is flat, then M is flat and the converse is true also.*

We have from the Equations (4.3) the relation

$$[K^S]_{\delta\gamma\beta}{}^\alpha + [K^S]_{\gamma\delta\beta}{}^\alpha = 0 \tag{4.8}$$

immediately, into which we substitute (4.4) corresponding to the range of different indices to obtain the sole relation (H. RUND [15], p. 105).

$$K_{kji}{}^h + K_{jki}{}^h = 0. \tag{4.9}$$

Also, by taking account of (2.4), we have

$$[K^S]_{\delta\gamma\beta\alpha} + [K^S]_{\delta\gamma\alpha\beta} = 0, \tag{4.10}$$

where we have put

$$[K^S]_{\delta\gamma\beta\alpha} = \overset{S}{g}_{\varepsilon\alpha}[K^S]_{\delta\gamma\beta}{}^\varepsilon,$$

into which we substitute (4.4) to obtain the sole relation (H. RUND, [15], p. 105)

$$K_{khji} + K_{khij} + 2C_{jib}K_{kha}{}^b y^a = 0 \tag{4.11}$$

If in (4.1) we change X, Y and Z cyclically and sum up the results, we have so-called Bianchi's identity, which has in terms of the adapted frame the expression

$$\begin{aligned} &[K^S]_{\delta\gamma\beta}{}^\alpha + [K^S]_{\gamma\beta\delta}{}^\alpha + [K^S]_{\beta\delta\gamma}{}^\alpha \\ &= \delta_\delta\Omega_{\gamma\beta}{}^\alpha + \delta_\gamma\Omega_{\beta\delta}{}^\alpha + \delta_\beta\Omega_{\delta\gamma}{}^\alpha \\ &+ \Omega_{\delta\varepsilon}{}^\alpha\Omega_{\gamma\beta}{}^\varepsilon + \Omega_{\gamma\varepsilon}{}^\alpha\Omega_{\beta\delta}{}^\varepsilon + \Omega_{\beta\varepsilon}{}^\alpha\Omega_{\delta\gamma}{}^\varepsilon \end{aligned} \tag{4.12}$$

and by substituting each term of (4.4) in accordance with the various indices that belong to (4.12), we deduce the following four relations (H. RUND, [15], p. 105 and p. 110).

$$K_{kji}{}^h + K_{jik}{}^h + K_{ikj}{}^h = 0 \tag{4.13}$$

$$(4.14)\qquad \nabla_l K_{kji}{}^h + \nabla_k K_{jl}{}^h + \nabla_j K_{lki}{}^h$$

$$+ (K_{lkc}{}^b y^c \partial_{\bar b}\Gamma^a_{ji} + K_{kic}{}^b y^c \partial_{\bar b}\Gamma^a_{li} + K_{jlc}{}^b y^c \partial_{\bar b}\Gamma^a_{kc}) = 0,$$

$$(4.15)\qquad (\nabla_l K_{kja}{}^h + \nabla_k K_{jia}{}^h + \nabla_j K_{lka}{}^h) y^a$$

$$+ (K_{lkb}{}^c \nabla_0 A_{jc}{}^h + K_{kjb}{}^e \nabla_0 A_{lc}{}^h + K_{jlb}{}^c \nabla_0 A_{kc}{}^h) y^b = 0$$

and

$$\partial_{\bar k} \nabla_0 A_{ji}{}^h = \nabla_k C_{ji}{}^h + y^b \nabla_b C_{kji}{}^h$$

$$(4.16)\qquad - (C_{ia}{}^h \nabla_b C_{kj}{}^h + C_{ja}{}^h \nabla_b C_{ki}{}^a + C_{ji}{}^a \nabla_b C_{ka}{}^h + 2C_{ka}{}^h \nabla_b C_{ji}{}^a) y^b.$$

The formulas (4.13) and (4.14), or equivalently (4.15) are called by H. RUND *Bianchi's identitites of the first and second kinds,* respectively.

Taking account of the theorem 3 we have from (4.11) and (4.14)

THEOREM 13. - *If the tangent bundle $T(M)$ over a Finslerian manifold is Kaehlerian, the curvature tensor of the base Finsler manifold satisfies*

$$(4.17)\qquad K_{khji} + K_{hkji} = 0,$$

$$\nabla_l K_{kji}{}^h + \nabla_k K_{jli}{}^h + \nabla_j K_{lki}{}^h = 0.$$

§ 5. - Ricci and scalar curvature of ∇^S.

We now compute the RICCI curvature of ∇^S. It is given by

$$(5.1)\qquad [K^S]_{\gamma\beta} = [K^S]_{\delta\gamma\beta}{}^\delta$$

and on using (4.4) those components $[K^S]_{\gamma\beta}$ will be

$$(a)\quad [K^S]_{ji} = K_{ji} - \frac{1}{4}(K^a{}_{jc}{}^d K_{aibd} + 2K^a{}_{ic}{}^d K_{ajbd} + K_{jac}{}^d K^a{}_{\cdot ibd}) y^c y^b$$

$$+ \frac{1}{2}(C_j{}^{ac} K_{ciba} + C_i{}^{ac} K_{cjba} - C_c{}^{ac} K_{jiba}) y^b$$

$$- 2(C_{ba}{}^b C_{ji}{}^a + C_{ja}{}^b C_{bi}{}^a)$$

$$- \frac{1}{2}\partial_{\bar b}(K_{jia}{}^b y^b) - \partial_{\bar a} C_{ji}{}^a + \nabla_j \nabla_0 A_i - \nabla_0 A_{bj}{}^a \nabla_0 A_{ai}{}^b,$$

$$(b)\qquad [K^S]_{\bar j i} = -\frac{1}{2}\nabla_b (K^b{}_{jai} y^b) + \frac{1}{2}(K_{dja}{}^b \nabla_0 A_{ib}{}^d + K^a{}_{jbi} y^b \nabla_0 A_a)$$

$$(5.2)\qquad + \nabla_i C_{aj}{}^a - \nabla_j C_{ai}{}^a,$$

(c) $$[K^S]_{j\bar{i}} = -\frac{1}{2}\nabla_b(K^b{}_{jai}y^b) + \frac{1}{2}K_{dja}{}^b\nabla_0 A_{ib}{}^d + K^a{}_{jbi}y^b\nabla_0 A_a) + \nabla_i C_{aj}{}^a - \nabla_j C_{ai}{}^a,$$

(d) $$[K^S]_{\bar{j}\bar{i}} = \frac{1}{4}K^{ca}{}_{cj}K_{cabi}y^c y^b + (\partial_{\bar{a}}C_{ji}{}^a - 2\partial_{\bar{j}}C_{ia}{}^a + 2C_{ab}{}^b C_{ji}{}^a - 2C_{ja}{}^b C_{bi}{}^a) + \nabla_a\nabla_0 A_{ji}{}^a - \nabla_0 A_a \nabla_0 A_{ji}{}^a.$$

Now, if $T(M)$ is an Einstein space, that is to say, if

(5.3) $$[K^S)_{\beta\alpha} = k\overset{S}{g}_{\beta\alpha},$$

or, by (1.24)

(5.4) $$[K^S]_{ji} = kg_{ji},\ [K^S]_{\bar{j}i} = [K^S]_{j\bar{i}} = 0\ [K^S]_{\bar{j}\bar{i}} = kg_{ji},$$

then, by taking account of the homogeneity properties of the functions involved, we have from the fourth equations of (5.2) and of (5.4)

(5.5) $$K^{ea}{}_{cj}K_{eabi}y^c y^b = 0,$$

(5.6) $$\partial_{\bar{a}}C_{ji}{}^a - 2\partial_{\bar{j}}C_{ia}{}^a + 2C_{ab}{}^b C_{ji}{}^a - 2C_{ja}{}^b C_{bi}{}^a = 0,$$

(5.7) $$\nabla_a\nabla_0 A_{ji}{}^a - \nabla_0 A_a \nabla_0 A_{ji}{}^a = 0.$$

Multiplying (5.6) by y^j and taking account of (4.6) we find taht

$$C_{ia}{}^a = 0,$$

and consequently

$$A_i = 0,$$

which implies by DEIKE's theorem stated in § 2 that the base manifold M is Riemannian, while we have from (5.5)

$$K_{jia}{}^h y^a = 0$$

and consequently M is flat. Thus

THEOREM 14. - *If the tangent bundle over a Finslerian or Riemannian manifold with Sasakian lift g^S as metric is an Einstein space, then the base manifold is flat.*

Let us consider the scalar curvature of ∇^S:

$$[K^S] = [K^S]_\alpha{}^\alpha. \tag{5.8}$$

Then we have from (5.2)

$$\begin{aligned}[K^S] = K + 2\nabla_a\nabla_0 A^a - \nabla_0 A_a \nabla_0 A^a - \nabla_0 A_c{}^{ba}\nabla_0 A_{bac} \\ -\frac{1}{4} K^{ea}{}_c{}^d K_{eabd} y^c y^b - 2\partial_{\bar b} C_a{}^{ba}\end{aligned} \tag{5.9}$$

where

$$K = K_j{}^j, \quad C_a^{ba} = g^{cb} C_{ac}{}^a$$

Thus, if $[K^S]$ vanishes, then by taking account of the homogeneity properties of the functions involved again, we have

$$\begin{aligned} K = \nabla_0 A_c{}^{ba}\nabla_0 A_{ba}{}^c + \nabla_0 A_a \nabla_0 A^a - 2\nabla_a\nabla_0 A^a \\ K_{jia}{}^h y^a = 0, \\ \partial_{\bar a} C_b^{ab} = 0, \end{aligned} \tag{5.10}$$

from which we have

THEOREM 15. - *If in the tangent bundle $T(M)$ over a Finslerian manifold M with Sasakian metric g^S as metric its scalar curvature vanishes, the $T(M)$ is Kaehlerian and if M is a Riemannian manifold, M must be flat.*

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