Integrals involving E-functions and Kampe de Feriet's functions of higher order.

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Summary. - *An integral involving the product of three generalised hypergeometric functions is evaluated in terms of Kampe de Feriet's functions of higher order.*

Infinite integrals involving products of 3, 4, 5 and six Bessel functions are deduced as particular cases.

1. - Introduction.

The main theorem to be proved is:

$$\int_0^\infty x^{k-1}\,{}_\tau F_\varepsilon\left[\begin{matrix}\beta_1, \dots, \beta_\tau; -\dfrac{x}{a}\\ \delta_1, \dots, \delta_\varepsilon\end{matrix}\right]{}_\tau F_\varepsilon\left[\begin{matrix}\beta_1', \dots, \beta_\tau'; -\dfrac{x}{b}\\ \delta_1', \dots, \delta_\varepsilon'\end{matrix}\right]\times$$

$$\times \sum_{i,-i}\frac{1}{i}E\left(\begin{matrix}\alpha_1, \dots \alpha_\mu, \ 1 : e^{i\pi}cx\\ \gamma_1, \dots, \gamma_\lambda\end{matrix}\right)dx$$

$$= \frac{2\pi}{c^k}\frac{\prod\limits_{r=1}^{\mu}\Gamma(\alpha_r+k)}{\prod\limits_{t=1}^{\lambda}\Gamma(\gamma_t+k)}F\left[\begin{matrix}\mu\\ \tau\\ \lambda\\ \varepsilon\end{matrix}\left|\begin{matrix}\alpha_1+k, \dots, \alpha_\mu+k\\ \beta_1, \beta_1'; \dots, \beta_\tau, \beta_\tau'\\ \gamma_1+k, \dots, \gamma_\lambda+k\\ \delta_1, \delta_1', \dots, \delta_\varepsilon, \delta_\varepsilon'\end{matrix}\right| -\frac{1}{ac}, -\frac{1}{bc}\right], \tag{1}$$

where $\tau \leq \varepsilon+1$, $\mu > \lambda+1$, $\mu+\tau \leq \lambda+\varepsilon+1$, $R(\alpha_r+k) > 0$ $(r=1, 2, \dots, \mu)$, $R(\beta_j+\beta_j'-k) > 0$ $(j=1, 2, \dots, \tau)$ and $a, b, c > 0$.

The function appearing on the right hand side of (1) is Kampe de Feriet's function of higher order (in more parameters) in two variables whese properties are given in [1] pp. 401 and 489. This function is defined as:

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$$(2)\quad F\left[\begin{array}{c|c|c} \mu & \alpha_1, \ldots, \alpha_\mu & \\ \tau & \beta_1, \beta_1'; \ldots; \beta_\tau, \beta_\tau' & \\ \lambda & \gamma_1, \ldots, \lambda_\lambda & x, y \\ \varepsilon & \delta_1, \delta_1'; \ldots, \delta_\varepsilon \delta_\varepsilon' & \end{array}\right] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\alpha_j, m+n) \prod_{j=1}^{\tau} \{(\beta_j, m)(\beta_j', n)\}}{\prod_{j=1}^{\lambda} (\gamma_j, m+n) \prod_{j=1}^{\varepsilon} \{(\delta_j, m)(\delta_j', n)\}} \times$$

$$\times \frac{x^m y^n}{(1; m)(1; n)}$$

where $\mu + \tau \leq \lambda + \varepsilon + 1$ and neither of the quantities γ, δ, δ' is a negative integer.

By specialising the values of μ, τ, λ, ε; the function (2) reduces to the four functions of P. Appel F_1, F_2, F_3, F_4. Thus we have (see [2] p. 151):

$$(3)\quad F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta_1, \beta_1' & \\ 1 & \gamma & x, y \\ 0 & \ldots & \end{array}\right] = F_1[\alpha; \beta_1, \beta_1'; \gamma; x, y],$$

$$(4)\quad F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & \beta_1, \beta_1' & \\ 0 & \ldots \ \ldots & x, y \\ 1 & \delta, \ \delta' & \end{array}\right] = F_2[\alpha; \beta_1, \beta_1'; \delta, \delta'; x, y],$$

$$(5)\quad F\left[\begin{array}{c|c|c} 0 & \ldots \ \ldots \ \ldots \ \ldots & \\ 2 & \beta_1, \beta_1', \beta_2, \beta_2' & \\ 1 & \gamma & x, y \\ 0 & \ldots \ \ldots \ \ldots \ \ldots & \end{array}\right] = F_3[; \beta_1, \beta_1'; \beta_2, \beta_2'; \gamma; x, y]$$

$$(6)\quad F\left[\begin{array}{c|c|c} 2 & \alpha_1, \alpha_2 & \\ 0 & \ldots \ \ldots & \\ 0 & \ldots \ \ldots & x, y \\ 1 & \delta_1, \delta_1' & \end{array}\right] = F_4[\alpha_1, \alpha_2; \delta_1, \delta_1'; x, y]$$

Also we have [2] p. 151:

$$F\left[\begin{array}{c|c|c}\mu & \alpha_1, \dots, \alpha_\mu & \\ 0 & \dots\ \dots\ \dots & \\ \lambda & \gamma_1, \dots, \gamma_\lambda & x, y \\ 0 & \dots\ \dots\ \dots & \end{array}\right] = F\begin{pmatrix}\alpha_1, \dots, \alpha_\mu ; x+y \\ \gamma_1, \dots, \gamma_\lambda\end{pmatrix}; \tag{7}$$

$$F\left[\begin{array}{c|c|c}0 & \dots\ \dots\ \dots\ \dots\ \dots & \\ \tau & \beta_1, \beta_1'; \dots, \beta_\tau, \beta_\tau' & \\ 0 & \dots\ \dots\ \dots\ \dots\ \dots & x, y \\ \varepsilon & \delta_1, \delta_1'; \dots; \delta_\varepsilon, \delta_\varepsilon' & \end{array}\right] = F\begin{pmatrix}\beta_1, \dots, \beta_\tau; x \\ \delta_1, \dots, \delta_\varepsilon\end{pmatrix} F\begin{pmatrix}\beta_1', \dots, \beta_\tau'; y \\ \delta_1', \dots, \delta_\varepsilon'\end{pmatrix} \tag{8}$$

$$F\left[\begin{array}{c|c|c}\omega & \alpha_1, \dots, \alpha_\omega & \\ 1 & \beta, \beta' & \\ \omega & \gamma_1, \dots, \gamma_\lambda & x, x \\ 0 & \dots\ \dots & \end{array}\right] = F\begin{pmatrix}\alpha_1, \dots, \alpha_\omega, \beta+\beta'; x \\ \gamma_1, \dots, \gamma_\omega\end{pmatrix}. \tag{9}$$

Again Kampé de Feriet's function is expressed as a double complex integral in the form (see [2] p. 155).

$$F\left[\begin{array}{c|c|c}\mu & \alpha_1, \dots, \alpha_\mu & \\ \tau & \beta_1, \beta_1'; \dots, \beta_\tau, \beta_\tau' & \\ \lambda & \gamma_1, \dots, \gamma_\lambda & x, y \\ \varepsilon & \delta_1, \delta_1'; \dots, \delta_\varepsilon, \delta_\varepsilon' & \end{array}\right] = \frac{\prod_{j=1}^{\lambda} \Gamma(\gamma_j) \prod_{j=1}^{\varepsilon} \{\Gamma(\delta_j)\Gamma(\delta_j')\}}{\prod_{j=1}^{\mu} \Gamma(\alpha_\mu) \prod_{j=1}^{\tau} \{\Gamma(\alpha_\mu)\Gamma(\beta_j')\}} \times \tag{10}$$

$$\times \frac{1}{(2\pi i)^2} \int\int \frac{\Gamma(s)\Gamma(t) \prod_{j=1}^{\mu} \Gamma(\alpha_j - s - t) \prod_{j=1}^{\tau} \{\Gamma(\beta_j - s)\Gamma(\beta_j' - t)\}}{\prod_{j=1}^{\lambda} \Gamma(\gamma_j - s - t) \prod_{j=1}^{\varepsilon} \{\Gamma(\delta_j - s)\Gamma(\delta_j' - t)\}} \times$$

$$\times (-x)^{-s}(-y)^{-t}\, ds\, dt,$$

where the contours are of Barne's type and are curved (if necessary) to separate the increasing sequences of poles from the decreasing sequences of poles.

The function appearing in the left and side of (1) is Mac Robert E-function whose definitions and properties are given in [3], pp. 348-358. A brief account of the E-functions is given in [4] p. 393. Section 2 contains a treatment of the E-function and preliminary results and section [3] contains the derivation of our main theorem. Section 4 contains the derivation of infinite integrals involving the products of three, four, five and six Bessel functions; as particular cases.

The following formulae are required in the proofs:

The Mellin transform pair ([5], p. 7)

(11) $$g(s) = \int_0^\infty x^{s-1} f(x)\, dx,$$

(12) $$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} g(s)\, ds,$$

and ([3], p. 347)

(13) $$I_n(x) = \frac{1}{\Gamma(n+1)} \left(\frac{1}{2}x\right)^n e^{-x} {}_1F_1\left(\begin{matrix} n+1;\ 2x \\ 2n+1 \end{matrix}\right),$$

where $I_n(x)$ is the modifiec Bessel function of the first kind.

2. - Properties of the E-function: If $p \leq q$, then the E-function is defined as:

(14) $$E(p;\ \alpha_r:\ q,\ \lambda_s:\ z) = \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)}{\Gamma(\lambda_1) \ldots \Gamma(\lambda_q)} F \left(\begin{matrix} \alpha_1, \ldots, \alpha_p; -\frac{1}{z} \\ \lambda_1, \ldots, \lambda_q \end{matrix}\right),$$

when $p \geq q+1$, $|\arg z| < \pi$, then the E-function (see [3], p. 353) can be shown to be

(15) $$E(p;\ \alpha_r:\ q;\ \lambda_s: z) = \sum_{r=1}^{p} \prod_{s=1}^{p}{}' \Gamma(\alpha_s - \alpha_r) \{ \prod_{t=1}^{q} \Gamma(\lambda_t - \alpha_r)\}^{-1} \Gamma(\alpha_r) z^{\alpha_r} \times$$
$$\times {}_{q+1}F_{p-1}\left(\begin{matrix} \alpha_r,\ \alpha_r - \lambda_1 + 1, \ldots, \alpha_r - \lambda_q + 1;\ (-)^{p-q} z \\ \alpha_r - \alpha_1 + 1, \ldots \ldots \alpha_r - \alpha_p + 1 \end{matrix}\right),$$

where the dash in the product means that the factor $\alpha_r - \alpha_r$ is ommitted and the asterisk means that the term $\alpha_r - \alpha_r + 1$ is also ommitted. Also ([3] p. 37)4 the following formula is required:

(16) $$E(p;\ \alpha_r:\ q;\ \lambda_s: z) = \frac{1}{2\pi i} \int \Gamma(\Theta) \frac{\prod_{r=1}^{p} \Gamma(\alpha_r - \Theta)}{\prod_{t=1}^{q} \Gamma(\lambda_t - \Theta)} z^{\Theta}\, d\Theta,$$

where the integral is taken along the η-axis with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $\alpha_1, \alpha_2, \dots, \alpha_p$ lie to the right of the conteur. Zero and negative values of the α's and λ's are excluded. When $p < q+1$ the contour is bent to the left at both ends. Convergence is secured if:

$$|\arg z| < \frac{1}{2}(p-p+1) \quad \text{if } p > q+1 \text{ and } |z| > 1 \quad \text{if } p = q+1.$$

From (14) and (15) it is clear that the E-function is immediately related to the generalized hypergeometric function and reduces to simple expressions in the ordinary or Gauss hypergeometric function when $p=2$, $q=1$. For $p=q=1$ it is also evident that the E-function reduces to the confluent hypergeometric function or Kummer function. When $p=0$, $q=1$ then we obtain the Bessel function of the first kind.

Thus we have:

$$(17) \qquad \Gamma(\tau+1)\,{}_0F_1\left(;\tau+1;-\frac{-1}{x}\right) = \frac{\Gamma(\tau+1)}{\Gamma(\tau+1)}E(;\tau+1;x) = x^{\frac{\tau}{2}} J_\tau\left(2x^{\frac{-1}{2}}\right).$$

More complicated parameters in the E-functions lead to the equivalance of the E-functions, with Bessel functions and with Hankel functions.

Some examples of this are:

$$(18) \qquad K_\mu(z) = \frac{1}{4\pi}\sum_{i,-i}\frac{1}{i}E\left(1;\frac{1}{2}\mu,\frac{1}{2}\mu:\frac{1}{4}z^2 e^{i\pi}\right);$$

where $K_\mu(z)$ is the modified Bessel function of the second kind and the symbol $\sum\limits_{i,-i}$ means that in the expression following it i is to be replaced by $-i$ and the two expressions are to be added.

Also it can be shown that:

$$(19) \qquad W_{k,m}(x)\,W_{-k,m}(x) = \frac{1}{2\pi^{3/2}}\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}1,1,\frac{1}{2},\frac{1}{2}+m,\frac{1}{2}-m: \dfrac{e^{i\pi}x^2}{4}\\ 1+k,\ 1-k\end{matrix}\right];$$

$$(20) \qquad e^{\frac{-1}{2}x}\,W^{k,m}(x) = \frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}+m,\frac{1}{2}-m,1:1-k:e^{i\pi}x\right),$$

$$(21) \quad K_m(x)\,K_n(x) = \frac{1}{4x\sqrt{\pi}}\sum_{i,-i}\frac{1}{i}E\left(\frac{1+m+n}{2},\frac{1-m+n}{2},\frac{1+m-n}{2},\frac{1-m-n}{2}: \frac{1}{2}: e^{i\pi}x^2\right);$$

(22) $$x^{\mu}K_{\tau}^{2}(x)=\frac{\sqrt{\pi}}{4}\sum_{i,-i}\frac{1}{i}E\left(\tau+\frac{1}{2}\mu,-\tau+\frac{1}{2}\mu,\frac{1}{2}\mu,1:\frac{1}{2}\mu+\frac{1}{2}:e^{i\pi}x^{2}\right)$$

(23) $$J_{\tau}(x)\,J_{-\tau}(x)=\{\Gamma(1-\tau)\,\Gamma(1+\tau)\}^{-1}\,{}_1F_2\left(\frac{1}{2};\,1-\tau,\,1+\tau;\,-x^{2}\right);$$

(24) $$J_{\tau}^{2}(x)=\pi^{\frac{-1}{2}}\,x^{2\tau}\,\Gamma\left(\frac{1}{2}+\tau\right)\{\Gamma(1+\tau)\,\Gamma(1+2\tau)\}^{-1}\,{}_1F_2\begin{pmatrix}\frac{1}{2}+\tau;\,-x^{2}\\1+\tau,1+2\tau\end{pmatrix}.$$

3. - Proof of the main theorem : From (15) and (16), we arrive at the fosmula

(25) $$\sum_{i,-i}\frac{1}{i}E\begin{pmatrix}\alpha_1,\,\dots,\,\alpha_\mu,\,1:e^{i\pi}x\\\gamma_1,\,\dots,\,\lambda_\lambda\end{pmatrix}=\frac{-1}{i}\int\frac{\prod_{r=1}^{\mu}\Gamma(\alpha_r-\Theta)}{\prod_{s=1}^{\lambda}\Gamma(\lambda_s-\Theta)}\,x^{\Theta}\,d\Theta,$$

where $R(\alpha_r-\Theta)>0$ $(r=1,\,2,\,\dots,\,\mu)$.

Also from (11), (12) and (25), we deduce that

(26) $$\frac{1}{2\pi}\int_0^{\infty}x^{y-1}\sum_{i,-i}\frac{1}{i}E\begin{pmatrix}\alpha_1,\,\dots,\,\alpha_\mu,\,1,:\,e^{i\pi}x\\\gamma_1,\,\dots,\,\gamma_\lambda\end{pmatrix}dx=\frac{\prod_{r=1}^{\mu}\Gamma(\alpha_r+y)}{\prod_{j=1}^{\lambda}\Gamma(\gamma_j+y)};$$

where $R(\alpha_r+y)>0$, $(r=1,\,2,\,\dots,\,\mu)$.

To prove (1), change each ${}_{\tau}F_{\varepsilon}$ into an E-function by means of (14), write $\frac{x}{c}$ for x; then the L. H. S. of (1) becomes

$$\frac{\prod_{j=1}^{\varepsilon}\{\Gamma(\delta_j)\,\Gamma(\delta_j')\}}{\prod_{j=1}^{\tau}\{\Gamma(\beta_j)\,\Gamma(\beta_j')\}\cdot c^{k}}\int_0^{\infty}x^{k-1}E\begin{pmatrix}\beta_1,\,\dots,\,\beta_\tau;\frac{\alpha c}{x}\\\delta_1,\,\dots,\,\delta_3\end{pmatrix}\times$$

$$\times E\begin{pmatrix}\beta_1',\,\dots,\,\beta_\tau';\frac{cb}{x}\\\delta_1',\,\dots,\,\delta_\varepsilon'\end{pmatrix}\sum_{i,-i}\frac{1}{i}E\begin{pmatrix}\alpha_1,\,\dots,\,\alpha_\mu,\,1:e^{i\pi}x\\\gamma_1,\,\dots,\,\gamma_\gamma\end{pmatrix}dx.$$

Here substitute for each of the first two E-functions from 16, change the order of integration so that the first integral becomes the last and so the last expression becomes

$$\frac{\prod_{j=1}^{\varepsilon}\{\Gamma(\delta_j)\Gamma(\delta_j')\}}{\prod_{j=1}^{\tau}\{\Gamma(\beta_j)\Gamma(\beta_j')\}c^k}\frac{1}{(2\pi i)^2}\int\int\Gamma(s)\Gamma(t)\frac{\prod_{j=1}^{\tau}\{\Gamma(\beta_j-s)\Gamma(\beta_j'-t)\}}{\prod_{j=1}^{\varepsilon}\{\Gamma(\delta_j-s)\Gamma(\delta_j'-t)\}}\times$$

$$\times(ac)^s(bc)^t\int_0^\infty x^{k-s-t-1}\sum_{i,-i}\frac{1}{i}E\begin{pmatrix}\alpha_1,\ldots,\alpha_\mu, 1:x\\ \gamma_1,\ldots,\gamma_\lambda\end{pmatrix}dx.$$

Now evaluate this last integral by means of (26) and the L.H.S. of (1) becomes

$$\frac{\prod_{j=1}^{\varepsilon}\{\Gamma(\delta_j)\Gamma(\delta_j')\}}{\prod_{j=1}^{\tau}\{\Gamma(\beta_j)\Gamma(\beta_j')\}}\cdot\frac{2\pi}{c^k}\times\frac{1}{(2\pi i)^2}\int\int\Gamma(s)\Gamma(t)\times$$

$$\times\frac{\prod_{j=1}^{\mu}\Gamma(\alpha_j+k-s-t)\prod_{j=1}^{\tau}\{\Gamma(\beta_j-s)\Gamma(\beta_j'-t)\}}{\prod_{j=1}^{\lambda}\Gamma(\gamma_j+k-s-t)\prod_{j=1}^{\varepsilon}\{\Gamma(\delta_j-s)\Gamma(\delta_j'-t)\}}\left(\frac{1}{ac}\right)^{-s}\left(\frac{1}{bc}\right)^{-t}ds\,dt.$$

Here apply (10) and so obtain the R.H.S. of (1).

4. - **Infinite integrals involving products of Bessel and Whittakers functions**: We are now in a position to obtain a large number of particular cases of (1).

Thus (1) in combination with (17) and (18) gives

$$\int_0^\infty x^{q-1}J_\tau(ax)J_\sigma(bx)K_\mu(cx)\,dx=\frac{2^{q-2}a^\tau b^\sigma}{\Gamma(1+\tau)\Gamma(1+\sigma)}\times$$

$$\times\frac{\Gamma\left(\frac{1}{2}\tau+\frac{1}{2}\sigma+\frac{1}{2}q+\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}\tau+\frac{1}{2}\sigma+\frac{1}{2}q-\frac{1}{2}\mu\right)}{\tau+\sigma+q}$$

$$\times F\left[\begin{array}{l|ll|l} 2 & \frac{\tau+\sigma+q+\mu}{2}, & \frac{\tau+\sigma+q-\mu}{2} & \\ 0 & \ldots & \ldots & -\frac{a^2}{c^2}, -\frac{b^2}{c^2} \\ 0 & \ldots & \ldots & \\ 1 & 1+\tau, & 1+\sigma & \end{array}\right].$$

Now apply (6) and get

$$(27)\qquad \int_0^\infty x^{q-1} J_\tau(ax)\, J_\sigma(bx)\, K_\mu(cx)\, dx = \frac{\Gamma\left(\frac{\tau}{2}+\frac{\sigma}{2}+\frac{q}{2}+\frac{\mu}{2}\right)\Gamma\left(\frac{\tau}{2}+\frac{\sigma}{2}+\frac{q}{2}-\frac{\mu}{2}\right)}{\Gamma(1+\tau)\,\Gamma(1+\sigma)} \times$$

$$2^{q-2} a^\tau b^\sigma c^{-\tau-\sigma-q} F_4\left[\frac{\tau+\sigma+q+\mu}{2},\ \frac{\tau+\sigma+q-\mu}{2};\ 1+\tau,\ 1+\sigma; \frac{-a_2}{c^2},\ \frac{-b^2}{c^2}\right],$$

where

$$R(q+\tau+\sigma\pm\mu)>0,\ a,\ b,\ c>0.$$

In (27), write $\frac{c}{i}$ for c, then, since

$$(28)\qquad K_\mu(t) = i^n\, G_\mu(it)$$

the formula becomes after utilizing the relation

$$(29)\qquad \pi\, i\, J_\mu(t) = G_\mu(t) - i^{2n}\, G_\mu(te^{i\pi}),$$

$$\int_0^\infty x^{q-1} J_\tau(ax)\, J_\sigma(bx)\, J_\mu(cx) = \frac{2q-1\ a^\tau b^\sigma \Gamma\left(\frac{\sigma}{2}+\frac{\mu}{2}+\frac{\tau}{2}+\frac{q}{2}\right)}{c^{\tau+\sigma+q}\,\Gamma(1+\tau)\,\Gamma(1+\sigma)}$$

$$\times \frac{1}{\Gamma\left(1-\frac{\sigma+\tau+q-\mu}{2}\right)} F_4\left[\frac{\tau+\sigma+q+\mu}{2},\ \frac{\tau+\sigma+q-\mu}{2};\ \tau+1,\ \sigma+1;\ \frac{a^2}{c^2},\ \frac{b^2}{c^2}\right];$$

where $R(q+\tau+\sigma+\mu)>0$, $R(q)\leq\frac{3}{2}$, $a,\ b,\ c>0$.

Again, using the formula

$$(30)\qquad K_\sigma(x) = \frac{\pi}{2\sin\sigma\pi}\left\{ i^n J_{-n}(ix) - i^{-n} J_n(ix)\right\},$$

it follows, from (27), that

$$(31)\qquad \int_0^\infty x^{q-1} J_\tau(ax)\, K_\sigma(bx)\, K_\mu(cx)\, dx = 2^{q-3}$$

$$\times \sum_{\sigma,-\sigma} \frac{a^\tau b^\sigma\, \Gamma(-\sigma)\,\Gamma\left(\frac{\tau}{2}+\frac{2}{\sigma}+\frac{q}{2}+\frac{\mu}{2}\right)\Gamma\left(\frac{\tau}{2}+\frac{\sigma}{2}+\frac{q}{2}-\frac{\mu}{2}\right)}{\Gamma(1+\tau)\ c^{q+\tau+\sigma}}$$

$$\times F_4\left[\frac{\tau+\sigma+q+\mu}{2},\ \frac{\tau+\sigma+q-\mu}{2};\ 1+\tau,\ 1+\sigma;\ \frac{a^2}{c^2},\ \frac{b^2}{c^2}\right];$$

where $R(q\pm\tau\pm\sigma\pm\mu)>0,\ a,\ b,\ c>0$.

In (1) write $x^2, \frac{1}{a^2}, \frac{1}{b^2}, \frac{c^2}{4}$ for x, a, b, c respectively.

Take $\tau=1,\ \varepsilon=2,\ \mu=2,\ \lambda=0$;
apply (18), (23) and get

(34)
$$\int_0^\infty x^{k-1} J_n^2(ax)\, J_m^2(bx)\, K_\mu(cx)\, dx = \frac{2^{2k+2m+2n-2}}{\pi c^{2k+2n+2m}}$$

$$\times \frac{\Gamma\left(\frac{\mu}{2}+\frac{k}{2}+m+n\right)\Gamma\left(\frac{-\mu}{2}+\frac{k}{2}+m+n\right)\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{1}{2}+n\right)}{\Gamma(1+m)\,\Gamma(1+n)\,\Gamma(1+2m)\,\Gamma(1+2n)}$$

$$\times F\left(\begin{array}{c|cc|c} 2 & \frac{1}{2}\mu+\frac{1}{2}k+m+n, & -\frac{1}{2}\mu+\frac{1}{2}k+m+n & \\ 1 & \frac{1}{2}+m, & \frac{1}{2}+n & -\frac{4a^2}{c^2},\ -\frac{4b^2}{c^2} \\ 0 & \cdots & \cdots & \\ 2 & 1+m,\quad 1+n, & 1+2m,\quad 1+2n & \end{array}\right),$$

where $R(k+2n+2m\pm\mu)>0,\ a,\ b,\ c>0$.

In (1) write $\frac{x^2}{4}, \frac{1}{a^2}, \frac{1}{b^2}, 4c^2$ for x, a, b, c respectively, take $\mu=5,\ \lambda=2$, apply (17) and (21) so getting

(35)
$$\int_0^\infty x^{k-1} J_\tau(ax)\, J_\mu(bx)\, K_m(cx)\, K_n(cx)\, dx$$

$$=\frac{\pi^{3/2}\, 2^{-\tau-\mu}\, a^\tau b^\mu\, \Gamma(1+\tau)\,\Gamma(1+\mu)}{c^{k+\tau+\mu+1}\,\Gamma\left(\frac{1}{2}k+\frac{1}{2}\tau+\frac{1}{2}\mu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}k+\frac{1}{2}\tau+\frac{1}{2}\mu+1\right)}$$

$$\times\Gamma\left(\frac{1+k+\tau+\mu+m+n}{2}\right)\Gamma\left(\frac{1+k+\tau+\mu+m-n}{2}\right)$$

$$\times\Gamma\left(\frac{1+k+\tau+\mu+n-m}{2}\right)\Gamma\left(\frac{1+k+\tau+\mu-m-n}{2}\right)$$

$$\times F\left[\begin{array}{c|l|c} 4 & \frac{1+k+\tau+\mu+m+n}{2}, \frac{1+k+\tau+\mu+n-m}{2}, \frac{1+k+\tau+\mu+m-n}{2}, \frac{1+k+\tau+\mu-n-m}{2} & \\ 0 & \ldots \quad \ldots \quad \ldots \quad \ldots & -\frac{a^2}{4c^2}, -\frac{b^2}{4c^2} \\ 2 & \frac{1+k+\tau+\mu}{2}, \frac{2+k+\tau+\mu}{2} & \\ 1 & 1+\tau, \quad 1+\mu & \end{array}\right],$$

where $R(k+\mu+\tau+m+n)>0$, $a, b, c>0$.

In (1) write x^2, $\frac{1}{a^2}$, $\frac{1}{b^2}$, c^2 for x, a, b, c respectively.

Take $\mu=5$, $\lambda=2$, apply (21) and (23), so getting

$$(36) \qquad \int_0^\infty x^{k-1} J_\tau(ax) J_{-\tau}(ax) J_\sigma(bx) J_{-\sigma}(bx) K_m(cx) K_n(cx)\, dx$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{k+m-n}{2}\right)\Gamma\left(\frac{k+n-m}{2}\right)\Gamma\left(\frac{k-m-n}{2}\right)}{4c^{k-1}\,\Gamma\left(\frac{1+k}{2}\right)\Gamma\left(\frac{k}{2}\right)\Gamma(1+\tau)\,\Gamma(1-\tau)\,\Gamma(1+\sigma)\,\Gamma(1-\sigma)}$$

$$\times F\left[\begin{array}{c|l|c} 4 & \frac{k+m+n}{2}, \frac{k+m-n}{2}, \frac{k+n-m}{2}, \frac{k-n-m}{2} & \\ 1 & \frac{1}{2}, \frac{1}{2} & -\frac{a^2}{c^2}, -\frac{b^2}{c^2} \\ 2 & \frac{1+k}{2}, \frac{k}{2} & \\ 2 & 1-\tau, 1-\sigma, 1+\tau, 1+\sigma & \end{array}\right],$$

where $R(k\pm m\pm n)>0$, $a, b, c>0$.

In (1) vrite x^2, $\frac{1}{a^2}$, $\frac{1}{b^2}$, c^2 for x, a, b, c respectively. Take $\mu=5$, $\lambda=2$, whith $\gamma_1=\frac{2}{1}$, $\gamma_2=1$, $\alpha_5=1$; apply (21) and (24) and so obtain

(37)
$$\int_0^\infty x^{k-1} J_\tau^2(ax)\, J_\sigma^2(bx)\, K_m(cx)\, K_n(cx)\, dx = 4\pi c^{-k-2\tau-2\sigma}$$
$$\times \frac{\Gamma\left(\frac{1}{2}+\tau\right)\Gamma\left(\frac{1}{2}+\sigma\right)\Gamma\left(\frac{k+m+n}{2}+\tau+\sigma\right)\Gamma\left(\frac{k+m-n}{2}+\tau+\sigma\right)}{\Gamma(1+\tau)\,\Gamma(1+2\tau)\,\Gamma(1+\sigma)\,\Gamma(1+2\sigma)}$$
$$\times \frac{\Gamma\left(\frac{k+n-m}{2}+\tau+\sigma\right)\Gamma\left(\frac{k-m-n}{2}+\tau+\sigma\right)}{\Gamma\left(\frac{k+1}{2}+\tau+\sigma\right)\Gamma\left(\frac{k}{2}+\tau+\sigma\right)}$$
$$\times F\left[\begin{array}{c|l} 4 & \frac{k+m+n+2\tau+2\sigma}{2},\ \frac{k+m-n+2\tau+2\sigma}{2},\ \frac{k+n-m+2\tau+2\sigma}{2},\ \frac{k-m-n+2\tau+2\sigma}{2} \\ 1 & \frac{1}{2}+\tau,\ \frac{1}{2}+\tau \\ 2 & \frac{1}{2}k+\frac{1}{2}+\tau+\sigma,\ \frac{1}{2}k+\tau+\sigma \\ 2 & 1+\tau,\ 1+\sigma;\ 1+2\tau,\ 1+2\sigma \end{array} \;\middle|\; -\frac{a^2}{c^2},\ \frac{b^2}{c^2}\right],$$

where $R(k+2\tau+2\sigma\pm m\pm n)>0$ and a, b, c are real and positive.

When $m=n$, the last formula becomes

(38)
$$\int_0^\infty x^{k-1} J_\tau^2(ax)\, J_\sigma^2(bx)\, K_n^2(cx)\, dx = 4\pi c^{-k-2\tau-2\sigma}$$

$$\times \frac{\Gamma\left(\frac{1}{2}+\tau\right)\Gamma\left(\frac{1}{2}+\sigma\right)\Gamma\left(\frac{2}{k}+n+\tau+\sigma\right)\Gamma\left(\frac{2}{k}+\tau+\sigma\right)\Gamma\left(\frac{2}{k}-n+\tau+\sigma\right)}{\Gamma(1+\tau)\,\Gamma(1+2\tau)\,\Gamma(1+\sigma)\,\Gamma(1+2\sigma)\,\Gamma\left(\frac{k}{2}+\frac{1}{2}+\tau+\sigma\right)}$$

$$\times F\left[\begin{array}{c|l|c} 3 & \frac{k}{2}+\tau+\sigma,\ \frac{k}{2}+n+\tau+\sigma,\ \frac{k}{2}-n+\tau+\sigma & \\ 1 & \frac{1}{2}+\tau,\ \frac{1}{2}+\sigma & ;\ \frac{-a^2}{c^2},\ \frac{-b^2}{c^2} \\ 1 & \frac{k}{2}+\frac{1}{2}+\tau+\sigma & \\ 2 & 1+\tau,\ 1+\sigma;\ 1+2\tau,\ 1+2\sigma & \end{array}\right];$$

where $R(k+2\tau+2\sigma\pm 2n)>0,\ a, b, c>0.$

Again in (1) write $\frac{x^2}{4}, \frac{1}{a^2}, \frac{1}{b_2}, 4c^2$ for x, a, b, c respectively, take $\mu=2$, $\lambda=1$, apply (17) and (20) so getting

$$(39)\qquad \int_0^\infty x^{q-1} J_\tau(ax)\, J_\sigma(bx)\, e^{-\frac{1}{2}x^2c^2}\, W_{k,m}(c^2x^2)\, dx = \frac{2^{q-2}}{c^{q+\tau+\sigma}}$$

$$\times \frac{\Gamma(1+\tau)\,\Gamma(\sigma+1)\,\Gamma\left(\frac{1}{2}+\frac{q}{2}+\frac{\tau}{2}+\frac{\sigma}{2}\pm m\right)}{\Gamma\left(1+\frac{q}{2}+\frac{\tau}{2}+\frac{\sigma}{2}-k\right)}$$

$$\times F\left[\begin{array}{c|l|c} 2 & \frac{1}{2}(1+q+\tau+\sigma+2m),\ \frac{1}{2}(1+q+\tau+\sigma-2m) & \\ 0 & \quad\cdots\qquad\qquad\cdots & \frac{-a^2}{4c^2},\ \frac{-b^2}{4c^2} \\ 1 & 1-k+\frac{q}{2}+\frac{1}{2}\tau+\frac{1}{2}\sigma & \\ 1 & 1+\tau,\ 1+\sigma & \end{array}\right],$$

where $R(q+\tau+\sigma\pm 2m+1)>0$ and a, b, c are real and positive.

In (1) write $x^2, \frac{1}{a^2}, c^2$ for x, a, b, c respectively. Take $\mu=2$, $\lambda=1$, apply (20) and (23) so getting

$$(40)\qquad \int_0^\infty x^{q-1} J_\tau(ax)\, J_{-\tau}(ax)\, J_\sigma(bx)\, J_{-\sigma}(bx)\, e^{-\frac{1}{2}c^2x^2}\, W_{k,m}(c^2x^2)\, dx$$

$$= \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}q+m\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}q-m\right)}{2c^q\,\Gamma\left(1+\frac{1}{2}q-k\right)\Gamma(1-\tau)\Gamma(1+\tau)\Gamma(1-\sigma)\Gamma(1+\sigma)}$$

$$\times F\left[\begin{array}{c|l|c} 2 & \frac{1}{2}+\frac{1}{2}q+m,\ \frac{1}{2}+\frac{1}{2}q-m & \\ 1 & \frac{1}{2},\ \frac{1}{2} & \frac{-a^2}{c^2},\ \frac{-b^2}{c^2} \\ 1 & 1+\frac{1}{2}q-k & \\ 2 & 1-\tau,\ 1-\sigma;\ 1+\tau,\ 1+\sigma & \end{array}\right],$$

where $R(q \pm 2m+1) > 0$ and a, b, c are real and positive.

In (1(write x^2, $\frac{1}{a^2}$, $\frac{1}{b^2}$, c^2 for x, a, b, c respectively. Take $\mu = 2$, $\lambda = 1$ apply (20) and (24) so getting

$$(41) \qquad \int_0^\infty x^{q-1} J_\tau^2(ax)\, J_\sigma^2(bx)\, e^{-\frac{1}{2}c^2x^2}\, W_{k,m}(c^2x^2)\,dx$$

$$= \frac{a^{2\tau}b^{2\sigma}\,\Gamma\left(\frac{1}{2}+\tau\right)\Gamma\left(\frac{1}{2}+\sigma\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}q+m+\sigma+\tau\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}q-m+\sigma+\tau\right)}{2\pi c^{q+2\sigma+2\tau}\,\Gamma(1+\tau)\Gamma(1+\sigma)\Gamma(1+2\tau)\Gamma(1+2\sigma)\Gamma\left(1-k+\sigma+\tau+\frac{1}{2}q\right)}$$

$$\times F\left[\begin{array}{c|l|c} 2 & \frac{1}{2}+\frac{1}{2}q+\sigma+\tau+m,\ \frac{1}{2}+\frac{1}{2}q+\sigma+\tau-m & \\ 1 & \frac{1}{2}+\tau,\ \frac{1}{2}+\sigma & -\frac{a^2}{c^2},\ -\frac{b^2}{c^2} \\ 1 & 1-k+\frac{1}{2}q+\sigma+\tau & \\ 2 & 1+\tau,\ 1+\sigma;\ 1+2\tau,\ 1+2\sigma & \end{array}\right],$$

where $R(q+2\tau+2\sigma+1 \pm 2m) > 0$ and a, b, c are real and positive. Finally (1) in combination with (20) and (13) gives integrals involving the product of three Bessel functions.

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