Weak solutions for nonlinear functional equations in Banach spaces. (*)

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Summary. - *Some existence theorems for abstract differential equations in Banach spaces are given.*

The paper is concerned with the perturbation of hyperdissipative mappings in a BANACH space X and with existence theory for the solutions of abstract functional equation

$$\text{(E)} \qquad \lambda x - Ax - Bx = y, \qquad \lambda > 0, \qquad y \in X$$

where A and B are (possibly) nonlinear dissipative operators from X into itself. Under different assumptions about A and B this problem was studied by several authors (see [1], [3], [6], [7], [17]).

In section 1 we gather together some results from the theory of dissipative mappings and nonlinear semigroups of contractions which we shall need later. For detailed informations one must refer to works of KOMURA [11], [12], KATO [10], BROWDER [7], CRANDALL and PAZY [5], BREZIS and PAZY [4].

In section 2 some results concerning the perturbation of hyperdissipative mappings are established. For the sake of simplicity we have considered only the case when A and B are single valued, but many of these results may be proved for multivalued mappings.

As application, in section 3 we give some existence results for the nonlinear evolution equation of the form

$$\frac{du}{dt} - B(t)u - Au + \lambda u = f(t); \qquad 0 \leq t \leq T, \qquad u(o) = 0.$$

where $B(t)$ is a family of generators of C_0–contraction semigroups on a BANACH space Y and A belongs to a certain class of hyperdissipative nonlinear operators defined in $L^p(0, T; Y)$. This problem was studied by many authors (see e.g. F. BROWDER [6], J. L. LIONS and A. STRAUS [13], C. BARDOS and H. BREZIS [3], G. DA PRATO [17]).

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Finally the author is indebt to Professor DA PRATO for having the chance to see his papers [16], [17] before pubblication.

1 §. - Dissipative mappings and nonlinear contraction semigroups.

Let X be a real BANACH space of norm $\| \|$ and let X^* its dual space. We denote by $(,)$ the pairing between X and X^* and by F the duality mapping of X into X^*, i.e.

$$F(x) = \{ y \in X^*, \ \|y\| = \|x\|, \ (x, y) = \|x\|^2 \}. \tag{1.1}$$

If X^* is strictly convex then $x \longrightarrow F(x)$ is a single valued demicontinuous mapping (i.e. continuous from X into X^*_w) and if in addition X^* is uniformly convex then F is uniformly continuous on every bounded subset of X (see e.g. [10]).

A mapping Δ from X to 2^X is said to be dissipative if for any $x, y \in D(\Delta)$ (the domain of Δ) and $u \in \Delta x$, $v \in \Delta y$,

$$(u - v, f) \leqq 0 \quad \text{for all} \quad f \in F(x - y). \tag{1.2}$$

Δ is said to be maximal dissipative if it has no properly dissipative extension in X.

LEMMA 1.1 (c.f. [10], [11]). - Let X be a real BANACH space and Δ be a dissipative mapping from X into 2^X. Assume that the range of $\alpha - \Delta$ is the whole space X for some $\alpha > 0$. Then Δ is maximal dissipative and for all $\lambda > 0$, $\lambda - \Delta$ has an inverse defined on all X which is lipschitzian of constant λ^{-1}.

A mapping Δ satisfying the conditions of Lemma 1.1 is called hyperdissipative. If X is strictly convex then Δ is hyperdissipative if and only if for all $\lambda > 0$,

$$\|(\lambda - \Delta)^{-1}\|_{\text{Lip}} \leqq \lambda^{-1}. \tag{1.3}$$

If X^* is uniformly convex then every dissipative and demicontinuous single valued mapping defined on all of X, which maps bounded sets of X into bounded sets is hyperdissipative (see F. BROWDER [8]). If X is a HILBERT space then according to well known result of MINTY [14], every maximal dissipative mapping in X is hyperdissipative.

Let Δ be maximal dissipative. Then as easily follows, for any $x \in D(\Delta)$ the set Δx is closed and convex, hence if X is a strictly convex there exists a unique element of minimum norm in Δx denoted by $\Delta^0 x$. If Δ is hyperdissipative we set

$$J_n(x) = (1 - n^{-1}\Delta)^{-1}x, \qquad x \in X; \qquad n = 1, 2, \ldots$$
$$A_n(x) = n(J_n(x) - x),$$

we give without proof the following elementary results (for a proof see e.g. [5], [10]).

LEMMA 1.2. – Let X be a real BANACH space with uniformly convex dual space X^* and Δ be a hyperdissipative mapping in X. Then

a) J_n is a contraction and Δ_n is dissipative on X.

b) For each $x \in D(\Delta)$, $\Delta_n x \in \Delta x$ and $\| \Delta_n x \| \leq \inf \| y \|;\ y \in \Delta x \}$.

c) For any $x \in \overline{D(\Delta)}$, $\lim J_n(x) = x$ as $n \longrightarrow \infty$.

d) Let $\{x_n\} \in D(\Delta)$ be strongly convergent to $x \in X$. If exist $y_n \in \Delta x_n$ such that $\| y_n \| \leq M < \infty$, then $x \in D(\Delta)$ and every weak clauster point of $\{y_n\}$ belongs to Δx.

e) If $x_n \longrightarrow x$ and $\| \Delta_n x_n \| \leq M < \infty$ as $n \longrightarrow \infty$, then $x \in D(\Delta)$ and any weak clauster point of $\{\Delta_n x_n\}$ belongs to Δx.

LEMMA 1.3. – Assume that X is uniformly convex. Then

a) For any $x \in D(\Delta)$, $\Delta_n x \longrightarrow \Delta^0 x$ as $n \longrightarrow \infty$

b) $\overline{D(\Delta)}$ is a convex subset of X.

we mention also the following useful result (see [10])

LEMMA 1.4. – Let X be a real BANACH space and $x(t)$ be a X-valued function defined on an interval of real axis. Suppose that $x(t)$ is weakly differentiable at $t = s$ and that $\| x(t) \|$ is also differentiable at $t = s$. Then

$$\| x(s) \| \frac{d}{dt} \| x(t) \|_{t=s} = (x'(s), f(s)) \tag{1.4}$$

for any $f(s) \in F(x(s))$.

DEFINITION. – Let Y be a subset of a BANACH space X. A function T: $:[0, \infty) \times Y \longrightarrow Y$ is said to be a semigroup of contractions on Y if it satisfies the following conditions

i) $T(t + s)x = T(t)T(s)x,\ t,\ s \geq 0,\ x \in Y$

ii) $\| T(t)x - T(t)y \| \leq \| x - y \|,\ t \geq 0,\ x,\ y \in Y$

iii) $T(0)x = x$ and $\lim\limits_{t \to 0} T(t)x = x$ for any $x \in Y$.

If T is a semigroup on a subset Y of X then for each $x \in Y$ and $h > 0$, set

$$L^h x = h^{-1}(T(h)x - x)$$

and denote by $D(L_s)$ ($D(L_w)$ respectively) the set of those $x \in Y$ for which $\lim_{h\to 0} L^h x$ (weak-$\lim_{h\to 0} L^h x$) exists. Define

$$L_s x = \lim_{h\to 0} L^h x, \qquad x \in D(L_s)$$

and

$$L_w x = \text{weak-}\lim_{h\to 0} L^h x, \qquad x \in D(L_w).$$

The operator L_s (L_w respectively) is called the strong (weak) generator of T. It is easy to see that L_s and L_w are dissipative in the space X. For every semigroup, T on $Y \subset X$ define

$$D_T = \{x \in Y \text{ such that } \varliminf_{t\to 0} \| T(t)x - x \| t^{-1} < \infty\}.$$

Obviously we have the inclusion relation $D(L_s) \subset (L_w) \subset D_T$.

LEMMA 1.5. - For any $x \in D_T$ the function $t \to T(t)x$ is lipschitz continuous on $[0, \infty)$.

PROOF. - It is a direct consequence of the fact that $t \to \| T(t)x - x \|$ is continuous and subadditive on positive axe. For a detailed proof see (5).

PROPOSITION 1. - Let T be a contraction semigroup on a reflexive BANACH space X. Then $\overline{D(L_s)} = \overline{D_T}$.

PROOF. - Let x be an an arbitrary point of D_T. According to Lemma 1.5 the function $t \to T(t)x$ is lipschitz continuous. Since the space X is reflexive it follows (see Y. KOMURA [11]) that it is a.e. differentiable on $(0, \infty)$. Hence $T(t)x \in D(L_s)$ for almost all $t > 0$ which implies that $x \in \overline{D(L_s)}$.

PROPOSITION 2. - Let X be a reflexive BANACH space and T be a nonlinear semigroup of contractions on X. If D_T is dense in X then the domain of every dissipative extension of L_w is contained in D_T.

PROOF. - It suffices to show that if for arbitrary $\tilde{x}$, $\tilde{y} \in X$, the following inequality

$$(L_w x - \tilde{y}, f) \leq 0 \tag{1.5}$$

holds fo every $f \in F(x - \tilde{x})$ and $x \in D(L_w)$, then $\tilde{x} \in D_T$. Let u be an arbitrary point of D_T. Since the function $t \longrightarrow T(t)x$ is almost everywhere differentiable on $(0, \infty)$ we have

$$\frac{d}{dt} T(t)u = L_w T(t)u, \qquad \text{a.e.}$$

Using Lemma 1.4 we get

$$2^{-1} \frac{d}{dt} \| T(t)u - x \|^2 = (L_w T(t)u, \ f(t)), \quad \text{a.e. on} \quad (0, \infty) \tag{1.6}$$

for any $f(t) \in F(T(t)u - x)$. Taking in the inequality (1.5) $x = T(t)u$, from (1.6) we obtain

$$\| T(t)u - \tilde{x} \| \leq \| u - \tilde{x} \| + t \| y \|$$

for all $t \geq 0$ and $u \in D_T$. Since D_T is dense in X this implies

$$\| T(t)\tilde{x} - \tilde{x} \| \leq t \| y \|, \qquad \forall\, t \geq 0 \tag{1.7}$$

completing the proof.

PROPOSITION 3. - Let X be a reflexive BANACH space and let T be a semigroup of contractions of X. If D_T is dense in X and if for any $y \in X$ the mapping $x \longrightarrow L_w x - y$ is also the weak generator of a semigroup of contractions on X then L_w is maximal dissipative and $D(L_w) = D_T$.

PROOF. - Let $\tilde{x}, \tilde{y} \in X$ such that

$$(L_w x - \tilde{y}, \ f) \leq 0 \tag{1.8}$$

for all $x \in D(L_w)$ and $f \in F(x - \tilde{x})$. If $T_{\tilde{y}}$ is the semigroup generated by $L_w - \tilde{y}$ then for any $u \in D(L_w)$ we have

$$\frac{d}{dt} T_{\tilde{y}}(t)u = L_w T_{\tilde{y}}(t)u - \tilde{y} \quad \text{a.e. on} \quad (0, \infty).$$

Taking in (1.8) $x = T_{\tilde{y}}(t)$, by the same argument as in the proof of Proposition 2 it follows

$$\| T_{\tilde{y}}(t)u - \tilde{x} \| \leq \| u - \tilde{x} \|$$

for all $u \in D(L_w)$ and $t \geq 0$. Since $\overline{D(L_w)} = \overline{D}_T = X$ this implies $T_{\tilde{y}}(t)\tilde{x} - \tilde{x} = 0$. Hence $L_w \tilde{x} = \tilde{y}$ which completes the proof.

The following result is essentialy due CRANDALL and PAZY [5].

PROPOSITION 4. - Let X be a stricly convex, reflexive BANACH space and T be a semigroup of nonlinear contractions on a subset Σ of X. Then $D(L_w) = D_T$. If in addition X is uniformly convex then $D(L_w) = D(L_s) = D_T$.

PROOF. - Let $\tilde{L}$ be the dissipative mapping defined by

$$\tilde{L}x = \text{weak-}\lim_{t \in \varphi} t^{-1}(T(t)x - x), \qquad x \in D_T$$

where the limit is considered for all filters φ convergent to 0. Denote by A maximal dissipative extension of $\tilde{L}$. Because for $x \in D_T$, $T(t)x \in D(L_s) \subset D(A)$ a.e. then as in the proof of inequality (17) we obtain

$$t^{-1}\| T(t)x - x \| \leq \| y \|, \qquad y \in Ax, \qquad x \in D_T, \qquad t \geq 0 \tag{1.9}$$

Since $\tilde{L}x \subset Ax$ for any $x \in D_T$ the inequality (1.9) implies $Lx = A^0 x$. Therefore the mapping $x \longrightarrow \tilde{L}x$ is single valued i.e. $D(L_w) = D_T$.

To prove the second statement we notice that from (1.9) it follows

$$t^{-1}\| T(t)x - x \| \leq \| L_w x \| \quad \text{for all} \quad x \in D(L_w) = D_T \quad \text{and} \quad t \geq 0 \tag{1.10}$$

Since as $t \longrightarrow 0$, $t^{-1}(T(t)x - x)$ is weakly convergent to $L_w x$ and X is uniformly convex, (1.10) implies that $t^{-1}(T(t)x - x)$ converges strongly to $L_w x$ as $t \longrightarrow 0$. Hence $D(L_w) = D(L_s)$ and the proof is complete.

As consequence of Proposition 2 and 4 we have

COROLLARY 1. - Let X be a strictly convex reflexive BANACH Space and let T be a contraction semigroup on X. If the the weak generator L_w of T is densely defined in X, then L_w is f-maximal i.e. no single valued dissipative mapping properly extends L_w.

2 §. - Perturbations of hyperdissipative mappings.

In what follows we shall assume that X is a real BANACH space with uniformly convex conjugate space X^*.

Let A and B be two dissipative single valued mappings from X into itself with the domain $D(A)$ and $D(B)$ respectively.

ASSUMPTIONS. - i) B is the generator of a C_0-contraction semigroup of bounded linear operators on X.

ii) A is hyperdissipative and there exist two non negative constants

ω and μ such that $(\lambda - A)^{-1}D(B) \subset D(B)$ for $\lambda > \omega$ and the following inequaliy

$$\| B(\lambda - A)^{-1}x \| \leq \| Bx \| ((\lambda - \omega)^{-1} + \mu(\lambda - \omega)^{-2}) \tag{2.1}$$

holds for any $x \in D(B)$ and $\lambda > \omega$.

THEOREM 1. - Assume that the hypotheses i) and ii) are satisfied. Then for any $\lambda > \omega + \mu$,

$$(\lambda - A - B)(D(A) \cap D(B)) \supset D(B) \tag{2.2}$$

and the inverse operator $(\lambda - A - B)^{-1}$ is a well defined lipschitzian mapping on $D(B)$ with $\| (\lambda - A - B)^{-1} \|_{Lip} \leq \lambda^{-1}$.

DEFINITION. - An element $x \in X$ is said to be a weak solution of the equation

$$\lambda x - Ax - Bx = y, \qquad \lambda \in R, \qquad y \in X \tag{E}$$

if there exists a sequence $\{x_n\} \subset D(A) \cap D(B)$ such that $x_n \to x$ and $\lambda x_n - Ax_n - Bx_n \to y$ as $n \to \infty$.

COROLLARY. 2. - Under the assumptions i) and ii) for any $y \in X$ and $\lambda > 0$ the equation (E) has a unique weak solution $x \in X$. Moreover the map $y \to x$ is lipschitzian of norm λ^{-1}.

PROOF. - Let $F(\lambda)$ be the operator $(\lambda - A - B)^{-1}$ defined on $D(B)$ for $\lambda > \omega + \mu$ and denote again by $F(\lambda)$ its estension on the whole space X. Let $\Delta : D(\Delta) \to 2^X$ be the dissipative mapping defined by

$$\Delta x = \lim_{x_n \to x} (A + B)x_n, \qquad x \in D(\Delta). \tag{2.3}$$

Where $D(\Delta)$ consists of the set of all $x \in X$ for which this limit exists. It is easily seen that $F(\lambda)x = (\lambda - \Delta)^{-1}x$ for any $\lambda > \omega + \mu$ and $x \in X$. According to Lemma 1.1 the operator $(\lambda - \Delta)^{-1}$ is well defined on the space X for all $\lambda > 0$ and $\| (\lambda - \Delta)^{-1} \|_{Lip} \leq \lambda^{-1}$. On the other hand it is easy to see that for any $y \in X$ and $\lambda > 0$, $x = (\lambda - \Delta)^{-1}y$ is a weak solution of (E) and conversely every weak solution of (E) may be written in this form. In this way Corollary 2 is proved.

In what follows we denote by $\overline{A + B}$ the closure of $A + B$ defined by the relation (2.3). Actually Theorem 1 and Corollary 2 assert that $\overline{A + B}$ is hyperdissipative.

THEOREM. 2. – If in addition to the hypotheses of Theorem 1 the space X is uniformly convex, then the operator $A+B$ is demiclosed on $D(A)\cap D(B)$ i.e. if $\{x_n\}\subset D(A)\cap D(B)$ is strongly convergent to $x_0\in D(A)\cap D(B)$ and $\{(A+B)x_n\}$ is weakly convergent to $y_0\in X$ then $(A+B)x_0=y_0$.

In order to prove Theorem 1 and 2 we note firstly the following

LEMMA 2.1. – Assume that the hypotheses i) and ii) are satisfied. Then for each $\lambda\geq 0$, $x\in D(A)\cap D(B)$ and $y\in D(B)$ there exists a lipschitz continuous function $u(t):[0,\infty)\to D(A)\cap D(B)$ such that

a) $(A+B)u(t)$ is weakly continuous on $[0,\infty)$ and the weak derivative of $u(t)$ exists and equals $(A+B-\lambda)u(t)+y$ for all $t\geq 0$.

b) $u(0)=x$ and the map $x\to u(t)$ is nonexpansive on $D(A)\cap D(B)$ for each $t\geq 0$.

PROOF. – Consider the approximate equations

$$\frac{d}{dt}u(t)=(A_n+B-\lambda)\,u(t)+y,\ u(0)=x,\ t\geq 0. \tag{2.4}$$

Using Banach's fixed point theorem it follows easily that the mapping $u\to(A_n+B-\lambda)\,u+y$ is hyperdissipative for any $n=1,2$. This implies (see e.g. [7], [10] that for any $x\in D(B)$ there exists a unique lipschitz continuous and weakly differentiable function $u_n(t):[0,\infty)\to D(B)$ such that $u_n(0)=x$ and which verifies the equation (2.4) in the weak sense. Let $T_n(t)$ be the linear semigroup $\exp(-\lambda-n+B)t$. According to ii) the solution u_n of (2.4) must satisfy the integral equation

$$u(t)=T_n(t)x+n\int_0^t T_n(t-s)\,(1-n^{-1}A)^{-1}u(s)ds+\int_0^t T_n(s)yds \tag{2.5}$$

In particular this implies that $u_n(t)$ is strongly continuous differentiable on $[0,\infty)$ and satisfies the equation (2.4) in the strong sense. It follows from (2.1) and (2.5)

$$\|Bu_n(t)\|\leq\exp(-(n+\lambda)\,t)\,\|Bx\|+(n^2(n-\omega)^{-1}+$$
$$\mu n^2(n-\omega)^{-2}\int_0^t\exp(-(n+\lambda)\,(t-s))\,\|Bu_n(s)\|\,ds+(n+\lambda)^{-1}\,\|By\|,\ t\geq 0.$$

Solving this integral equation we get

$$\|Bu_n(t)\|\leq\exp(\mu n^2+n\omega\,(n-\lambda)-\lambda(n-\omega)^2\,(n-\omega)^2\,t)\|\,Bx\|+M\,\|By\| \tag{2.6}$$

where M is a nonnegative constant independent of n and t. Similarly one obtains the estimate

$$(2.7)\quad \|u_n(t)\| \leq \exp(-\lambda t)\|x - u_0\| + (\|Au_0\| + \|Bu_0\| + \|y\|)\int_0^t \exp(-\lambda s)ds + \|u_0\|$$

for all $t \geq 0$, u_0 being an arbitrary point of $D(A) \cap D(B)$. Now applying Lemma 1.4 to the function $v(t) = u_n(t+h) - u_n(t)$ in virtue of dissipativity of A_n and B we get

$$\|u_n(t+h) - u_n(t)\| \frac{d}{dt}\|u_n(t+h) - u_n(t)\| \leq 0, \qquad \text{a. e}$$

If $x \in D(A) \cap D(B)$ then the proceding inequality implies

$$(2.8)\quad \|\frac{d}{dt}u_n(t)\| \leq \|Ax\| + \|Bx\| + \lambda\|x\|, \ \forall\, t \geq 0$$

Therefore $\|Bu_n(t)\|$ and $\|A_n u_n(t)\|$ are bounded as $n \to \infty$ on each bounded interval of $[0, \infty)$. Now following essentialy [10], [11] it follows that the strong limit $u(t) = \lim u_n(t)$ exists uniformly on every bounded subset. Thus according to Lemma 1.2, $u(t) \in D(A) \cap D(B)$, $\text{weak-}\lim\limits_{n\to\infty} A_n u_n(t) = Au(t)$ and $\text{weak-}\lim\limits_{n\to\infty} Bu_n(t) = Bu(t)$, for any $t \geq 0$. Since as easily follows (see [10]) the function $t \to (A + B)\,u(t)$ is weakly continuous, passing to limit in (2.5) we deduce that u verifies on $[0, \infty)$ the equation

$$(E_0)\quad \frac{d}{dt}u(t) + (\lambda - A - B)\,u(t) = y, \ u(0) = x$$

in the sense of weak differentiability. b) is a direct consequence of Lemma 1

PROOF OF THEOREM 1. - Let $u(t, y)$ be the solution of equation (E_0) with $y \in D(B)$ and $\lambda > \omega + \mu$. Then from (2.6) and (2.7) we get

$$(2.9)\quad \|u(t, y)\| + \|Bu(t, y)\| \leq M \text{ for all } t \geq 0.$$

where M is a nonnegative constant independent of t and λ. Let us remark that if $y, \tilde{y} \in D(B)$ then the function $v(t) = u(t, y) - u(t, \tilde{y})$ is weakly continuous differentiable and $t \to \|v(t)\|$ is lipschitz continuous on $[0, \infty)$. This follows from Lemma 2.1 since $v(t)$ is the limit of a sequence of functions wit these properties (see the proof of Lemma 2.1). Then the applying again Lemma 1.4 we get

$$2^{-1}\frac{d}{dt}\|v(t)\|^2\leq-\lambda\|v(t)\|^2+\|y-\tilde{y}\|\,\|v(t)\|$$

for almost all $t\geq 0$. Solving this differential inequality we get

$$\|u(t,y)-u(t,\tilde{y})\|\leq\lambda^{-1}\|y-\tilde{y}\|,\ t\geq 0 \tag{2.10}$$

for any $\lambda>\omega+\mu$. In a similar way it follows

$$\|u(t+h,y)-u(t,y)\|\leq\exp(-\lambda t)\|u(h,y)-u(0,y)\| \tag{2.11}$$

for any $y\in D(B)$ and $t\geq 0$. In particular the inequality (2.11) implies that the strong limit $u=\lim_{t\to\infty}u(t,y)$ exists for any $y\in D(B)$. According to Lemma 1.2, $u\in D(B)$ and

$$\text{weak-}\lim_{t\to\infty}Bu(t,y)=Bu$$

Since

$$\text{weak-}\lim_{h\to 0}h^{-1}(u(h,y)-u(0,y))<\infty,$$

from (2.11) we deduce

$$\lim_{t\to\infty}\frac{d}{dt}(u(t,y),x^*)=0,\ \forall x^*\in X^* \tag{2.12}$$

Hence $\|Au(t)\|$ is bounded on $[0,\infty)$. Then using again Lemma 1.2 it follows that $u\in D(A)$ and $\text{weak-}\lim_{t\to\infty}Au(t)=Au$. Passing to limit in (E_0) and taking account of (2.12) we obtain

$$(\lambda-A-B)u=y$$

and from (2.10)

$$\|(\lambda-A-B)^{-1}\|_{\text{Lip}}\leq\lambda^{-1},\ \forall\lambda>\omega+\mu \tag{2.13}$$

and the proof is complete.

PROOF OF THEOREM 2. - Let $x_0\in D(A)\cap D(B)$, $y_0\in X$ and $\{x_n\}$ be a sequence of $D(A)\cap D(B)$ such that $x_n\to x$ and $(A+B)x_n\rightharpoonup y_0$ ($\rightharpoonup$ denotes the weak convergence in X). From the continuity of duality mapping it follows

$$(y_0-(A+B)x,F(x_0-x))\leq 0,\qquad\forall x\in D(A)\cap D(B)$$

Hence

$$(y_0 - u, F(x_0 - x)) \leq 0 \tag{2.14}$$

for all $x \in D(\overline{A+B})$ and for any $u \in \overline{A+B}x$. On the other hand it follows from Lemma 2.1 that for any $y \in D(B)$ the mapping $x \to (A+B)x - y$ is the generator of a contraction semigroup on $D(A) \cap D(B)$. Denote by T_y the extension of this semigroup on $\overline{D(A) \cap D(B)}$ and by L_y its generator. Let $\{y_n\}$ be a sequence of $D(B)$ strongly convergent to y_0 and let T_{y_n} be the associated semigroups. Let $x \in D(A) \cap D(B)$. Noticing

$$\frac{d}{dt} T_{y_n}(t)x = (A+B)\, T_{y_n}(t)\, x - y_n, \quad \text{a.e. on } (0, \infty)$$

and applying Lemma 1.4 to the function $T_{y_n}(t) - T_{y_m}(t)x$, we get

$$\| T_{y_n}(t)x - T_{y_m}(t)x \| \leq t \| y_n - y_m \|, \quad \text{for all } t \geq 0 \text{ and } n, m = 1, 2. \tag{2.15}$$

Therefore $\lim_{n \to \infty} T_{y_n}(t)x = T_{y_0}(t)x$ exists for any $x \in D(A) \cap D(B)$ and $t \geq 0$. If L_{y_0} denotes the generator of the semigroup T_{y_0} then in virtue of Proposition 4, the inequality (2.15) implies $D(L_{y_0}) = D(L_{y_n})$ for any $n = 1, 2$ and

$$\| L_{y_0}x - L_{y_n}x \| \leq \| y_0 - y_n \|, \ \forall x \in D(L_{y_0})$$

Hence $L_{y_0}x = (A+B)x - y_0$ for all $x \in D(A) \cap D(B) \subset D(L_{y_0})$. Let us remark that $D(L_{y_0}) \subset D(\overline{A+B})$ and $L_{y_0}x \in (\overline{A+B})x - y_0$ for all $x \in D(L_{y_0})$. Indeed for any $x \in D(L_{y_0})$ and $\tilde{x} \in D(A) \cap D(B)$ we have obviously

$$(t^{-1}(T_{y_0}(t)\,x - x) - t^{-1}(T_{y_0}(t)\tilde{x} - \tilde{x}),\ F(x - \tilde{x})) \leq 0, \quad t > 0$$

Hence

$$(L_{y_0}x - (A+B)\tilde{x} + y_0,\ F(x - \tilde{x})) \leq 0$$

Since $\overline{A+B}$ is the closure of $A+B$, the above inequality implies

$$(L_{y_0}x - u, F(x - \tilde{x})) \leq 0$$

for all $\tilde{x} \in D(\overline{A+B})$ and $u \in \overline{A+B}\tilde{x} - y_0$. As the mapping $\tilde{x} \to \overline{A+B}\tilde{x} - y_0$ is hypermaximal dissipative it follows that $x \in D(\overline{A+B})$ and $L_{y_0}x \in \overline{A+B}x - y_0$.

Then we may take in the inequality (2.14) $x = T_{y_0}(t)\tilde{x}$ and $u = L_{y_0} T_{y_0}(t)\tilde{x} + y_0$ for any $\tilde{x} \in D(L_{y_0})$ and for almost all $t \geq 0$. Taking account that a.e. we have

$$\frac{d}{dt} T_{y_0}(t)\tilde{x} = L_{y_0} T_{y_0}(t)\tilde{x},$$

as in the Proof of Proposition 3 from (2.14) we deduce

$$T_{y_0}(t)x_0 = x_0, \quad \text{for all } t \geq 0$$

Therefore $L_{y_0}x_0 = 0$. Since $x_0 \in D(A) \cap D(B)$ this implies $(A + B)x_0 = y_0$ and Theorem 2 is proved.

REMARKS. 1° By an easy adaptation of the proof, Theorem 2 follows if we merely assume that X is strictly convex and X^* unifoımly convex.

2°. In particular Theorem 2 asserts that $\overline{A+B}x = (A+B)x$ for all $x \in D(A) \cap D(B)$. It then follows from a more generally result of R. T. Rockafeller [18] that if X is a Hilbert space then under hypotheses i) and ii) the operator $A+B$ is demicontinuous on $D(A) \cap D(B)$.

If A and B are linear we have a more precise result (see G. Da Prato [15] [16]). Namely,

COROLLARY 3. - Let X be a reflexive Banach space and A, B be two linear operators from X into itself. Assume

a) A and B are the generators of two linear semigroups of contraction on X

b) There exist two non negative constants ω and μ such that $(\lambda - A)^{-1} D(B) \subset D(B)$ and

$$\|B(\lambda - A)^{-1}x\| \leq \|Bx\|((\lambda - \omega)^{-1} + \mu(\lambda - \omega)^{-2}) \tag{2.16}$$

holds for any $\lambda > \omega$ and $x \in D(B)$.

Then

$$D(B) \subset (\lambda - A - B)(D(A) \cap D(B)) \text{ for all } \lambda > \omega + \mu \tag{2.17}$$

the operator $A + B$ is preclosed and its closure $\overline{A+B}$ is the generator of a C_0 semigroup of contractions on X.

PROOF. - As easily seen Lemma 2.1 remains valid in this case. Let T be the linear contraction semigroup generated by $A+B$ on $D(A) \cap D(B)$ and let L be its generator in X. As in the proof of Theorem 2 it follows that $L \subset \overline{A+B}$. Since L is maximal dissipative (see (19)) this implies $L = \overline{A+B}$ completing the proof.

THEOREM 3. - Let X be a real Banach space and A, B be two hyperdissipative nonlinear mappings from X into itself. Assume that X^* is uniformly convex and

a) For any bounded sequence $\{x_n\} \subset D(A) \cap D(B)$, $\|(A_n + B)x_n\| \leq M < \infty$, implies that $\|Ax_n\|$ is bounded as $n \to \infty$.

b) There exists a subset X_0 of X, $X_0 \supset D(A)$ and a non negative constant ω such that $(\lambda - B)^{-1} X_0 \subset D(A)$ for all $\lambda > \omega$.

Then for al $\lambda > 0$,

$$X_0 \subset (\lambda - A - B)\,(D(A) \cap D(B)) \tag{2.18}$$

and $(\lambda - A - B)^{-1}$ is a well defined lipschitzian mapping from X_0 into X. Moreover for any $y \in X_0$ and $\lambda > 0$ the equation (E) has a unique weak solution $x \in X$.

PROOF. - Consider the approximate equations

$$(\lambda - A_n - B)x = y, \quad y \in X, \quad n = 1, 2. \dots \tag{2.19}$$

which are equivalent to

$$x = (\lambda + n - B)^{-1}(y + n(I - n^{-1}A)^{-1}x) \tag{2.20}$$

Notice that by assumption b) for any $y \in X$ the equation (2.19) (or (2.20)) has a unique solution $x = R_n(\lambda)y \in D(A) \cap D(B)$. It holds moreover

$$\|R_n(\lambda)y - R_n(\lambda)\tilde{y}\| \leq \lambda^{-1}\|y - \tilde{y}\| \tag{2.21}$$

for any y, $\tilde{y} \in X$ and $\lambda > 0$. If x_0 is an arbitrary point of $D(A) \cap D(B)$ from (2.19) one obtains

$$\lambda\|R_n(\lambda)y - x_0\| \leq \|y\| + \|Ax_0\| + \|Bx_0\| + \lambda\,\|x_0\|$$

for any $\lambda > \omega - n$. From the inequality

$$(\lambda - A_n - B)\,R_n(\lambda)\,y = y, \quad n = 1, 2, \dots \tag{2.22}$$

it follows in virtue of the assumption a) that $AR_n(\lambda)y$ is bounded as $n \to \infty$ for each $\lambda > 0$. Since $\|A_n x\| \leq \|Ax\|$ for all $x \in D(A)$ this implies that also $A_nR_n(\lambda)$ and $BR_n(\lambda)y$ are bounded for any $\lambda > 0$ and $y \in X_0$. On the other hand from (2.22) we have

$$\|R_n(\lambda)y - R_m(\lambda)y\|^2 \leq (A_nR_n(\lambda)y - A_mR_m(\lambda)y,\ F(R_n(\lambda)y - R_m(\lambda)y)),\ n, m = 1.$$

Using the dissipativity of A we get

$$\|R_n(\lambda)y - R_m(\lambda)y\|^2 \leq \|A_n R_n(\lambda)y - A_m R_m(\lambda)y\| \, \|F(R_n(\lambda)y - R_m(\lambda)y) -$$

$$- F(J_n R_n(\lambda)y - J_m R_m(\lambda)y)\|$$

Noticing

$$R_n(\lambda)y - J_n R_n(\lambda)y = n^{-1} A_n R_n(\lambda) y, \; n = 1, 2, .$$

it follows from the continuity of the duality mapping F,

$$\|R_n(\lambda)y - R_m(\lambda)y\| \leq \varepsilon(n, m)$$

where $\lim\limits_{n,m\to\infty} \varepsilon(n, m) = 0$. Thus $R(\lambda)y = \lim\limits_{n\to\infty} R_n(\lambda)y$ exists for any $y \in X_0$ and $\lambda > 0$.

As $A_n R_n(\lambda)y$ and $BR_n(\lambda)y$ are bounded, according to Lemma 1.2, $R(\lambda)y \in D(A) \cap D(B)$, $BR_n(\lambda)y \to BR(\lambda)y$ and $A_n R_n(\lambda)y \to AR(y)y$ as $n \to \infty$. Hence

$$(\lambda - A - B) R(\lambda)\tilde{y} = y, \; \tilde{y} \in X_0, \; \lambda > 0.$$

It follows from (2.21)

$$\|R(\lambda)y - R(\lambda)\tilde{y}\| \leq \lambda^{-1} \|y - \tilde{y}\|$$

for any $\tilde{y}$, $y \in X_0$ and $\lambda > 0$. Let us denote again by $R(\lambda)$ the extension of the map $y \to R(\lambda)y$ on the set $\overline{X}_0$. It is easily see that for any $y \in \overline{X}_0$, $x = R(\lambda)y$ is the weak solution of the equation (E). Thus the last part of Theorem 3 follows.

COROLLARY 3. - Let A and B be two hyperdissipative mappings defined in a real Banach space X with the dual X^* uniformly convex If $D(B) \subset D(A)$ and the assumption a) of Theorem 3 is satisfied then the operator $A + B$ is hyperdissipative. If in addition we suppose X uniformly convex and

c) $\qquad \|(A + B)x\| \to \infty$ as as $\|x\| \to \infty$, $x \in D(B)$

then the range $R(A + B)$ of $A + B$ is the whole space X.

PROOF. - Since the hypothesis $b)$ is implied by $D(B) \subset D(A)$, we refer to Theorem 3 to conclude that $R(\lambda - A - B) = X$ for all $\lambda > 0$. To prove the second part consider y an arbitrary point of X and denote by x_λ the solution of the equation

$$(\lambda - A - B)x = y, \; \lambda > 0$$

Let x_0 be an arbitrary point of $D(B)$. In virtue of the dissipativity of $A+B$ we obtain

$$\|x_\lambda - x_0\| \leq \|\lambda x_0\| + \|Ax_0\| + \|Bx_0\| + \|y\|$$

According to c) $\{x_\lambda\}$ is bounded as $\lambda \to 0$. We may assume $y=0$ without loss of generality. Hence we have

$$\lim_{\lambda \to 0} (A+B)x_\lambda = 0 \tag{2.23}$$

Let T be the contraction semigroup generated by $A+B$ on the convex set $\overline{D(B)}$. Using a standard argument we deduce

$$\|T(t)x_\lambda - x_\lambda\| \leq t\|(A+B)x_\lambda\|, \quad \lambda,\, t>0.$$

We may suppose that x_λ is weakly convergent to $x \in X$. Since for any $t>0$, $T(t)x_\lambda - x_\lambda$ is strongly convergent to 0 and X is uniformly convex, according to a well know result of Browder [7] we heve $T(t)x - x = 0$ for all $t>0$. Hence $(A+B)x=0$ which completes the proof.

In particular we have (see [5] and [7]).

COROLLARY 4. - Let A and B be hyperdissipative operators defined in a Banach space X with its dual X^* uniformly convex. Assume that $D(B) \subset D(A)$ and that for every $r>0$ there exist constants $b(r)<1$ and $c(r)>0$ such that

$$\|Ax\| \leq b(r)\|Bx\| + c(r), \text{ for } x \in D(B),\ \|x\| \leq r \tag{2.24}$$

Then the operator $A+B$ is hyperdissipative. Moreover if the condition c) is satisfied then $R(A+B)=X$.

COROLLARY 5. - Let X be a real Banach space with dual space uniformly convex and let A, B be two nonlinear dissipative operators with domain and range in X. Assume that

i) $D(A)=X$, A is demicontinuous and maps bounded sets into bounded sets.

ii) B is hyperdissipative.

Then the operator $A+B$ is hyperdissipative. If in addition the condition c) is verified then $R(A+B)=X$.

Let A be a nonlinear operator with the domain and range in X and let $\delta \geq 1$.

DEFINITION. - A is said to be δ-quasilinear if $D(A)$ is a linear subspace of X and the following two conditions

i) $\|A(u+v)\|^{1/\delta} \leq \|Au\|^{1/\delta} + \|Av\|^{1/\delta}$

ii) $\|A(\lambda u)\| \leq \|Au\| |\lambda|^{\delta}$

hold for any real λ and $u, v \in D(A)$.

THEOREM 4. - Assume that X^* is uniformly convex and let A, B be hyperdissipative operators in X. Suppose that A is δ-quasilinear and that there exist nonnegative constants ω, μ such that $(\lambda - B)^{-1} D(A) \subset D(A)$ and

$$\|A(\lambda - B)^{-1} x\| \leq \|Ax\|((\lambda - \omega)^{-\delta} + \mu(\lambda - \omega)^{-2\delta}) \tag{2.25}$$

hold for any $x \in \overline{D(A)}$ and $\lambda > \omega$.

Then for each $\lambda > \omega + \mu^{1/\delta}$, $(\lambda - A - B)(D(A) \cap D(B)) \supset D(A)$. Moreover for each $\lambda > 0$ and $y \in \overline{D(A)}$ the equation (E) has a unique weak solution $x \in X$.

PROOF. - As above consider the approximate equations

$$(\lambda - A_n - B)x = y, \quad y \in D(A), \quad n = 1, 2, . \tag{2.26}$$

and denote by $R_n(\lambda)y$ the corresponding solutions. It is easy to see that for any $\lambda > \omega$, $R_n(\lambda)y \in D(A) \cap D(B)$. Recalling that $\|A_n R_n(\lambda) y\| \leq \|AR_n(\lambda)y\|$, from (2.25) we get the estimate

$$\|AR_n(\lambda)y\|^{1/\delta} \leq (\|Ay\|^{1/\delta} + n\|AR_n(\lambda)y\|^{1/\delta})((\lambda + n - \omega)^{-1} + \mu^{1/\delta}(\lambda + n - \omega)^{-2}) \tag{2.27}$$

which implies

$$\|AR_n(\lambda)y\| \leq M(\lambda)\|Ay\|, \quad \lambda > \omega + \mu^{1/\delta} \tag{2.28}$$

where $M(\lambda)$ is independent of n. Then following essentially the proof of Theorem 3 one deduces that for any $\lambda > \omega + \mu^{1/\delta}$, $R_n(\lambda)y$ is strongly convergent to $x = R(\lambda)y$ whic satisfies the equation (E) for any $y \in D(A)$ and $\lambda > \omega + \mu^{1/\delta}$. The last statement of Theorem follows by the same argument as in the proof of Corollary 2.

§ 3. - Nonlinear initial value problems.

In this section we consider the evolution equation

$$\frac{du}{dt} - B(t)u(t) - Au(t) = f(t); \quad 0 \leq t \leq T < \infty \tag{3.1}$$

in a real Banach Y of norm $\| \; \|$, where $B(t)$ is a family of linear operators from Y into itself and A belongs to a certain class of nonlinear dissipative operators in Y. Assume that the dual space Y^* is uniformly convex and denote by $(,)$ the pairing between Y and Y^*. About $B(t)$ we make the following assumptions

j) For any $t \in [0, T]$, $B(t)$ is a densely defined closed operator of domain $D(B(t))$. The resolvent of $B(t)$ satisfies

$$\|(\lambda - B(t))^{-1}\| \leq \lambda^{-1}, \ \lambda > 0, \ t \in [0, T] \tag{3.2}$$

jj) $D(B(s)) \subset D(B(t))$ for all $t \geq s$ and

$$\|B(t)x - B(s)x\| \leq M(t-s)\|B(s)x\|, \ \forall x \in D(B) \tag{3.3}$$

where M is a nonnegative constant independent of t and x.

Let $X = L^p(0, T; Y)$ be the space of all Y-valued measurable functions defined on $(0, T)$ normed by

$$\|u\|_p = \left(\int_0^T \|u(t)\|^p \, dt \right)^{1/p} \qquad 1 \leq p < \infty$$

and

$$W^{1,p}(0, T; Y) = \left\{ u \in L^p(0, T; Y) \text{ such that } \frac{du}{dt} \in L^p(0, T; Y) \right\}$$

where $\frac{d}{dt}$ is considered in the sense of Y-valued vectorial distributions on $(0, T)$. Let us introduce the linear operator B defined in X as follows

$$Bu(t) = B(t)u(t) \quad \text{a.e. on } (0, T) \quad \text{for } u \in D(B)$$

where

$$D(B) = \{ u \in L^p(0, T; Y) \text{ such that } u(t) \in D(B(t)) \text{ a.e and } B(t)u(t) \in L^p(0, T; Y) \}$$

It follows (see G. Da Prato [16]) that $D(B)$ is dense in X and j) implies that B generates a linear contraction semigroup on X. Denote by Λ the operator $-\frac{dt}{d}$ with the domain

$$D(\Lambda) = \{ u \in W^{1,p}(0, T; Y), \ u(0) = 0 \}$$

We see easily that for any $\lambda > 0$ and $f \in L^p(0, T; Y)$,

$$(\lambda - \Lambda)^{-1} f(t) = \int_0^t \exp(-\lambda(t-s)) f(s) ds, \qquad t \in [0, T]$$

In particular the preceding formula implies that is the generator of a linear contraction semigroup on X. Finally let L_0 be the operator defined by

$$L_0 u = \Lambda u + Bu, \quad u \in D(B) \cap D(\Lambda). \tag{3.4}$$

According to Corollary 3, L_0 is a densely defined preclosed operator and its closure $L: D(L) \to X$ generates a linear contraction gemigroup on the space X.

In what follows we denote by F the duality mapping between Y and Y^* and by $C(0, T; Y)$ the space of all continuous Y-valued functions defined on $[0, T]$.

LEMMA 3.1. - $D(L) \subset C(0, T; Y)$ and the following inequality

$$\|u(t)\|^p \leq -p \int_0^t (Lu(s), F(u(s)) \|u(s)\|^{p-2} ds, \ t \in [0, T] \tag{3.5}$$

holds for all $u \in D(L)$.

PROOF. - Firstly we assume that $u \in C^1(0, T; Y)$ and $u(0) = 0$. Here $C^1(0, T; Y)$ denotes the space of all Y-valued differentiable functions on $[0, T]$. Thus by Lemma 1.4 we have

$$\|u(t)\|^p = -p \int_0^t \frac{du}{dt}, F(u(s)) \|u(s)\|^{p-2} ds, \quad t \in [0, T] \tag{3.7}$$

Since the mapping $u(t) \to F(u(t)) \|u(t)\|^{p-2}$ is demicontinuous from $L^p(0, T; Y)$ in its dual, the equality (3.7) may be extended oubviously for all $u \in D(\Lambda)$. In particular this implies that $D(\Lambda) \subset C(0, T; Y)$. On the other hand because $B(t)$ is dissipative for any $t \in [0, T]$, from (3.7) we get

$$\|u(t)\|^p \leq -p \int_0^t (L_0 u(s), F(u(s))) \|u(s)\|^{p-2} ds \tag{3.8}$$

for all $u \in D(L_0)$. Hence

$$\|u(t)\|^p \leq p \| \varphi_t L_0 u \|_p \|u\|_p^{p-1}, \quad \forall u \in D(L_0), \ t \in [0, T] \tag{3.9}$$

where φ_t deotes the characteristic function of the interval $[0, t]$ with $0<t<T$.

Let u be a point of $D(L)$. By definition there exists a sequence $\{u_n\}\subset D(L_0)\subset C(0, T; Y)$ such that $u_n\to u$ and $L_0u_n\to Lu$ in the strong topology of $L^p(0, T; Y)$. It follows from (3.9) that $\{u_n(t)\}$ converges uniformly to $u(t)$ on $[0, T]$. In this way it follows that $u\in C(0, T; Y)$ and (3.6) holds for all $u\in D(L)$.

EXEMPLE 1. - Let $\{A(t)\}_{t\in[0,T]}$ be a family of nonlinear operators from Y into itself with $D(A(t))=Y$ for all $t\in[0, T]$. Assume

jjj) For any $t\in[0, T]$, $A(t)$ is demicontinuous and dissipative on the space Y.

jv) For any $u\in Y$ the function $t\to A(t)u$ is continuous and there exist nonnegative constants a, b and α such that

$$\|A(t)u\|\leq a\|u\|^{\alpha+1}+b \tag{3.10}$$

for all $u\in Y$ and $t\in[0, T]$.

Let A be the nonlinear operator defined in $L^p(0, T; Y)$ as follows

$$D(A)=L^{p(1+\alpha)}(0, T; Y),\quad p>1$$

$$Au(t)=A(t)u(t),\qquad \text{a.e. on } (0, T),\ u\in D(A)$$

Let us notice that A is hyperdissipative in $X=L^p(0, T; Y)$. Indeed as we have remarked above (see F. Browder [8]) the condition jjj) together (3.10) imply that $A(t)$ is hyperdissipative for all $t\in[0, T]$. Hence for any $f\in L^p(0, T; Y)$, $u(t)=(\lambda-A(t))^{-1}f(t)$ is well defined for almost all $t\in(0, T)$ and from jv) it follows that it is measurable on $(0, T)$. Since for each t, $(\lambda-A(t))^{-1}$ is lipschitzian, this implies that $u\in L^p(0, T; Y)$. Hence $(\lambda-A)^{-1}$ is well defined on X for all $\lambda>0$ and we have obviously $\|(\lambda-A)^{-1}\|\leq\lambda^{-1}$.

THEOREM 5. - Assume that the hypotheses j) ~ jv) are satisfied. Then for any $f\in L^p(0, T; Y)$, $p>1$ and $\lambda>0$ the evolution equation

$$\begin{cases}\lambda u(t)+\dfrac{du}{dt}-B(t)u(t)+A(t)u(t)=f(t),\ 0\leq t\leq T\\ u(0)=0\end{cases}\tag{3.11}$$

has a unique weak solution $u\in L^p(0, \mathrm{T}; Y)$ i.e. there exists a sequence $\{u_n\}\subset W^{1,p}(0, T; Y)\cap D(B)$, $u_n(0)=0$ such that $u_n\to u$ and

$$\lambda u^n(t)+\frac{du_n}{dt}-B(t)u_n(t)-Au_n(t)\to f(t)$$

in the strong topology of $L^p(0, T; Y)$. Moreover the solution $u(t)$ is continuous on $[0, T]$, $u(0)=0$ and the mapping $f \to u$ is lipschitzian on the space $L^p(0, T; Y)$.

PROOF. - We shall verify the hypotheses of Theorem 3 where $X = L^p(0, T; Y)$, $X_0 = D(A)$ and $B = L$. Since $D(L) \subset C(0, T; Y)$ the condition jjj) implies $D(L) \subset D(A)$. In order to verify a) consider a bounded sequence $\{u_n\} \subset D(L)$ such that $\|(L + A_n) u_n\| \leq M < \infty$ as $n \to \infty$. We may assume $A0 = 0$ without loos of generality. Then taking account that A is dissipative we get from (3.6)

$$\|u_n(t)\|^p \leq p\|(L + A_n)u_n\|_p \|u_n\|_p^{p-1}, \quad 0 \leq t \leq T, \ n = 1, 2,. \tag{3.12}$$

Combining jjj) and (3.12) it follows that $\|Au_n\|_p$ is bounded and so the condition a) of Theorem 3 is verified. Therefore for any $f \in L_p(0, T; Y)$ and $\lambda > 0$ the equation

$$\lambda u - Lu - Au = f \tag{3.13}$$

has a unique weak solution $u \in L^p(0, T; Y)$, i.e. there exists $\{u_n\} \subset D(L)$ such that $u_n \to u$ and $u_n - Lu_n - Au_n \to f$ in $L^p(0, T; Y)$. Using again Lemma 3.1 and the dissipativity of A we get

$$\|u_n(t) - u_m(t)\|^p \leq p\|(A + L)u_n - (A + L)u_m\|_p \|u_n - u_m\|_p^{p-1}, \ m, n = 1, 2 \ldots \tag{3.14}$$

Therefore $\{u_n(t)\}$ converges uniformly to $u(t)$ which implies $u \in C(0, T; Y)$ and $u(0) = 0$. In particular (3.14) implies $\|Au_n\|_p \leq M < \infty$. According to Lemma (1.2) it follows that $u \in D(A) \cap D(L)$, $Au_n \rightharpoonup Au$ and $Lu_n \rightharpoonup Lu$ as $n \to \infty$. This shows that u is an ordinary solution of equation (3.13) and the conclusi. sion of Theorem 5 follows from the definition of the operator L repeating the above argument.

EXEMPLE 2. - We take $Y = L^q(\Omega)$ where Ω is an open bounded subset of R^n and $q \geq 1$. Let α be a nonnegative number and let A be the operator defined by

$$Au(t, \cdot) = \gamma\,|u(t, \cdot)|^\alpha u(t, \cdot), \quad \text{a.e. on } (0, T) \tag{3.15}$$

for all $u \in D(A) = L^{p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega))$. Here γ is a nonnegative constant. In what follows we denote by $\|u\|_{p,q}$ the norm

$$\|u\|_{p,q} = \left(\int^T dt\left(\int_\Omega |u(t, x)|^q\, dx\right)^{p/q}\right)^{1/p}, \quad u \in L^p(0, T; L^q(\Omega))$$

Since the duality mapping of $L^p(0, T; Y)$ into its dual is given by $K(u) = u(t, x) \, |u(t, x)|^{q-2} \|u(t, \cdot)\|^{p-q}_{L_p(\Omega)} \|u\|^{2-p}_{p,q}$ it is easily seen that $-A$ is dissipative.

Let us show that if $p, q \geq (\alpha+2)(\alpha+1)^{-1}$ then $-A$ is hyperdissipative in $L^p(0, T; Y)$. Denote by A_λ the operator $\lambda u + Au$. Wee see easily that for any $\lambda > 0$, A_λ is monotone, demicontinuous and coercive from $L^{\alpha+2}(0, T; L^{\alpha+2}(\Omega))$ into its dual $L^{\alpha+2/\alpha+1}(0, T; L^{\alpha+2/\alpha+1}(\Omega))$. Thus according to a well known result (see F. Browder [9]), $A\lambda$ is surjective. Hence for any $f \in L^p(0, T; L^q(\Omega)) \subset L^{\alpha+2/\alpha+1}(0, T; L^{\alpha+2/\alpha+1}(\Omega))$ the equation

$$\lambda u + Au = f$$

has a unique solution $u \in L^{\alpha+2}(0, T; L^{\alpha+2}(\Omega))$. But actually $u \in L^p(0, T; L^q(\Omega))$ because a.e. on $(0, T) \times (\Omega)$ we have $|u(t, x)| \leq |f(t, x)|/\lambda + \gamma |u(t, x)|^\alpha$

Let $\{B(t)\}$ be a set of linear operators in $L^q(\Omega)$ satisfyng to assumptions j) and jj). Denote by $\tilde{B}(t)$ the restrictions of $B(t)$ at the space $L^{p(1+\alpha)}(\Omega)$. Assume

jjj') For any $t \in [0, T]$, $\tilde{B}(t)$ is the generator of a linear semigroup of contractions on $L^{q(1+\alpha)}(\Omega)$ satisfyng to condition jj) in this space.

Then as above we may define $\tilde{B}u(t) = \tilde{B}(t)u(t)$ a.e. on $[0, T]$ with $D(\tilde{B}) = \{u;\ u \in L^{p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega)),\ u(t) \in D(\tilde{B}(t))$ a.e. and $\tilde{B}(t)u(t) \in L^{p(1+\alpha)}(0, T; L^{q(1+\alpha}(\Omega)\}$. Similarly we consider

$$\tilde{L}_0 u = -\frac{du}{dt} + \tilde{B}u, \quad u \in D(L_0)$$

where $D(\tilde{L}_0) = D(\tilde{B}) \cap \{u \in W^{1,p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega));\ u(0) = 0\}$ and $\frac{d}{dt}$ is taken in the sense of $L^{q(1+\alpha)}(\Omega)$-valued distributions on $(0, T)$. Let $\tilde{L}$ be the closure of $\tilde{L}_0$ in the space $L^{p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega))$. It follows easily that $\tilde{L} \subset L$.

In particular this implies

$$(\lambda - L)^{-1} L^{p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega)) \subset L^{p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega))$$

and

$$\|(\lambda - L)^{-1} f\|_{p(1+\alpha),\, q(1+\alpha)} \leq \lambda^{-1} \|f\|_{p(1+\alpha),\, q(1+\alpha)} \tag{3.16}$$

for any $\lambda > 0$ and $f \in L^{p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega))$. Noticing

$$\|Af\|_{p,q} = \|f\|^{\alpha+1}_{p(1+\alpha),\, q(1+\alpha)}$$

it follows from (3.16) that $\|A(\lambda - L)^{-1} f\|_{L^p(0,T;Y)} \leq \lambda^{-(1+\alpha)} \|Af\|_{L^p(0,Y;T)}$.

Since A is obviously $\alpha + 1$-quasilinear we reffer to Theorem 4 to deduce that for any $f \in L^p(0, T; L^q(\Omega))$ and $\lambda > 0$ there exists a sequence $\{u_n\} \subset D(L) \cap L^{p(1+\alpha)}(0, T; L^{q(1+\alpha)}(\Omega))$ such that $u_n \to u$ and

$$\lambda u_n(t, \cdot) - Lu_n(t, \cdot) - \gamma |u_n(t, \cdot)|^\alpha u_n(t, \cdot) \to f(t, \cdot)$$

in $L^p(0, T; L^q(\Omega))$-norm. As in Exemple 1 it follows that $\{u_n(t)\}$ is uniformly convergent to $u(t)$ on $[0, T]$ which implies that $u \in C(0, T; L^q(\Omega))$ and $u(0) = 0$. In this sense $u(t)$ may be considered as a weak solution of the equation

$$(3.17) \qquad \begin{cases} \lambda u(t, \cdot) + \dfrac{du(t, \cdot)}{dt} - B(t) u(t, \cdot) + \gamma |u(t, \cdot)|^\alpha u(t, \cdot) = f(t, \cdot), \ 0 \leq t \in T \\ u(0, \cdot) = 0 \end{cases}$$

Summarising all there results we get

THEOREM 6. - Let $\{B(t)\}_{t \in [0, T]}$ be a family of linear operators in $L^q(\Omega)$ satisfyng the assumptions j), jj) and jjj'). Suppose $p, q \geq (\alpha + 2)(\alpha + 1)^{-1}$. Then for each $\lambda > 0$ and $f \in L^p(0, T; L^q(\Omega))$ the equation (3.17) has a unique solution $u \in L^q(0, T; L^q(\Omega))$. Moreover u belongs to $C(0\ T; L^q(\Omega))$ and $u(0) = 0$.

REMARKS 1°. - In particular the hypotheses of Theorem 6 are satisfied for $\{B(t)\}$ defined as follows (see G. Da Prato [16]).

Suppose that the boundary of the open domain Ω is sufficiently smooth and denote by $b(t, u, v)$ the bilinear form

$$b(t, u, v) = -\sum_{i,j=i}^{n} \int_\Omega b_{ij}(x, t)\, \partial u / \partial x_i\, \partial v / \partial x_j\, dx$$

Assume that there exists $\mu > 0$ such that $\mu^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} b_{ij}(x, t) \xi_i \xi_j \leq \mu |\xi|^2$ and consider V a closed subspace of $H^1(\Omega)$ such tht $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ and the following conditions are satisfied

I) If $u, v \in C(\bar{\Omega}) \cap V$ then $|u| \in V$ and $uv \in V$

II) There exist $\delta \in R$ and $K \in R_+$ such that $b(u, u) + \|u\|_2^2 \geq K \|u\|_v^2$.

Then the variational problem

$$(u, v) - b(t, u, v) := (f, v), \ v \in V$$

has a unique solution $u = B(t) f \in L^q(\Omega)$ for any $f \in L^q(\Omega)$ $q > 1$. Moreover the operator $B(t)$ is for every $t \in [0, T]$ the generator of a C_0 contraction semigroup on $L^q(\Omega)$ (for the proof see the paper above mentioned).

Thus under suitable regularity assumption about $b_{ij}(t, x)$ the condition jj) yelds on every $L^q(\Omega)$ with $q > 1$. Hence the hypotheses of Theorem 6 are satisfied;

2°. - For the applicability of Theorem 3 and 4 we must notice that the duality mapping of $L^p(0, T; Y)$ into its dual space is uniformly continuous on every bounded set of $L^p(0, T; Y)$. Indeed this mapping may be expressed in the following form

$$Ku(t) = F(u(t))\|u(t)\|^{p-2}\|u\|_p^{2-p}$$

where F is the duality mepping of Y into Y^*. Since Y^* is uniformly convex F is continuous uniformly on the bounded set of Y which implies the continuity of K.

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