Totally Real Submanifolds of a Quaternion Projective Space (*).

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Dedicated to Professor T. Otsuki on his sixtieth birthday

Summary. – *See Introduction.*

1. – Introduction.

A quaternion manifold is defined as a Riemannian manifold whose holonomy group is a subgroup of $\text{Sp}(1) \cdot \text{Sp}(m)$. The quaternion projective space $P^m(\mathbf{H})$ and its noncompact dual are two important examples of quaternion manifolds. It is well-known that they have constant positive and negative quaternion sectional curvature, respectively. We simply call quaternion manifolds with constant quaternion sectional curvature the quaternion-space-forms.

Let N be a submanifold of a quaternion manifold M . N is called an invariant submanifold of M if N is invariant under the curvature operator of M , i.e., $\tilde{R}(X, Y) \cdot (TN) \subset TN$ for all vector fields X, Y in TN , where TN is the tangent bundle of N and $\tilde{R}$ is the curvature tensor of M . In [1], it is proved that if N is an n -dimensional ($n \geq 4$) invariant submanifold of a quaternion-space-form of nonzero quaternion sectional curvature, then N is either a totally real manifold or a half-quaternion submanifold, or a quaternion submanifold.

In this paper we shall study some fundamental properties of totally real submanifolds in a quaternion-space-form. The other submanifolds will be studied in another paper.

In § 2, some fundamental formulas and properties of totally real submanifolds will be obtained.

In § 3, we shall establish a sufficient condition for a totally real submanifold in a quaternion projective space $P^m(\mathbf{H})$ to be totally geodesic in terms of scalar curvature.

In § 4, some pinching theorems on sectional curvature for totally real submanifolds in $P^m(\mathbf{H})$ and $P^m(\mathbf{C})$ are obtained. Some of these results are independent of codimension.

In the last section, we shall study minimal totally real immersion of real-space-forms in quaternion-space-form.

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2. – Totally real submanifolds.

Let M be a $4m$ -dimensional quaternion manifold with metric g . Then there exists a 3-dimensional vector space V of tensors of type $(1, 1)$ with local basis of almost Hermitian structures I, J, K such that

- (a) $IJ = -JI = K, JK = -KJ = I, KI = -IK = J, I^2 = J^2 = K^2 = -1,$
- (b) for any local cross-section φ of V , $\nabla_x \varphi$ is also a cross-section of V , where X is an arbitrary vector field in M and ∇ the Riemannian-connection on M .

Condition (b) is equivalent to the following condition:

- (b)' there exist local 1-forms p, q and r such that

$$(2.1) \quad \begin{aligned} \nabla_x I &= r(X)J - q(X)K, \\ \nabla_x J &= -r(X)I + p(X)K, \\ \nabla_x K &= q(X)I - p(X)J. \end{aligned}$$

Let X be a unit vector on the quaternion manifold M . Then X, IX, JX and KX form an orthonormal frame on M . We denote by $Q(X)$ the 4-plane spanned by them. We call $Q(X)$ the *quaternion-section* determined by X . For any two vectors X, Y in M , if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X, Y)$ spanned by X, Y is called a *totally real plane*. Any 2-plane in a quaternion-section is called a *quaternion-plane*. The sectional curvature of a quaternion-plane π is called the *quaternion sectional curvature* of π . A quaternion manifold is a *quaternion-space-form* if its quaternion sectional curvatures are equal to a constant c .

It is well-known that a quaternion manifold M is a quaternion-space-form if and only if its curvature tensor $\tilde{R}$ is of the following form [4]:

$$(2.2) \quad \begin{aligned} \tilde{R}(X, Y)Z &= \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y + g(IY, Z)IX - g(IX, Z)IY \\ &\quad + 2g(X, IY)IZ + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ \\ &\quad + g(KY, Z)KX - g(KX, Z)KY + 2g(X, KY)KZ \}, \end{aligned}$$

for vector fields tangent to M .

In the following, let $M^m(c)$ denote a $4m$ -dimensional quaternion-space-form of quaternion sectional curvature c , and let $P^m(H)$ denote the $4m$ -dimensional quaternion projective space of constant quaternion sectional curvature $c > 0$.

Let N be an n -dimensional Riemannian manifold and $x: N \rightarrow M^m(c)$ be an isometric immersion of N into a quaternion manifold $M^m(c)$. We call N a *totally real*

submanifold of $M^m(c)$ if each tangent 2-plane of N is mapped by x into a totally real plane in $M^m(c)$.

Let ∇ and $\tilde{\nabla}$ be the Riemannian connections on N and $M^m(c)$, respectively. The second fundamental form σ of the immersion is given by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X, Y are vector fields tangent to N and σ is a normal-bundle-valued symmetric 2-form on N . For a normal vector field ξ on N , we write $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ (respectively, $D_X \xi$) is the tangential (respectively, normal) component of $\tilde{\nabla}_X \xi$. We have $g(\sigma(X, Y), \xi) = g(A_\xi X, Y)$.

The mean-curvature vector H is defined by $H = \text{trace } \sigma/n$. N is said to be *minimal* if $H = 0$ identically. For the second fundamental form σ , we define the covariant differentiation $\bar{\nabla}$ with respect to the connection in the (tangent bundle) $\oplus$ (normal bundle) by

$$(2.3) \quad (\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

for all vector fields X, Y, Z tangent to N .

Let R be the curvature tensor of N . Then the equation of Gauss is given by

$$(2.4) \quad g(R(X, Y)Z, W) = g(\tilde{R}(X, Y)Z, W) + g(\sigma(X, W), \sigma(Y, Z)) \\ - g(\sigma(X, Z), \sigma(Y, W)),$$

for X, Y, Z, W tangent to N .

Assume that N is a totally real submanifold of M . For any orthogonal vectors X, Y in N , $\pi(X, Y)$ is totally real in $M^m(c)$. Hence, $Q(X)$ and $Q(Y)$ are also orthogonal. We thus have $g(X, \varphi Y) = g(\psi X, Y) = 0$ for $\varphi, \psi = I, J$, or K . By (2.2) we have

$$\tilde{R}(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y\},$$

for $X, Y, Z \in TN$. Hence (2.4) reduces to

$$(2.4)' \quad g(R(X, Y)Z, W) = \frac{c}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ + g(\sigma(Y, Z), \sigma(X, W)) - g(\sigma(X, Z), \sigma(Y, W)).$$

Since $\dim N = n$ and N is totally real in $M^m(c)$, we have $n \leq m$. Let $p = 4m - n$. In $M^m(c)$ we choose a local field of orthonormal frames:

$$e_1, \dots, e_n, e_{n+1}, \dots, e_m; \quad e_{I(1)} = Ie_1, \dots, e_{I(m)} = Ie_m; \\ e_{J(1)} = Je_1, \dots, e_{J(m)} = Je_m; \quad e_{K(1)} = Ke_1, \dots, e_{K(m)} = Ke_m,$$

in such a way that, restricted to N , $e_1, \dots, e_n$ are tangent to N . With respect to this

frame field, I, J, K have the following forms;

$$(2.5) \quad I = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix},$$

where E denotes the $(m \times m)$ -identity matrix

We shall use the following convention on the range of indices unless otherwise stated:

$$A, B, C, D = 1, \dots, m, \quad I(1), \dots, I(m), \quad J(1), \dots, J(m), \quad K(1), \dots, K(m),$$

$$k, i, j, l = 1, \dots, n,$$

$$\alpha, \beta = n + 1, \dots, m, \quad I(1), \dots, K(m),$$

$$\lambda, \mu = n + 1, \dots, m.$$

$$\varphi, \psi = I, J, K.$$

With respect to the frame field of $M^m(c)$ chosen above, let

$$\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^m, \omega^{I(1)}, \dots, \omega^{I(m)}, \omega^{J(1)}, \dots, \omega^{J(m)}, \omega^{K(1)}, \dots, \omega^{K(m)}$$

be the field of dual frames. Then the structure equations of $M^m(c)$ are given by

$$(2.6) \quad d\omega^A = - \sum \omega_B^A \wedge \omega^B, \quad \omega_B^A + \omega_A^B = 0, \\ d\omega_B^A = - \sum \omega_C^A \wedge \omega_B^C + \tilde{Q}_B^A, \quad \tilde{Q}_B^A = \frac{1}{2} \sum \tilde{R}_{ABCD} \omega^C \wedge \omega^D.$$

Since N is totally real, $Q(e_i)$ and $Q(e_j)$, $i \neq j$ are orthogonal. Thus, $g(\varphi(e_i), \psi(e_j)) = 0$ for $i \neq j$. From these and condition (2.1) for $M^m(c)$, we have

$$(2.7) \quad \begin{aligned} \omega_j^i + \omega_i^j &= 0, & \omega_j^i &= \omega_{\varphi(j)}^{\varphi(i)}, & \omega_j^{\varphi(i)} &= \omega_i^{\varphi(j)}, \\ \omega_\mu^\lambda + \omega_\lambda^\mu &= 0, & \omega_\mu^\lambda &= \omega_{\varphi(\mu)}^{\varphi(\lambda)}, & \omega_\mu^{\varphi(\lambda)} &= \omega_\lambda^{\varphi(\mu)}, \\ \omega_\lambda^i + \omega_i^\lambda &= 0, & \omega_\lambda^i &= \omega_{\varphi(\lambda)}^{\varphi(i)}, & \omega_\lambda^{\varphi(i)} &= \omega_i^{\varphi(\lambda)}. \end{aligned}$$

Restricting these forms to N we have the following structure equations of the immersion:

$$\begin{aligned}\omega^\alpha &= 0, \\ d\omega_j^i &= -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i, \quad \Omega_j^i = \frac{1}{2} \sum R_{ijkl} \omega^k \wedge \omega^l, \\ \Omega_j^i &= \bar{\Omega}_j^i - \sum \omega_\alpha^i \wedge \omega_j^\alpha \quad (\text{the equation of Gauss}).\end{aligned}$$

If we write $\omega_i^\alpha = \sum h_{ij}^\alpha \omega^j$, then

$$h_{ij}^\alpha = h_{ji}^\alpha, \quad h_{ij}^\alpha = g(A_\alpha e_i, e_j),$$

where $A_\alpha = A_{e_\alpha}$ and the equation (2.4)' of Gauss is written as

$$(2.8) \quad R_{ijkl} = \tilde{R}_{ijkl} + \sum (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha).$$

Moreover, from (2.7), we have

$$(2.9) \quad h_{jk}^{q(i)} = h_{ik}^{q(j)} = h_{ji}^{q(k)}.$$

If the second fundamental tensors commute, i.e., $[A_\xi, A_\eta] = 0$ for any vector fields ξ, η normal to N . Then, by a suitable choice of $e_1, \dots, e_n$, we have $h_{jk}^{q(i)} = 0$ for $j \neq k$. Thus, by using (2.9), we find

$$(2.10) \quad h_{jk}^{q(i)} = 0 \quad \text{for } i \neq j \text{ or } j \neq k.$$

In particular, if $n = m$, (2.4)' and (2.10) imply the following.

PROPOSITION 1. — *Let N be an n -dimensional totally real submanifold of a quaternion-space-form $M^n(c)$. If the second fundamental tensors commute, then N is of constant sectional curvature $c/4$.*

If N is totally umbilical and totally real, the second fundamental tensors commute. Thus Proposition 1 implies the following.

COROLLARY 1. — *Let N be an n -dimensional totally umbilical and totally real submanifold of $M^n(c)$. Then N is totally geodesic and of constant sectional curvature $c/4$.*

With respect to the frame fields chosen above, the curvature tensor $\tilde{R}$ of (1.2) is written as

$$(2.11) \quad \begin{aligned}\tilde{R}_{ABCD} &= c/4(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + I_{AC}I_{BD} - I_{AD}I_{BC} + 2I_{AB}I_{CD} \\ &\quad + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD} + K_{AC}K_{BD} - K_{AD}K_{BC} + 2K_{AB}K_{CD}),\end{aligned}$$

where δ_{AB} is the Kronecker delta.

By the equation (2.4)' of Gauss, the sectional curvature $K(X, Y)$ of N for a plane determined by orthonormal vectors X, Y is given by

$$K(X, Y) = g(R(X, Y)Y, X) = \frac{c}{4} + \sum \{g(A_\alpha X, X)g(A_\alpha Y, Y) - g(A_\alpha X, Y)^2\}.$$

The Ricci tensor S of N is given by

$$\begin{aligned} S(X, Y) &= \sum g(R(e_i, X)Y, e_i) \\ &= \frac{(n-1)c}{4} g(X, Y) + \sum \{(\operatorname{tr} A_\alpha) g(A_\alpha X, Y) - g(A_\alpha X, A_\alpha Y)^2\}, \end{aligned}$$

and the scalar curvature ϱ of N is given by

$$\varrho = \frac{n(n-1)}{4} c + \sum (\operatorname{tr} A_\alpha)^2 - \|\sigma\|^2,$$

where $\|\sigma\|^2 = \sum h_{ij}^\alpha h_{ij}^\alpha$: In particular, if N is minimal, then

$$(2.12) \quad S - \frac{1}{4}(n-1)cg \quad \text{is negative semidefinite,}$$

$$(2.12)' \quad \varrho \leq \frac{1}{4}(n-1)nc.$$

As an immediate consequence of these relations, we have the following characterizations of totally real, totally geodesic submanifolds.

PROPOSITION 2. — *Let N be an n -dimensional totally real minimal submanifold of a quaternion-space-form $M^m(c)$. Then N is totally geodesic if and only if it satisfies one of the following conditions:*

$$(a) \ K = \frac{c}{4}, \quad (b) \ S = \frac{1}{4}(n-1)cg, \quad (c) \ \varrho = \frac{1}{4}n(n-1)c.$$

REMARK 1. — Since an m -dimensional real projective space $P^m(\mathbf{R})$ of constant sectional curvature $c/4$ can be isometrically imbedded in the $4m$ -dimensional quaternion projective space $P^m(\mathbf{H})$ of quaternion sectional curvature c as a totally real totally geodesic submanifold, every submanifold N of $P^m(\mathbf{R})$ can be regarded as a totally real submanifold of $P^m(\mathbf{H})$. If N is minimal in $P^m(\mathbf{R})$, N is a totally real, minimal submanifold in $P^m(\mathbf{H})$.

Since there exist totally umbilical hypersurfaces in $P^{n+1}(\mathbf{R})$ which is not totally geodesic, there exist n -dimensional totally umbilical, totally real submanifold in $P^{n+1}(\mathbf{H})$ which is not totally geodesic in $P^{n+1}(\mathbf{H})$. Thus the codimensions in Proposition 1 and Corollary 1 are best possible.

3. – Pinchings on scalar curvature.

In [3], the Laplacian of $\|\sigma\|^2$ was calculated for a minimal submanifold in a locally symmetric manifold, i.e., the following formula holds:

$$(3.1) \quad \frac{1}{2} \Delta \|\sigma\|^2 = \|\bar{\nabla} \sigma\|^2 + \sum \operatorname{tr} (A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum (\operatorname{tr} A_\alpha A_\beta)^2 \\ + \sum (4\tilde{R}_{\alpha\beta ij} h_{jk}^\alpha h_{ik}^\beta - \tilde{R}_{\alpha k \beta k} h_{ij}^\alpha h_{ij}^\beta + 2\tilde{R}_{ijkj} h_{ii}^\alpha h_{kl}^\alpha + 2\tilde{R}_{ijkl} h_{il}^\alpha h_{jk}^\alpha).$$

Since N is minimal, (2.5) and (3.1) imply

$$(3.2) \quad 4\tilde{R}_{\alpha\beta ij} h_{jk}^\alpha h_{ik}^\beta = c \sum_{i,\varphi} \operatorname{tr} A_{\varphi(i)}^2,$$

$$(3.3) \quad \tilde{R}_{\alpha k \beta k} h_{ij}^\alpha h_{ij}^\beta = \frac{nc}{4} \|\sigma\|^2 + \frac{3}{4} c \sum_{i,\varphi} \operatorname{tr} A_{\varphi(i)}^2,$$

$$(3.4) \quad \tilde{R}_{ijkj} h_{il}^\alpha h_{kl}^\alpha = \frac{1}{4} (n-1) c \|\sigma\|^2,$$

$$(3.5) \quad \tilde{R}_{ijkl} h_{il}^\alpha h_{jk}^\alpha = \frac{c}{4} \|\sigma\|^2.$$

Moreover, by using (2.8) we find

$$(3.6) \quad \sum (\tilde{R}_{ijkj} h_{il}^\alpha h_{kl}^\alpha + \tilde{R}_{ijkl} h_{il}^\alpha h_{jk}^\alpha) = \sum (R_{ijkj} h_{il}^\alpha h_{kl}^\alpha + R_{ijkl} h_{il}^\alpha h_{jk}^\alpha) \\ - \frac{1}{2} \sum \operatorname{tr} (A_\alpha A_\beta - A_\beta A_\alpha)^2 + \sum (\operatorname{tr} A_\alpha A_\beta)^2.$$

Combining (3.4), (3.5) and (3.6), we obtain

$$(3.7) \quad \sum (R_{ijkj} h_{il}^\alpha h_{kl}^\alpha + R_{ijkl} h_{il}^\alpha h_{jk}^\alpha) = \frac{1}{2} \sum \operatorname{tr} (A_\alpha A_\beta - A_\beta A_\alpha)^2 \\ - \sum (\operatorname{tr} A_\alpha A_\beta)^2 + \frac{nc}{4} \|\sigma\|^2.$$

Substituting (3.2), (3.3), (3.4) and (3.5) into (3.1) we obtain

$$(3.8) \quad \frac{1}{2} \Delta \|\sigma\|^2 = \|\bar{\nabla} \sigma\|^2 + \sum \operatorname{tr} (A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum (\operatorname{tr} A_\alpha A_\beta)^2 \\ + \frac{nc}{4} \|\sigma\|^2 + \frac{c}{4} \sum_{i,\varphi} (\operatorname{tr} A_{\varphi(i)}^2).$$

On the other hand, by (3.1)-(3.7), we also find

$$(3.9) \quad \frac{1}{2} \Delta \|\sigma\|^2 = \|\bar{\nabla} \sigma\|^2 + \frac{1-a}{2} \sum \operatorname{tr} (A_\alpha A_\beta - A_\beta A_\alpha)^2 + a \sum (\operatorname{tr} A_\alpha A_\beta)^2 \\ - \frac{nac}{4} \|\sigma\|^2 + \frac{c}{4} \sum_{i,\varphi} \operatorname{tr} A_{\varphi(i)}^2 \\ + (1+a) \sum (R_{ijkj} h_{il}^\alpha h_{kl}^\alpha + R_{ijkl} h_{il}^\alpha h_{jk}^\alpha).$$

Now, we need the following algebraic lemma proved in [3].

LEMMA 1. — Let A and B be symmetric $(n \times n)$ -matrices. Then

$$-\operatorname{tr}(AB - BA)^2 \leq 2(\operatorname{tr} A^2)(\operatorname{tr} B^2)$$

and the equality holds for nonzero matrices A and B if and only if A and B can be transformed by an orthogonal matrix simultaneously into scalar multiples of $\bar{A}$ and $\bar{B}$, respectively, where

$$\bar{A} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right] \quad \bar{B} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right].$$

Moreover, if A_1, A_2, A_3 are symmetric $(n \times n)$ -matrices such that

$$-\operatorname{tr}(A_a A_b - A_b A_a)^2 = 2(\operatorname{tr} A_a^2)(\operatorname{tr} A_b^2), \quad 1 \leq a, b \leq 3, \quad a \neq b,$$

then at least one of the matrices A_a must be zero.

Now, we shall prove the following.

THEOREM 3. — Let N be an n -dimensional compact totally real minimal submanifold of the quaternion projective space $P^m(\mathbf{H})$. If

$$\varrho \geq \frac{n[(n-1)(2p-1)-p]c}{4(2p-1)} \quad \text{or equivalently} \quad \|\sigma\|^2 \leq \frac{nc}{4(2-1/p)},$$

where $p = 4m - n$, then N is totally geodesic.

PROOF. — By (3.8) and Lemma 1, we have

$$\frac{1}{2} \Delta \|\sigma\|^2 \geq \|\bar{\nabla} \sigma\|^2 - 2 \sum_{\alpha \neq \beta} (\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2) - \sum (\operatorname{tr} A_\alpha A_\beta)^2 + \frac{nc}{4} \|\sigma\|^2 + \frac{c}{4} \sum_{i, \varphi} (\operatorname{tr} A_{\varphi(i)}^2).$$

Let $S_{\alpha\beta} = \sum \operatorname{tr}(A_\alpha A_\beta) = \sum h_{ij}^\alpha h_{ji}^\beta$. Then $(S_{\alpha\beta})$ is a symmetric $(p \times p)$ matrix and it can be diagonalized for a suitable choice of $\{e_\alpha\}$. Hence we may assume that $\operatorname{tr} A_\alpha A_\beta = 0$ for $\alpha \neq \beta$. Furthermore, if we put

$$p\sigma_1 = \sum \operatorname{tr} A_\alpha^2 = \|\sigma\|^2, \quad p(p-1)\sigma_2 = 2 \sum_{\alpha < \beta} \operatorname{tr} A_\alpha^2 \operatorname{tr} A_\beta^2,$$

we then have

$$\begin{aligned} (3.10) \quad \frac{1}{2} \Delta \|\sigma\|^2 &= \|\bar{\nabla} \sigma\|^2 - p(p-1)\sigma_2 - p^2\sigma_1^2 + \frac{nc}{4} \|\sigma\|^2 + \frac{c}{4} \sum (\operatorname{tr} A_{\varphi(i)}^2) \\ &= \|\bar{\nabla} \sigma\|^2 + p(p-1)(\sigma_1^2 - \sigma_2) - (2p^2 - p)\sigma_1^2 + \frac{nc}{4} \|\sigma\|^2 + \frac{c}{4} \sum (\operatorname{tr} A_{\varphi(i)}^2) \\ &= \|\bar{\nabla} \sigma\|^2 + p(p-1)(\sigma_1^2 - \sigma_2) - \left(2 - \frac{1}{p}\right) \|\sigma\|^4 + \frac{nc}{4} \|\sigma\|^2 + \frac{c}{4} \sum (\operatorname{tr} A_{\varphi(i)}^2). \end{aligned}$$

Since

$$\begin{aligned} p(p-1)(\sigma_1^2 - \sigma_2) &= (p-1) \left(\sum \operatorname{tr} A_\alpha^2 \right) - 2p \sum_{\alpha < \beta} \operatorname{tr} A_\alpha^2 \operatorname{tr} A_\beta^2 \\ &= \sum_{\alpha < \beta} (\operatorname{tr} A_\alpha^2 - \operatorname{tr} A_\beta^2)^2 \geq 0, \end{aligned}$$

(3.10) implies

$$\frac{1}{2} \Delta \|\sigma\|^2 \geq \frac{nc}{4} \|\sigma\|^2 - \left(2 - \frac{1}{p}\right) \|\sigma\|^4 = \left[\frac{nc}{4} - \left(2 - \frac{1}{p}\right) \|\sigma\|^2\right] \|\sigma\|^2.$$

Hence, if $\|\sigma\|^2 \leq nc/4(2 - 1/p)$, then $\Delta \|\sigma\|^2 \geq 0$. Thus, by a well-known lemma of E. Hopf, we have $\Delta \|\sigma\|^2 = 0$, consequently we have either $\|\sigma\| = 0$, i.e. N is totally geodesic or $\|\sigma\|^2 = nc/(4(2 - 1/p))$.

If $\|\sigma\|^2 = nc/(4(2 - 1/p))$, all inequality signs above become equalities. Especially, we have

$$-\operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 = 2(\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2).$$

Thus Lemma 1 implies that at most two of the A_α 's are nonzero. Now, in the equality

$$\sum_{\alpha < \beta} (\operatorname{tr} A_\alpha^2 - \operatorname{tr} A_\beta^2) = 0,$$

there are more than three terms in the summation, we have that all A_α should be zero. This leads to a contradiction.

If $m = n$, then $\sum_{\varphi, i} \operatorname{tr} A_{\varphi(i)}^2 = \|\sigma\|^2$ and $p = 3n$. Thus (3.10) gives

$$\frac{1}{2} \Delta \|\sigma\|^2 \leq \frac{(n+1)c}{4} \|\sigma\|^2 - \frac{6n-1}{3n} \|\sigma\|^4.$$

By using this inequality and a similar argument as given in the proof of Theorem 3, we obtain the following.

THEOREM 4. — *Let N be an n -dimensional compact totally real minimal submanifold of the quaternion projective space $P^n(\mathbf{H})$. If*

$$\varrho \leq \frac{n(3n^2 - 5n - 1)}{2(6n - 1)} c \quad \text{or equivalently} \quad \|\sigma\|^2 \leq \frac{3n(n+1)c}{4(6n-1)}$$

then N is totally geodesic and $\varrho = \frac{1}{4}n(n-1)c$.

4. — Pinchings on sectional curvature.

In this section we shall use (3.9) and a method in [7] to obtain a sufficient condition for a totally real minimal submanifold in $P^m(\mathbf{H})$ to be totally geodesic in terms of sectional curvature.

Let $\alpha_1, \dots, \alpha_n$ be the eigenvalues of A_α . Then we have

$$(4.1) \quad \sum_{i,j,k,l} (R_{ijkj} h_{il}^\alpha h_{kl}^\alpha + R_{ijkl} h_{il}^\alpha h_{jk}^\alpha) = \frac{1}{2} \sum_{i,k} (\alpha_i - \alpha_k)^2 R_{ikik}.$$

Now, we assume that the sectional curvature of N is greater than or equal to γ . (4.1) gives

$$(4.2) \quad \sum_{i,j,k,l} (R_{ijkj} h_{il}^\alpha h_{kl}^\alpha + R_{ijkl} h_{il}^\alpha h_{jk}^\alpha) \geq \frac{1}{2} \sum_{i,k} (\alpha_i - \alpha_k)^2 \gamma.$$

Since N is minimal, $\sum_{i,k} (\alpha_i - \alpha_k)^2 = 2n \operatorname{tr} A_\alpha^2$. Thus by (3.9) and (4.2) we find

$$(4.3) \quad \frac{1}{2} \Delta \|\sigma\|^2 \geq \|\bar{\nabla} \sigma\|^2 + \frac{1-a}{2} \sum \operatorname{tr} (A_\alpha A_\beta - A_\beta A_\alpha)^2 + a \left(\sum \operatorname{tr} A_\alpha A_\beta \right)^2 - \frac{nac}{4} \|\sigma\|^2 + \frac{c}{4} \sum \operatorname{tr} A_{\varphi(i)}^2 + (1+a)n\gamma \|\sigma\|^2,$$

for $a \geq -1$. By using the arguments as before, we may assume that $\operatorname{tr} A_\alpha A_\beta = 0$ for $\alpha \neq \beta$. Thus by using Lemma 1 again, we get

$$(4.4) \quad \frac{1}{2} \Delta \|\sigma\|^2 \geq (a-1) \sum_{\alpha \neq \beta} (\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2) + a \left(\sum \operatorname{tr} A_\alpha^2 \right)^2 - \frac{nac}{4} \|\sigma\|^2 + \frac{c}{4} \sum \operatorname{tr} A_{\varphi(i)}^2 + (1+a)n\gamma \|\sigma\|^2.$$

Since $\sum_{\alpha \neq \beta} (\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2) + \sum_\alpha (\operatorname{tr} A_\alpha^2)^2 = \left(\sum_\alpha \operatorname{tr} A_\alpha^2 \right)^2 = \|\sigma\|^4$ and $\sum (\operatorname{tr} A_\alpha^2) \geq \|\sigma\|^4/n$, (4.4) gives

$$\frac{1}{2} \Delta \|\sigma\|^2 \geq \left\{ \frac{1}{n} - (1-a) \right\} \|\sigma\|^4 + \left\{ (1+a)n\gamma - \frac{nac}{4} \right\} \|\sigma\|^2 + \frac{c}{4} \sum \operatorname{tr} A_{\varphi(i)}^2.$$

In particular, putting $a = 1 - 1/n$, we obtain

$$(4.5) \quad \frac{1}{2} \Delta \|\sigma\|^2 \geq \left\{ (2n-1)\gamma - \frac{n-1}{4} c \right\} \|\sigma\|^2 + \frac{c}{4} \sum \operatorname{tr} A_{\varphi(i)}^2.$$

Therefore, if $\gamma \geq (n-1)c/4(2n-1)$, then the right-hand side of (4.5) is nonnegative. Thus, by using Hopf's lemma again, we obtain $\Delta \|\sigma\|^2 = 0$, $\sum \operatorname{tr} A_{\varphi(i)}^2 = 0$ and the inequality signs in this section are actually equalities. Thus, in particular, we have

$$-\operatorname{tr} (A_\alpha A_\beta - A_\beta A_\alpha)^2 = 2(\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2),$$

for $\alpha \neq \beta$. Thus at most two of the A_α 's are nonzero and, with respect to suitable

frame, these two matrices are scalar multiples of $\bar{A}$ and $\bar{B}$ in lemma 1. Without loss of generality, we may assume that

$$(4.6) \quad A_{n+1} = a \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \quad A_{n+2} = b \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

$$A_\alpha = 0 \quad \text{for } \alpha \neq n+1, n+2.$$

If N is not totally geodesic in $P^m(\mathbf{H})$, then $\gamma = (n-1)c/4(2n-1)$. On the other hand, by (2.4)' and (4.6), we find that the sectional curvature $K(\pi(e_1, e_i))$ of the plane $\pi(e_1, e_i)$ satisfies

$$(4.7) \quad K(\pi(e_1, e_i)) = R_{1i1i} = c/4 \quad \text{for } i > 2.$$

Since N is not totally geodesic, we may assume that $A_{n+2} \neq 0$. For $A_\alpha = A_{n+2}$, we have $\alpha_1 = b$, $\alpha_2 = -b$, $\alpha_i = 0$, $i > 2$. Thus $(\alpha_1 - \alpha_i)^2 \neq 0$ for $i > 2$. Therefore, from (4.1) and the equality of (4.2), we find $R_{1i1i} = \gamma = (n-1)c/4(2n-1)$ which is smaller than $c/4$. Consequently, we have $n = 2$. Since $\sum \text{tr } A_{\varphi(i)}^2 = 0$, $m > 2$ and $K = c/12$.

If $n = m$, then $\sum \text{tr } A_{\varphi(i)}^2 = \|\sigma\|^2$. Thus (4.5) reduces to

$$(4.8) \quad \frac{1}{2} \Delta \|\sigma\|^2 \geq \left\{ (2n-1)\gamma - \frac{n-2}{4}c \right\} \|\sigma\|^2.$$

Thus if $\gamma \geq (n-2)c/4(2n-1)$, then $\Delta \|\sigma\|^2 = 0$. By a similar argument, we may prove that either N is totally geodesic or $n = m = 2$ and $K = 0$. Consequently, we have the following.

THEOREM 5. — *Let N be an n -dimensional compact totally real minimal submanifold of the quaternion projective space $P^m(\mathbf{H})$. If the sectional curvature of N satisfies*

$$(4.9) \quad K \geq \frac{(n-1)c}{4(2n-1)},$$

then either (i) N is totally geodesic in $P^m(\mathbf{H})$ or (ii) $n = 2$, $m > 2$ and N is of positive constant curvature $c/12$.

THEOREM 6. — *Let N be an n -dimensional compact, totally real, minimal submanifold of the quaternion projective space $P^n(\mathbf{H})$. If the sectional curvature of N satisfies*

$$(4.10) \quad K \geq \frac{(n-2)c}{4(2n-1)},$$

then either (i) N is totally geodesic in $P^n(\mathbf{H})$ or (ii) $n = 2$ and N is a flat surface.

REMARK 2. – Let (x, y, z) be the natural coordinate system in $\mathbf{R}^3$ and $(u^1, u^2, u^3, u^4, u^5)$ the natural coordinate system in $\mathbf{R}^5$. We consider the mapping defined by

$$\begin{aligned} u^1 &= \frac{1}{\sqrt{3}}yz, & u^2 &= \frac{1}{\sqrt{3}}xy, & u^3 &= \frac{1}{\sqrt{3}}xz, \\ u^4 &= \frac{1}{2\sqrt{3}}(x^2 - y^2), & u^5 &= \frac{1}{6}(x^2 + y^2 - 2z^2). \end{aligned}$$

This defines an imbedding of the real projective plane into the unit hypersphere $S^4 \subset R^5$. This real projective plane in S^4 is called the *Veronese surface*. It is a minimal submanifold of S^4 with sectional curvature $K = \frac{1}{3}$. Now, we consider the immersion of S^4 into the real projective space $P^4(\mathbf{R})$ of sectional curvature 1 and the canonical totally geodesic totally real isometric imbedding of $P^4(\mathbf{R})$ into the quaternion projective space $P^m(\mathbf{H})$ of constant quaternion sectional curvature $4 = c$. Then the composition immersion of the Veronese surface into $P^m(\mathbf{H})$ gives us a compact, totally real, minimal surface in $P^4(\mathbf{H})$ with $K = c/12$ and $\bar{\nabla}\sigma = 0$.

REMARK 3. – Let T^2 be a flat, totally real, minimal surface in the complex projective plane $P^2(\mathbf{C})$ of constant holomorphic sectional curvature c . Then since $P^2(\mathbf{C})$ can be isometrically imbedded in $P^2(\mathbf{H})$ as a totally geodesic, totally real submanifold, the composition immersion gives us a compact totally real minimal surface in $P^2(\mathbf{H})$ with $K = 0$ and $\bar{\nabla}\sigma = 0$.

From Remarks 2 and 3, we see that Theorems 5 and 6 are best possible. By using similar arguments as given in [3], we may prove that the Veronese surface in $P^4(\mathbf{H})$ is the only compact totally real minimal submanifold in $P^4(\mathbf{H})$ with $K = ((n-1)/4(2n-1))c$.

THEOREM 7. – *Let N be an n -dimensional compact totally real minimal submanifold of the complex projective space $P^m(\mathbf{C})$ of constant holomorphic sectional curvature c . If the sectional curvature of N satisfies*

$$(4.11) \quad K \geq \frac{(n-1)c}{4(2n-1)},$$

then either (i) N is totally geodesic in $P^m(\mathbf{C})$ or (ii) $n = 2$, $m > 2$ and N is of positive constant curvature $c/12$.

This theorem follows from Theorem 5 and the fact that $N \subset P^m(\mathbf{C}) \subset P^m(\mathbf{H})$ can be regarded as a totally real minimal submanifold of $P^m(\mathbf{H})$.

REMARK 4. – If we use the similar argument as in [3], we may prove that N is in fact the Veronese surface given in Remark 2.

Similarly, by using Theorem 6, we have the following.

THEOREM 8. — *Let N be an n -dimensional compact totally real minimal submanifold of the complex projective space $P^n(\mathbf{C})$ of constant holomorphic sectional curvature c . If we have*

$$(4.12) \quad K \geq \frac{(n-2)c}{4(2n-1)},$$

then either (i) N is totally geodesic in $P^n(\mathbf{C})$ or (ii) $n = 2$ and N is a flat surface.

Theorem 8 improves slightly the main theorem of [5].

REMARK 5. — It shall be noticed that Theorems 5 and 7 are independent of codimension of N in the ambient spaces.

5. — Real-space-forms in quaternion-space-forms.

In this section, we shall assume that N is of constant sectional curvature c' isometrically immersed in a $4n$ -dimensional quaternion-space-form $M^n(c)$ of quaternion sectional curvature c .

LEMMA 2. — *Let N be an n -dimensional totally real minimal submanifold in a quaternion-space-form $M^n(c)$. If N has constant sectional curvature c' , then*

$$(5.1) \quad \operatorname{tr} \left(\sum_{\varphi, i} A_{\varphi(i)}^2 \right)^2 = \sum_{\varphi, i, j} (\operatorname{tr} A_{\varphi(i)} A_{\varphi(j)})^2,$$

$$(5.2) \quad c' \leq c/4,$$

$$(5.3) \quad \sum_{\varphi} \operatorname{tr} (A_{\varphi(i)} A_{\varphi(j)}) = \|\sigma\|^2 \frac{\delta_{ij}}{n},$$

$$(5.4) \quad \sum_{\varphi, i, j} \operatorname{tr} (A_{\varphi(i)} A_{\varphi(j)})^2 - \sum_{\varphi, i, j} (\operatorname{tr} A_{\varphi(i)} A_{\varphi(j)})^2 = \left(c' - \frac{c}{4} \right) \|\sigma\|^2.$$

PROOF. — (1) From (2.9), we have

$$\begin{aligned} \operatorname{tr} \left(\sum_{\varphi(i)} A_{\varphi(i)}^2 \right)^2 &= \sum \operatorname{tr} A_{\varphi(i)}^2 A_{\varphi(j)}^2 = \sum h_{kl}^{\varphi(i)} h_{lm}^{\varphi(i)} h_{mh}^{\varphi(j)} h_{hk}^{\varphi(j)} \\ &= \sum h_{il}^{\varphi(k)} h_{li}^{\varphi(m)} h_{jh}^{\varphi(m)} h_{hj}^{\varphi(k)} = \sum (\operatorname{tr} A_{\varphi(k)} A_{\varphi(m)})^2. \end{aligned}$$

(2) Let the scalar curvature of N be ϱ . Then $\varrho = n(n-1)c'$. Thus

$$n(n-1)c' = \frac{1}{4}n(n-1)c - \|\sigma\|^2 \leq \frac{1}{4}n(n-1)c.$$

This implies $c' \leq c/4$.

(3) Since N is of constant sectional curvature c' . N is Einsteinian and the Ricci tensor S of N is $S = \rho g/n$. Thus by the equation of Gauss, we find

$$\sum A_{\varphi(i)}^2 = \left(\frac{1}{4}(n-1)c - \frac{\rho}{n} \right) E = \frac{\|\sigma\|^2}{n} E,$$

when E is the identity transformation. Hence

$$\frac{\|\sigma\|^2}{n} \delta_{ij} = \sum h_{il}^{\varphi(k)} h_{lj}^{\varphi(k)} = \sum h_{lk}^{\varphi(i)} h_{ki}^{\varphi(j)} = \sum_{\varphi} \operatorname{tr} (A_{\varphi(i)} A_{\varphi(j)}).$$

(4) The curvature tensor of N satisfies

$$R_{ijkl} = c'(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

On the other hand, the curvature tensor $\tilde{R}$ of M satisfies

$$\tilde{R}_{ijkl} = \frac{c}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

along N . Thus by the equation of Gauss, we have

$$\sum (h_{ik}^{\varphi(m)} h_{jl}^{\varphi(m)} - h_{il}^{\varphi(m)} h_{jk}^{\varphi(m)}) = \left(c' - \frac{c}{4} \right) (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Multiplying $\sum h_{il}^{\varphi(r)} h_{ik}^{\varphi(r)}$ on both sides and sum up, we get (4).

THEOREM 9. — *Let N be an n -dimensional totally real minimal submanifold in the $4n$ -dimensional quaternion projective space $P^n(\mathbf{H})$. If the sectional curvature of N is constant, say c' , then either (i) N is totally geodesic, (i.e., $c' = c/4$) or (ii) N is non-positively curved, i.e., $c' \leq 0$.*

PROOF. — If N has constant sectional curvature c' , then its scalar curvature $\rho = n(n-1)c'$ is also constant. Thus $\|\sigma\|^2 = \frac{1}{4}n(n-1)c - \rho$ is constant, too. By (3.8) we find

$$0 = \|\bar{\nabla}\sigma\|^2 + \sum \operatorname{tr} (A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha})^2 - \sum (\operatorname{tr} A_{\alpha}A_{\beta})^2 + \frac{(n+1)c}{4} \|\sigma\|^2.$$

Notice that $\alpha, \beta, \dots$ run on $\varphi(1), \dots, \varphi(n)$ for $\varphi = I, J, K$. From these we get

$$\begin{aligned} 0 &= \|\bar{\nabla}\sigma\|^2 + 2 \sum \operatorname{tr} (A_{\varphi(i)}A_{\varphi(j)})^2 - 2 \sum (\operatorname{tr} A_{\varphi(i)}^2 A_{\varphi(j)}^2) - \sum (\operatorname{tr} A_{\varphi(i)}A_{\varphi(j)})^2 + \frac{(n+1)c}{4} \|\sigma\|^2 \\ &= \|\bar{\nabla}\sigma\|^2 + 2 \sum \operatorname{tr} (A_{\varphi(i)}A_{\varphi(j)})^2 - 2 \operatorname{tr} (\sum A_{\varphi(i)}^2)^2 - \sum (\operatorname{tr} A_{\varphi(i)}A_{\varphi(j)})^2 + \frac{(n+1)c}{4} \|\sigma\|^2. \end{aligned}$$

As in the proof of our previous theorems, we may assume that $\operatorname{tr} A_{\varphi(i)}A_{\varphi(j)} = 0$ for $i \neq j$. Using Lemma 2, we have

$$0 = \|\bar{\nabla}\sigma\|^2 + 2 \sum \operatorname{tr} (A_{\varphi(i)}A_{\varphi(j)})^2 - 3 \operatorname{tr} (\sum A_{\varphi(i)}^2)^2 + \frac{(n+1)c}{4} \|\sigma\|^2.$$

Substituting (3) and (4) of Lemma 2 into this equation, we find

$$\begin{aligned}\sum \operatorname{tr} (A_{\varphi(i)} A_{\varphi(j)})^2 &= \sum (\operatorname{tr} A_{\varphi(i)} A_{\varphi(j)})^2 + \left(c' - \frac{c}{4}\right) \|\sigma\|^2 \\ &= \sum (\operatorname{tr} A_{\varphi(i)}^2)^2 + \left(c' - \frac{c}{4}\right) \|\sigma\|^2 \\ &= \frac{\|\sigma\|^4}{n} + \left(c' - \frac{c}{4}\right) \|\sigma\|^2.\end{aligned}$$

Thus, we have

$$0 = \|\bar{\nabla}\sigma\|^2 + \frac{(n-1)c + 8c'}{4} \|\sigma\|^2 - \frac{\|\sigma\|^4}{n}.$$

Combining this with $\|\sigma\|^2 = n(n-1)(c/4 - c')$, we get

$$\|\bar{\nabla}\sigma\|^2 = n(n^2-1)c' \left(c' - \frac{c}{4}\right).$$

From this and (2) of Lemma 2, we deduce that $c' = c/4$ or $c' \leq 0$.

The following result is an immediate consequence of Theorem 9.

THEOREM 10. — *Let N be an n -dimensional totally real minimal submanifold in the $4n$ -dimensional quaternion projective space $P^n(\mathbf{H})$. If (1) the sectional curvature of N is constant and (2) the second fundamental form σ is parallel, then either (i) N is totally geodesic or (ii) N is flat.*

REMARK 6. — Since the Veronese surface in $P^4(\mathbf{H})$ is a minimal totally real surface of constant curvature $c/12$ with parallel second fundamental form. Theorems 9 and 10 are not true for arbitrary codimension.

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