Spaces of germs and jets.

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Summary. - *In this paper we present some results on the theory of jets and germs concerning the paralel transport and the linear structure.*

0. - Let M be a C^∞ manifold and $E \xrightarrow{p} M$ a real, or complex, vector bundle over M.

Let $T_{x_0}(M)$ denote the space of tangent vectors to M in the point $x_0 \in M$.

Let $s \in C^\infty(M, E)$ a section. I denote by $j^k_{x_0}s$ the k-jet of s in the point x_0, and by $g_{x_0}s$ the germ of this section.

Then $J^k(E) = \bigcup_{\substack{x_0 \in M \\ s \in C^\infty(M, E)}} j^k_{x_0}s$ admits in a natural way a C^∞-vector bundle-structure.

I introduce in the space $\mathcal{G}(E) = \bigcup_{\substack{x_0 \in M \\ s \in C^\infty(M, E)}} g_{x_0}s$ a structure of vector bundle with fiber, a locally-convex space. I suppose that M has a Riemannian structure Γ and let L be a linear connection in E.

For $0 < r$, sufficiently small, let $B(r, x_0) \subset M$ be the ball centered in the point $x_0 \in M$ with radius r. The space $C^\infty(B(r, x_0), E)$ is a Fréchét-space. Then $\mathcal{G}_{x_0}(E) = \lim_{r \to 0} C^\infty(B(r, x_0), E)$ is a locally-convex space; this is the fiber of the space $\mathcal{G}(E)$. Let $\pi^k : \mathcal{G}_{x_0}(E) \to J^k_x(E)$ be the natural epimorphism.

We will prove in this paper:

THEOREM 1.

If M is a Riemannian manifold and $E \xrightarrow{p} M$ is a vector bundle with a linear connection L, then there exists a linear connection $\mathcal{L}$ in the space of germs of sections $\mathcal{G}(E)$ which prolongates L. The connection $\mathcal{L}$ is compatible with the connection in $J^k(E)$ [5].

THEOREM 2.

Let M and N be Riemannian manifolds. Let $\mathcal{G}(M, N)$ be the space of germs of maps from M to N, and $J^k(M, N)$ the space of the k-jets of maps.

Then $\mathcal{G}(M, N)$ and $J^k(M, N)$ admit a *vector* bundle structure on $M \times N$.

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COROLLARY 2.1.

$$J^k(M, N) \simeq [S^1(T^*M) \oplus S^2(T^*M) \oplus \ldots \oplus S^k(T^*M)] \otimes TN$$

($S^l(T^*M)$ denotes the l-simetric tensorial product).

1. - We will represent the germs and the k-jets in a different manner.

Let $\gamma : I \to M$ be a C^∞-path in M, $\gamma(0) = x_0$, $\gamma(1) = x_1$.

The connections Γ and L determine two isomorphisms:

$$\mathcal{T}^\Gamma(\gamma) : T_{x_0}(M) \to T_{x_1}(M)$$

$$\mathcal{T}^L(\gamma) : E_{x_0} \to E_{x_1}.$$

If $s \in \mathcal{G}_{x_0}(M, E)$, we associate an element $\tilde{s} \in \mathcal{G}_{0x_0}(T_{x0}(M), E_{x_0})$ in the following manner:

$$\tilde{s}(\theta) = (\mathcal{T}^L(\gamma^\Gamma(\theta)))(s(\exp_{x_0}\theta)), \quad \text{for}$$

$\theta \in T_{x_0}(M)$, and $\gamma^\Gamma(\theta)$ is the geodesic from $\exp_{x_0}(\theta)$ to x_0.

The element $\tilde{s}$ is well defined because exp is a local diffeomorphism.

The application $\sim : s \to \tilde{s}$ is an isomorphism.

From this moment we will identify the germs s with $\tilde{s}$.

Now we can define what we mean by a C^∞-section in $\mathcal{G}(E)$.

Let ω be the projection $\omega : T(M) \to M$, and $U \subset M$ a open set in M.

Let V, $W \subset T(M)|_u$ be two open neighborhoods of the zero-section in $T(M)$ over U.

If $\Sigma_v \in C^\infty(V, \omega^*E)$, then $g(\Sigma_v)|_{T_x(M)}$, $(x \in U)$ is a germ in $\mathcal{G}_{x_0}(T_x, E_x)$.

We say that $\Sigma \in \Gamma(U, \mathcal{G}(M, E))$ is a C^∞-section on U with values in $\mathcal{G}(M, E)$, if there exists an open set V as above, and $\Sigma_v \in C^\infty(V, \omega^*E)$ such that

$$\Sigma(x) = g_{0_x}(\Sigma_v|_{T_x}).$$

Clearly, two C^∞ sections Σ_V, Σ_W determine the same section in $\mathcal{G}(M, E)$ over U, if and only if, there exists an open set R as V and W, $R \subset V \cap W$, such that $(E_V)|_R = (\Sigma_W)|_R$.

Now, we can define the paralel transport $\mathcal{T}^\mathfrak{L}$ in $\mathcal{G}(M, E)$ and the corresponding covariant derivative $\nabla^\mathfrak{L}$.

If $\gamma : I \to M$ is a C^∞-path in M, then $\mathcal{T}^\mathfrak{L}_{(\gamma)} : \mathcal{G}_{\gamma(0)}(M, E) \to \mathcal{G}_{\gamma(1)}(M, E)$ is defined in the following manner:

(1) $$(\mathcal{T}^\mathfrak{L}_{(\gamma)}\tilde{s})(\theta) = \mathcal{T}^L(\gamma)\{\tilde{s}[\mathcal{T}^\Gamma(\gamma^{-1})(\theta)]\}, \quad \theta \in T_{\gamma(1)}(M), \ s \in \mathcal{G}_{\gamma(0)}(M, E).$$

If X is a C^∞-section in $T(M)$ and $\Sigma \in C^\infty(M, \mathcal{G}(M, E))$, we will define $\nabla^{\mathcal{L}}_X\Sigma$. Let $x \in M$, and $\gamma : I \to M$, $\gamma(0) = x$, $\dot\gamma(0) = X(x)$.

Then, by definition:

$$(2) \qquad (\nabla^{\mathcal{L}}_X\Sigma)(x) = \frac{d}{dt}\left\{[\mathcal{T}^{\mathcal{L}}(\gamma(t)^{-1})](\Sigma(\gamma(t))\right\}_{t=0}.$$

We will prove that the definition (2) is correct. To verify this afformation, we will calculate in a local chart.

Let U be a local chart of coordinates, $(x^1, \dots, x^n)$ in M and $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$ a local chart of coordinates in $T(M)$ $\left(\text{if } t \in T(M),\ t = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\right)$.

By (2), and (1), for $\theta \in T_x(M)$, we have:

$$(3) \qquad [(\nabla^{\mathcal{L}}_X\Sigma)(x)](\theta) = \frac{d}{dt}\left\{\mathcal{T}^L(\gamma(t)^{-1}[\Sigma(\mathcal{T}^\Gamma(\gamma(t))(\theta))]\right\}_{t=0}.$$

We suppose that $E|_u \simeq U \times \mathbb{R}^m$ (or $E|_u \simeq U \times \mathbb{C}^m$), and we identify $E|_u$, via this isomorphism, with $U \times \mathbb{R}^m$ (or $U \times \mathbb{C}^m$).

Then $\Sigma(\mathcal{T}^\Gamma(\gamma(t))(\theta)) = s(t, \theta)$ is a C^∞-section on $\gamma(t)$ with values in $\mathbb{R}^m$ (or $\mathbb{C}^m$).

By (3)

$$[(\nabla^{\mathcal{L}}_X)(x)](\theta) = \frac{d}{dt}\left\{\mathcal{T}^L(\gamma(t)^{-1}s(t, \theta)\right\}_{t=0} =$$

$$= \nabla^L_X s(t, \theta) =$$

$$= (X\, s(t, \theta))_x + L_u(X)(s(t, \theta))_{t=0},$$

where L_u is a linear, omogen, operator determined by L and U.

Let $X(x) = \xi^1 \frac{\partial}{\partial x^1} + \dots + \xi^n \frac{\partial}{\partial x^n}$, $\tilde\theta(t) = \mathcal{T}^\Gamma(\gamma(t))(\theta)$; then $s(t, \theta) = \Sigma(\gamma(t), \tilde\theta(t))$.

We have:

$$(\nabla^{\mathcal{L}}_X\Sigma)(x)(\theta) = (X\,\Sigma(\gamma(t), \tilde\theta(t)))_x + L_u(X)(\Sigma(x, \theta)) =$$

$$(4) \qquad = \frac{\partial\Sigma}{\partial x^1}\xi^1 + \dots + \frac{\partial\Sigma}{\partial x^n}\xi^n + \frac{\partial\Sigma}{\partial\tilde\theta^1}\left(\frac{d\tilde\theta^1}{dt}\right)_{t=0} + \dots \frac{d\Sigma}{\partial\theta^n}\left(\frac{d\tilde\theta^n}{dt}\right)_{t=0} +$$

$$+ L_u(x)(\Sigma(x, \theta)) = \sum_{i=1}^{n} \frac{\partial\Sigma}{\partial x^i}\xi^i + \sum_{\substack{i,\, k=1 \dots n \\ j=1 \dots m}}^{n} \frac{\partial\Sigma}{\partial\theta^i}\Gamma^i_{jk}\theta^j\xi^k + L_u(x)(\Sigma(x, \theta))$$

Γ^i_{jk} being the connection coefficients of the Riemannian connection in the chart U.

The last formula allows us to conclude that $\nabla^{\mathcal{L}}_X$ is well defined.

We must verify the relations:

i°. $\nabla^{\mathfrak{L}}_{fX+Y} = f\,\Delta^{\mathfrak{L}}_X + \nabla^{\mathfrak{L}}_Y \qquad f \in C^\infty(M),\ X,\ Y \in C^\infty(M,\ T(M))$

ii°. $\nabla^{\mathfrak{L}}_X(\Sigma_1 + \Sigma_2) = \nabla^{\mathfrak{L}}_X\Sigma_1 + \nabla^{\mathfrak{L}}_X\Sigma_2\,,\ \Sigma_1\,,\ \Sigma_2 \in C^\infty(M,\ \mathcal{G}(M,\ E))$

iii°. $\nabla^{\mathfrak{L}}_X(f\,\Sigma_1) = (Xf)\Sigma_1 + f\,\nabla_X\Sigma_1\,.$

But these properties of V_X follows easy from (4).

This proof allows us to affirme that the assertion of the Theorem 1 relative to $\mathcal{G}(M,\ E)$ is established.

We have considered the projectiou $\pi^k : \mathcal{G}(M,\ E) \to J^k(M,\ E)$. But we can constract a smoth monomorphism:

$$i^k : J^k(M,\ E) \to \mathcal{G}(M,\ E)$$

such that $\pi^k i^k = 1$.

If $\alpha \in J^k_{x_0}(M,\ E)$, let $\tilde{s} \in \mathcal{G}_{x_0}(M,\ E)$ such that $j^k_{x_0}\tilde{s} = \alpha$.

Then, by definition:

$i^k\alpha =$ the k-order polinomial approximation of $\tilde{s}$ in $\theta_x \in T(x)$ by TAYLOR formula.

It is easy to verify that i^k is well defined.

From this moment a k-jet in E is a k-order polinomial on $T_x(M)$ with values in E_x.

It is easy to see that $\mathcal{T}^{\mathfrak{L}}$ *invariantes* $J^k(M,\ E)$ and the formula (4) proves that $\nabla^{\mathfrak{L}}$ induces an operator ∇^k on $J^k(M,\ E)$.

The proof the Theorem 1 is complete.

2. - It is known that $\mathcal{G}(M,\ N)$ is a bundle on $M \times N$, [1].

It remains only to define a linear structure on $\mathcal{G}_{(x,\,y)}(M,\ N)$.

Let $s \in \mathcal{G}_{(x,\,y)}(M,\ N)$; we define $\tilde{s} \in \mathcal{G}_{(0_x,\,0_y)}(T_x(M),\ T_y(M))$

$$\tilde{s}(\theta) = \exp_y^{-1}[s(\exp_x(\theta))].$$

The function $\sim : s \to \tilde{s}$,

$$\sim\, : \mathcal{G}_{(x,\,y)}(M,\ N) \to \mathcal{G}_{(0_x,\,0_y)}(T_x(M),\ T_y(N))$$

is clearly bijective.

$\mathcal{G}_{(0_x,\,0_y)}(T_x(M),\ T_y(N))$ has a natural structure of real vector space. Then we can introduce a real vector structure in $\mathcal{G}_{(x,\,y)}(M,\ N)$ by $\sim$. From this moment we will identify $\mathcal{G}_{(x,\,y)}(M,\ N)$ with $\mathcal{G}_{(0_x,\,0_y)}(T_x(M),\ T_y(N))$.

If $\alpha \in J^k_{(x,y)}(M, N)$, then let $\tilde{s} \in \mathcal{G}_{(0_x, 0_y)}(T_x(M), T_y(N))$ be an element such that $j^k\tilde{s} = \alpha$.

We construct the monomorphism

$$i^k : J^k(M, N) \to \mathcal{G}(M, N)$$

by the definition:

$i^k\alpha =$ the k-order polinomial approximation of $\tilde{s}$ in 0_x (by TAYLOR formula).

It is an easy consequence from the definition that $j^k i^k = 1$, and that the operations of the linear structure in $\mathcal{G}(M, N)$ invariantes $i^k J^k(M, N) \subset (M, N)$. We must verify that operations of the linear structure in $\mathcal{G}(M, N)$ and $J^k(M, N)$ are continuous and smooht. This is standard.

The theorem 2 is proved.

As a corollary, we can deduce the structure of $J^k(M, N)$.

If V and W are two real vectos spaces, then the space of the k-order polinomials P on V, with values in W, such that $P(0) = 0$, is naturally isomorphic to

$$\left(\sum_{l=1}^{k} S^l V^* \right) \otimes W.$$

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