Integral transforms related to heat conduction.

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Summary. - *Relations between such transcendental differential operators as $\exp tD_x^2$ and $\exp xD_t^{1/2}$ are obtained. A new characterization of temperature functions represented by the classical POISSON integral is obtained, as well as one for positive temperature functions.*

Introduction. - In the study of heat conduction there are several different basic methods of attack upon a given boundary value problem; for example a) series expansions, b) the POISSON integral, c) convolution integrals. The worker in the field learns by experience which is likely to be useful for a given problem. But no study seems to have been made of the relations between the various methods in the cases where all are applicable. Such a study is one of the purposes of the present note. For example, a temperature function $u(x, t)$,

$$(1) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

is determined by its CAUCHY data, $u(0, t) = f(t)$, $u_x(0, t) = g(t)$. For certain functions $f(t)$ and $g(t)$ this temperature function will also have a POISSON integral representation,

$$(2) \quad u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) F(y) dy$$

$$k(x, t) = (4\pi t)^{-1/2} e^{-x^2/4t} \quad t > 0.$$

What relations exist between the three functions $f(t)$, $g(t)$, $F(y)$?

The key to the study is the basic transform

$$(3) \quad f(t) = \int_0^{\infty} k(y, t) d\alpha(y).$$

When this integral converges absolutely for $0 < t < \infty$ we say that $f(t)$ belongs to class A and indicate the relation between the functions $f(t)$ and $\alpha(y)$ by the symbol $[f(t), d\alpha(y)]$. Or if $\alpha(y)$ is absolutely continuous,

$$(4) \quad \alpha(y) = \int_0^y \varphi(r) dr,$$

we write $[f(t), \varphi(y)]$. One convenient physical interpretation of the latter relation is that $f(t)$ may be regarded as the temperature at the origin of an infinite bar at time t , the initial temperature having been zero to the right of the origin and defined by $\varphi(y)$ to the left. Explicitly, $u(x, t)$ is defined by (2) with $F(y) = 0$ for $y > 0$, $F(y) = \varphi(-y)$ for $y < 0$. Then assuming the relation (3) (4) we have $u(0, t) = f(t)$.

A further purpose of the paper is to investigate the use of the operator $\sqrt{D}$, where D stands for differentiation with respect to t , in the theory of heat conduction. The need for such a study is clearly indicated by the following considerations. A familiar LAPLACE transform is

$$(5) \quad e^{-x\sqrt{s}} = \int_0^{\infty} e^{-sr} h(x, r) dr \quad x > 0, s > 0,$$

where $h(x, t) = -2k_x(x, t)$. Replacing s by D in (5) and adopting the usual interpretation of e^{aD} as a translation,

$$e^{aD}f(t) = f(a + t),$$

we obtain

$$e^{-x\sqrt{D}}f(t) = \int_0^{\infty} f(t-r)h(x, r)dr.$$

If $f(t)$ vanishes for negative t we thus obtain a familiar convolution representation of temperature functions,

$$(6) \quad u(x, t) = e^{-x\sqrt{D}}f(t) = \int_0^t h(x, t-r)f(r)dr.$$

That the function (6) should satisfy (1) is symbolically evident,

$$u_{xx} = e^{-x\sqrt{D}}(-\sqrt{D})^2 f(t) = u_t = e^{-x\sqrt{D}}f'(t).$$

It turns out to be, for $x > 0$, the function $u(x, t)$ of the previous paragraph. In this note we give the obvious series definitions of the operators $\sinh x\sqrt{D}$, $\cosh x\sqrt{D}$ and $e^{x\sqrt{D}}$, and show that these operators are applicable to any function of class A . Moreover, the resulting functions satisfy (1) and have POISSON integral representations.

A new characterization of functions $u(x, t)$ with absolutely convergent POISSON integral representations is obtained. The criterion is in terms of $u(0, t)$ and $u_x(0, t)$. It is then used to show the relation between positive solutions of (1) and completely monotonic functions. In particular it is noted that positive solutions of (1), for $t > 0$, can be constructed from absolutely monotonic functions $f(t)$, for $t > 0$, by the operator $\cosh x\sqrt{D}f(t)$.

1. A basic integral transform. Let us introduce notations for several functions of frequent occurrence in the theory of heat conduction.

DEFINITION 1.1. - For $-\infty < x < \infty$, $0 < t < \infty$

$$\begin{aligned} k(x, t) &= \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \\ h(x, t) &= (x/t)k(x, t) \\ l(x, t) &= \operatorname{erfc}(x/\sqrt{4t}) = \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-r^2} dr. \end{aligned}$$

The function $k(x, t)$ is the familiar « source solution » of the heat equation (1). We call attention to the following relation between these functions:

$$l_t = l_{xx} = h, \quad l_x = -2k, \quad 2k_x = -h.$$

We now introduce two useful classes of functions.

DEFINITION 1.2. - $f(t) \in A \iff t^{-1/2}f(t^{-1})$ has a Laplace-Stieltjes transform which is absolutely convergent for $0 < t < \infty$:

$$\begin{aligned} (1.1) \quad \frac{1}{\sqrt{t}} f\left(\frac{1}{t}\right) &= \int_0^{\infty} e^{-t\nu} d\alpha(y) \\ \int_0^{\infty} e^{-t\nu} |d\alpha(y)| &< \infty \quad 0 < t < \infty. \end{aligned}$$

For example, $t^c \in A$ if $c \geq -1/2$, $t^c \notin A$ if $c < -1/2$. Throughout this paper, and without further mention, we shall assume that in any STIELTJES integral transform the integrator function is a normalized function of bounded variation in every finite interval:

$$\alpha(0) = 0, \quad \alpha(t+) + \alpha(t-) = 2\alpha(t) \quad t \neq 0.$$

DEFINITION 1.3. - $f(t) \in A^* \iff t^{-1/2}f(t^{-1})$ is a Laplace-Lebesgue transform which is absolutely convergent for $0 < t < \infty$:

$$\begin{aligned} (1.2) \quad \frac{1}{\sqrt{t}} f\left(\frac{1}{t}\right) &= \int_0^{\infty} e^{-t\nu} \varphi(y) dy \\ \int_0^{\infty} e^{-t\nu} |\varphi(y)| dy &< \infty \quad 0 < t < \infty. \end{aligned}$$

For example, $t^c \in A^*$ if $c > -1/2$; $t^{-1/2} \notin A^*$. It is clear that $A^* \subset A$ but not conversely. In (1.2) and in other LEBESGUE transforms of the present paper it will be assumed that the transformed function (here $\varphi(y)$) belongs to L in every finite interval.

The basic integral transform to be studied here is

$$(1.3) \quad f(t) = \int_0^\infty k(y, t) d\alpha(y)$$

or the special case

$$(1.4) \quad f(t) = \int_0^\infty k(y, t) \varphi(y) dy.$$

Our first result follows.

THEOREM 1.1.

$$f(t) \in A \iff (1.3) \text{ holds,}$$

$$f(t) \in A^* \iff (1.4) \text{ holds,}$$

the integrals converging absolutely for $0 < t < \infty$.

The proof follows simply from the following changes of variables. Equations (1.3) and (1.4) are equivalent to

$$(1.5) \quad \frac{1}{\sqrt{t}} f\left(\frac{1}{4t}\right) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\nu t} d\alpha(\sqrt{y})$$

$$(1.6) \quad \frac{1}{\sqrt{t}} f\left(\frac{1}{4t}\right) = \frac{1}{\sqrt{4\pi}} \int_0^\infty e^{-\nu t} \frac{\varphi(\sqrt{y})}{\sqrt{y}} dy,$$

respectively.

The transforms (1.3) and (1.4) will be abbreviated by bracket notation as follows.

DEFINITION 1.4.

$$[f(t), d\alpha(y)] \iff f(t) \in A \text{ and (1.3) holds;}$$

$$[f(t), \varphi(y)] \iff f(t) \in A^* \text{ and (1.4) holds;}$$

$f(t)$ is the generating function, $\alpha(y)$ or $\varphi(y)$ is the determining function, $k(y, t)$ is the kernel.

For example, if $U(y) = 0$ for $y \leq 0$ and $= 1$ for $y > 0$, then $[1/\sqrt{4\pi t}, dU(y)]$. Also [1, 2]. Further examples follow in § 2.

2. A brief table of transforms. We give here a brief table of the more important transforms. It will serve to show what kinds of functions can be represented and will illustrate later theory. It may be checked by a table of LAPLACE transforms using (1.5) (1.6).

$$f(t) = \int_0^{\infty} k(y, t) \varphi(y) dy = \int_0^{\infty} k(y, t) d\alpha(y)$$

	$f(t)$	$\varphi(y)$	$\alpha(y)$
1.	1	2	$2y$
2.	t	y^2	$\frac{y^3}{3}$
3.	t^n	$\frac{2(n!)}{(2n)!} y^{2n}$	$\frac{2(n!)}{(2n+1)!} y^{2n+1}$
4.	$\frac{\sqrt{t}}{\sqrt{\pi}}$	y	$\frac{y^2}{2}$
5.	$\frac{t^n \sqrt{t}}{\sqrt{\pi}}$	$\frac{y^{2n+1}}{4^n n!}$	$\frac{y^{2n+2}}{2^{2n+1} (n+1)!}$
6.	$\frac{1}{\sqrt{4\pi t}}$		$U(y)$
7.	$k(x, t)$		$U(y-x)$
8.	$\frac{b}{2} e^{b^2 t} \operatorname{erfc} b \sqrt{t}$	$b e^{-b^2 y}$	$1 - e^{-b^2 y}$
9.	$\frac{1}{2} (1 + 4tb)^{-1/2}$	$e^{-b^2 y^2}$	$\frac{\sqrt{\pi}}{\sqrt{4b}} \operatorname{erf} y \sqrt{b}$
10.	$l(x, t)$	$2U(y-x)$	$2(y-x)U(y-x)$

In formulas 3 and 5 $n = 0, 1, 2, \dots$. In 7 and 10, $x > 0$. In 8, $-\infty < b < \infty$; in 9, $b \geq 0$. The functions 6 and 7 belong to A but not to A^* .

3. Order of increase of determining function. For $f(t)$ to belong to A it is necessary that the corresponding determining function should not increase too rapidly. To obtain the relation we appeal to the following known result. See I. I. HIRSCHMANN and D. V. WIDDER (*) [1955; 121].

LEMMA 3.1.

1. $\varphi(y) > 0, \varphi'(y) \leq 0$ $0 < y < \infty$
2. $\int_0^\infty \varphi(y) d\alpha(y)$ converges
3. $\varphi(\infty) = 0$
- $\Rightarrow$ $\alpha(y) = o(1/\varphi(y))$ $y \rightarrow \infty$.

THEOREM 3.1.

$$[f(t), d\alpha(y)] \Rightarrow \lim_{y \rightarrow \infty} y^{-2} \log |\alpha(y)| \leq 0.$$

By hypothesis the integral (1.3) converges for any fixed $t = t_0 > 0$. The function $k(y, t_0)$ is positive and tends monotonically to zero with $1/y$. Hence by the lemma

$$\alpha(y) = o(e^{y^2/4t_0}) \quad y \rightarrow \infty,$$

and from this the conclusion is evident. A proof could also be given using (1.5) and D. V. WIDDER [1946; 39].

We may also draw conclusions about $f(t)$ as $t \rightarrow 0$.

THEOREM 3.2.

$$\begin{aligned} [f(t), d\alpha(y)] &\Rightarrow f(t) \sim \alpha(0+) / \sqrt{4\pi t} & t \rightarrow 0+, \\ [f(t), \varphi(y)] &\Rightarrow f(t) = o_+(1/\sqrt{t}) & t \rightarrow 0+. \end{aligned}$$

For, from equation (1.5) and D. V. WIDDER [1946; 182]

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} f\left(\frac{1}{4t}\right) = \frac{\alpha(0+)}{\sqrt{\pi}}.$$

This is equivalent to the first conclusion of the theorem. The second is a consequence of the first if

$$\alpha(y) = \int_0^y \varphi(r) dr.$$

4. Properties of the basic transform. By virtue of equations (1.5) (1.6), many of the properties of the transforms (1.3) and (1.4) can be inferred from the theory of the LAPLACE transform. First, it is convenient to observe that

(*) The notation $[m; n]$ means page n in article dated m of the bibliography.

either integral may be differentiated with respect to t , under the integral sign, as often as desired; and the resulting integral will remain absolutely convergent for $0 < t < \infty$. This follows from a corresponding property of LAPLACE transforms. Also it is important to observe that either transform can be inverted so that either function of the pair $[f(t), d\alpha(y)]$ or $[f(t), \varphi(y)]$ can be obtained from the other by integration. The precise results follow.

THEOREM 4.1.

$$[f(t), d\alpha(y)] \Rightarrow$$

$$(4.1) \quad \alpha(y) = \lim_{R \rightarrow \infty} \frac{1}{i\sqrt{4\pi}} \int_{c-iR}^{c+iR} \frac{e^{sy}}{\sqrt{s^3}} f\left(\frac{1}{4s}\right) ds \quad 0 < y < \infty,$$

for any constant $c > 0$.

The integration is along a vertical line in the complex s -plane. That branch of $\sqrt{s^3}$ is intended which is positive on the positive real axis. We recall that $\alpha(y)$ is normalized. From (1.5) we have

$$\frac{\alpha(\sqrt{y})}{\sqrt{\pi}} = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{e^{sy}}{\sqrt{s^3}} f\left(\frac{1}{4s}\right) ds$$

by the standard inversion of the LAPLACE transform. The conclusion of the theorem follows at once.

THEOREM 4.2.

$$[f(t), \varphi(y)], \varphi(y) \text{ of bounded variation,}$$

$$\Rightarrow$$

$$(4.2) \quad \lim_{R \rightarrow \infty} \frac{y}{i\sqrt{\pi}} \int_{c-iR}^{c+iR} \frac{e^{sy}}{\sqrt{s}} f\left(\frac{1}{4s}\right) ds = \frac{\varphi(y+) + \varphi(y-)}{2} \quad y > 0$$

for any $c > 0$.

This result follows from (1.6) and a standard inversion. But observe that the assumed absolute convergence implied in $[f(t), \varphi(y)]$ is an essential hypothesis for the validity of (4.2), though not for (4.1).

These two theorems of course imply the uniqueness of the representations (1.3), (1.4). That is, $\alpha(y)$ is uniquely determined by $f(t)$ and $\varphi(y)$ is determined almost everywhere.

5. Related transforms. In view of the importance of the functions $h(x, t)$ and $l(x, t)$ in the theory of heat conduction it seems desirable to consider

in addition to the transform (1.3) the related ones

$$(5.1) \quad f(t) = \int_0^{\infty} h(y, t) d\alpha(y)$$

$$(5.2) \quad f(t) = \int_0^{\infty} l(y, t) d\alpha(y).$$

We first discuss convergence.

LEMMA 5.1. - *For any $t_0 > 0$ and $y \rightarrow \infty$*

$$k(y, t_0) = O(e^{-y^2/4t_0})$$

$$h(y, t_0) = O(ye^{-y^2/4t_0})$$

$$l(y, t_0) = O(y^{-1}e^{-y^2/4t_0}).$$

The first two results are immediate consequences of Definition 1.1. The last may be proved by l'Hospital's rule.

THEOREM 5.1. - *If any one of the integrals (1.3) (5.1) (5.2) converges absolutely for $0 < t < \infty$, so do the others.*

From the lemma it is clear that for $0 < t_0 < t_1 < t_2$

$$(5.3) \quad l(y, t_0) \leq k(y, t_0) \leq h(y, t_0) \leq k(y, t_1) \leq l(y, t_2),$$

at least for all y sufficiently large. Hence the result can be proved by direct comparison of the integrals involved. For example,

$$\int_0^{\infty} h(y, t_0) |d\alpha(y)| \ll \int_0^{\infty} l(y, t_2) |d\alpha(y)|,$$

so that the absolute convergence of (5.2) at t_2 implies that of (5.1) for any smaller positive value of t . Since t_2 may be taken arbitrarily large the desired result is immediate.

Differentiation under the integral signs in (5.1) and (5.2) is permissible.

THEOREM 5.2. - *If (5.2) converges absolutely for $0 < t < \infty$. then*

$$(5.4) \quad f'(t) = \int_0^{\infty} h(y, t) d\alpha(y)$$

$$(5.5) \quad f^{(n+1)}(t) = \int_0^{\infty} \frac{\partial^n}{\partial t^n} h(y, t) d\alpha(y) \quad n = 0, 1, \dots$$

the integrals converging absolutely for $0 < t < \infty$.

By (5.3) the absolute convergence of (5.2) implies the uniform convergence of the integral (5.4) in a neighborhood of every $t_0 > 0$. This establishes equation (5.4), which is equivalent to

$$\frac{1}{\sqrt{t^3}} f' \left(\frac{1}{4t} \right) = \frac{4}{\sqrt{\pi}} \int_0^\infty e^{-\nu t} \sqrt{y} d\alpha(\sqrt{y}).$$

And now (5.5) may be inferred from properties of the LAPLACE transform.

THEOREM 5.3. - *If (5.2) holds, the integral converging absolutely for $0 < t < \infty$, then*

$$\alpha(y) = \lim_{R \rightarrow \infty} \frac{y}{i\sqrt{4\pi}} \int_{c-iR}^{c+iR} \frac{e^{sy^2}}{\sqrt{s}} ds \quad 0 < y < \infty$$

for any $c > 0$.

If we integrate (5.2) by parts we obtain

$$(5.6) \quad f(t) = 2 \int_0^\infty k(y, t) \alpha(y) dy.$$

The integrated term has vanished by virtue of Lemma 5.1 and Theorem 3.1, which also establish the absolute convergence of the integral (5.6). An application of Theorem 4.2 now yields the desired inversion.

THEOREM 5.4. - *If (5.1) holds, the integral converging absolutely for $0 < t < \infty$, then*

$$\int_0^y r d\alpha(r) = \lim_{R \rightarrow \infty} \frac{1}{8i\sqrt{\pi}} \int_{c-iR}^{c+iR} \frac{e^{sy^2}}{\sqrt{s^3}} f \left(\frac{1}{4s} \right) ds \quad y > 0$$

for any constant $c > 0$.

For, equation (5.1) is equivalent to

$$(5.7) \quad tf(t) = \int_0^\infty k(y, t) y d\alpha(y).$$

The conclusion now results from an application of Theorem 4.1. Observe that for the transform (5.2) the determining function is uniquely determined by the generating function. For (5.1) $\alpha(y)$ is determined up to an additive constant by $f(t)$.

6. LAPLACE transforms of the kernels. It is convenient to record the LAPLACE transforms of the three kernels of Definition 1.1.

THEOREM 6.1.

$$(6.1) \quad \int_0^{\infty} e^{-st} l(x, t) dt = \frac{e^{-x\sqrt{s}}}{s} \quad 0 < s < \infty, 0 \leq x < \infty$$

$$(6.2) \quad \int_0^{\infty} e^{-st} k(x, t) dt = \frac{e^{-x\sqrt{s}}}{\sqrt{4s}} \quad 0 < s < \infty, 0 \leq x < \infty$$

$$(6.3) \quad \int_0^{\infty} e^{-st} h(x, t) dt = e^{-x\sqrt{s}} \quad 0 < s < \infty, 0 < x < \infty.$$

See, for example, H. BATEMAN [1954; 177], formula 11, for equation (6.1). The second two may be obtained from the first by differentiation with respect to x .

By means of the composition theorem for LAPLACE transforms we can now obtain certain useful convolution relations between the kernels.

THEOREM 6.2. - For $x > 0$, $t > 0$

$$(6.4) \quad l(x, t) = \frac{2}{\sqrt{\pi t}} * k(x, t) = \frac{2}{\sqrt{\pi}} \int_0^t \frac{k(x, r)}{\sqrt{t-r}} dr$$

$$(6.5) \quad 2k(x, t) = \frac{1}{\sqrt{\pi t}} * h(x, t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{h(x, r)}{\sqrt{t-r}} dr.$$

By (6.3) and (6.2) equation (6.5) follows from the identity

$$\left(\frac{1}{\sqrt{s}}\right) e^{-x\sqrt{s}} = \frac{e^{-x\sqrt{s}}}{\sqrt{s}}.$$

For

$$\int_0^{\infty} \frac{e^{-st}}{\sqrt{t}} dt = \frac{\sqrt{\pi}}{\sqrt{s}} \quad 0 < s < \infty.$$

Likewise (6.4) is equivalent to

$$\left(\frac{1}{\sqrt{s}}\right) \frac{e^{-x\sqrt{s}}}{\sqrt{s}} = \frac{e^{-x\sqrt{s}}}{s}.$$

THEOREM 6.3. - For $x > 0$, $y > 0$, $t > 0$

$$(6.6) \quad k(x, t) * h(y, t) = \int_0^t k(x, r) h(y, t-r) dr = k(x+y, t)$$

$$(6.7) \quad 4k(x, t) * k(y, t) = l(x, t) * h(y, t) = l(x+y, t).$$

These two results follow, respectively, from the two identities

$$\begin{aligned} \left(\frac{e^{-x\sqrt{s}}}{\sqrt{4s}} \right) e^{-y\sqrt{s}} &= \frac{e^{-(x+y)\sqrt{s}}}{\sqrt{4s}} \\ \left(\frac{e^{-x\sqrt{s}}}{\sqrt{s}} \right) \left(\frac{e^{-y\sqrt{s}}}{\sqrt{s}} \right) &= \left(\frac{e^{-x\sqrt{s}}}{s} \right) e^{-y\sqrt{s}} = \frac{e^{-(x+y)\sqrt{s}}}{s}. \end{aligned}$$

7. Fractional derivatives and integrals. We adopt the RIEMANN-LIOUVILLE definition of fractional derivative and integral.

DEFINITION 7.1. - For $c > 0$

$$\begin{aligned} D^{-c}f(t) &= f^{(-c)}(t) = \int_0^t \frac{(t-r)^{c-1}}{\Gamma(c)} f(r) dr \\ D^c f(t) &= f^{(c)}(t) = D^n D^{c-n} f(t), \end{aligned}$$

where $n-1$ is the largest integer contained in c .

For example,

$$\begin{aligned} \sqrt{D} t^{-1/2} &= D \frac{1}{\sqrt{\pi}} \int_0^t \frac{dr}{\sqrt{r(t-r)}} = D \sqrt{\pi} = 0 \\ \sqrt{D} t^n &= \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} t^{n-\frac{1}{2}} \end{aligned} \quad n > -1/2.$$

THEOREM 7.1. - For $x > 0$, $t > 0$

$$(7.1) \quad \frac{1}{\sqrt{D}} k(x, t) = \frac{1}{2} l(x, t)$$

$$(7.2) \quad \frac{1}{\sqrt{D}} h(x, t) = 2k(x, t)$$

$$(7.3) \quad \sqrt{D} l(x, t) = 2k(x, t)$$

$$(7.4) \quad \sqrt{D} k(x, t) = \frac{1}{2} h(x, t).$$

The first two of these results are restatements of (6.4) and (6.5), respectively. Equation (7.4) comes from (7.1) by differentiation. From (7.1)

$$\frac{1}{\sqrt{D}} u(x, t) = \frac{2}{D} k(x, t) = 2 \int_0^t k(x, r) dr.$$

And now (7.3) comes by differentiation of the latter equation.

THEOREM 7.2.

$$[f(t), d\alpha(y)] \Rightarrow [f^{(-1/2)}(t), \alpha(y)].$$

That is, if $f(t) \in A$, then $f^{(-1/2)}(t) \in A^*$. By hypothesis

$$f(t) = \int_0^\infty k(y, t) d\alpha(y),$$

so that

$$f^{(-1/2)}(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{dr}{\sqrt{t-r}} \int_0^\infty k(y, r) d\alpha(y).$$

By FUBINI's theorem we may interchange the order of integration provided

$$(7.5) \quad \int_0^t \frac{g(r)}{\sqrt{t-r}} dr < \infty, \quad g(r) = \int_0^\infty k(y, r) |d\alpha(y)|.$$

By Theorem 3.2

$$g(r) \sim \frac{|\alpha(0+) |}{\sqrt{4\pi r}} \quad r \rightarrow 0+,$$

so that (7.5) is evident. Hence by (7.1)

$$(7.6) \quad \begin{aligned} f^{(-1/2)}(t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty d\alpha(y) \int_0^t \frac{k(y, r)}{\sqrt{t-r}} dr \\ &= \frac{1}{2} \int_0^\infty u(y, t) d\alpha(y) \quad 0 < t < \infty. \end{aligned}$$

Now integration by parts gives

$$f^{(-1/2)}(t) = \int_0^\infty k(y, t) \alpha(y) dy,$$

the integrated term vanishing by Theorem 3.1. The integral converges absolutely, $0 < t < \infty$, by the same theorem.

COROLLARY 6.2.

$$[f(t), \varphi(y)] \Rightarrow [f^{(-1/2)}(t), \varphi^{(-1)}(y)].$$

By successive application of this result we could show that

$$[f(t), \varphi(y)] \Rightarrow [f^{(-n/2)}(t), \varphi^{(-n)}(y)].$$

It should be noted that $[f(t), \varphi(y)]$ does not imply $[\sqrt{D}f(t), \varphi'(y)]$ even when $\varphi'(y)$ exists. For example, if $f(t) = 1$, $\varphi(y) = 2$, then $\sqrt{D}f(t) = (\pi t)^{-1/2}$, $\varphi'(y) = 0$. But $t^{-1/2}$ does not belong to A^* .

THEOREM 7.3. - If $\alpha(0+) = \alpha(0) = 0$,

$$\begin{aligned} [f^{(-1/2)}(t), d\alpha(y)] &\Leftrightarrow \int_0^t f(r) dr = \frac{1}{2} \int_0^\infty l(y, t) d\alpha(y) \\ &\Leftrightarrow f(t) = \frac{1}{2} \int_0^\infty h(y, t) d\alpha(y), \end{aligned}$$

the integrals converging absolutely for $0 < t < \infty$.

For, if

$$\frac{1}{\sqrt{D}} f(t) = \int_0^\infty k(y, t) d\alpha(y),$$

we have by (7.6)

$$\int_0^t f(r) dr = \frac{1}{2} \int_0^\infty l(y, t) d\alpha(y).$$

Differentiation with respect to t gives

$$(7.7) \quad f(t) = \frac{1}{2} \int_0^\infty l(y, t) d\alpha(y) = \frac{1}{2} \int_0^\infty h(y, t) d\alpha(y) \quad 0 < t < \infty.$$

Conversely, assume (7.7), the integral converging absolutely. Integrating fractionally we have

$$\begin{aligned} \frac{1}{\sqrt{D}} f(t) &= \frac{1}{\sqrt{4\pi}} \int_0^t \frac{dr}{\sqrt{t-r}} \int_0^\infty h(y, r) d\alpha(y) \\ &= \frac{1}{\sqrt{4\pi}} \int_0^\infty d\alpha(y) \int_0^t \frac{h(y, r)}{\sqrt{t-r}} dr. \end{aligned}$$

By (7.2) the inner integral is $k(y, t)$ for $y > 0$ but is zero for $y = 0$. Since $\alpha(y)$ is assumed continuous at $y = 0$ the discontinuity of the inner integral there does not prevent the existence of the iterated STIELTJES integral. In fact

$$\int_0^\infty |\alpha(y)| \int_0^t \frac{h(y, r)}{\sqrt{t-r}} dr = \int_0^\infty k(y, t) |\alpha(y)| < \infty.$$

The latter integral is finite by the assumed absolute convergence of (7.7) and by Theorem 5.1. Hence FUBINI's theorem is applicable and the theorem is proved.

8. CAUCHY's problem, by series. Let us introduce a notation for solutions of the heat equation.

DEFINITION 8.1. - $u(x, t) \in H$ in a region $\Leftrightarrow u(x, t) \in C^2$ and $u_{xx}(x, t) = u_t(x, t)$ there.

CAUCHY's problem may be stated as follows. Given functions $f(t)$ and $g(t)$; find $u(x, t) \in H$ so that $u(x_0, t) = f(t)$, $u_x(x_0, t) = g(t)$. We shall suppose, without essential loss of generality, that $x_0 = 0$.

THEOREM 8.1.

1.	$u(x, t) \in H$	$ x < \delta, 0 < t < \infty$
2.	$u(0, t) = u_x(0, t) = 0$	$0 < t < \infty$
$\Rightarrow$	$u(x, t) = 0$	$ x < \delta, 0 < t < \infty.$

If $u(x, t) \in H$ in a region then for fixed t it is analytic in x and for fixed x it is of class C^∞ in t . See E. GOURSAT [1915; 314]. Choose $t_0 > 0$. From the differential equation we have

$$\frac{\partial^{2k} u}{\partial x^{2k}} = \frac{\partial^k u}{\partial t^k}, \quad \frac{\partial^{2k+1} u}{\partial x^{2k+1}} = \frac{\partial^k}{\partial t^k} \left(\frac{\partial u}{\partial x} \right)$$

for $|x| < \delta$ and $0 < t < \infty$. By hypothesis the right-hand sides of these equations are zero for $x = 0, t = t_0$. That is, $u(x, t_0)$ and all its derivatives are zero at $x = 0$. Hence by the analyticity of $u(x, t)$ as a function of x , $u(x, t_0)$ is identically zero for $|x| < \delta$. Since t_0 is arbitrary the proof is complete.

THEOREM 8.2.

1.	$f(t)$ and $g(t)$ are analytic for $\operatorname{Re} t > 0$	
2.	$u(x, t) \in H$	$-\infty < x < \infty, 0 < t < \infty$
3.	$u(0, t) = f(t), u_x(0, t) = g(t)$	
$\Rightarrow$		
A.	$u(x, t) = \sum_{k=0}^{\infty} f^{(k)}(t) \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} g^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!}$	
B.	$u(x_0, t), -\infty < x_0 < \infty$, is analytic for $\operatorname{Re} t > 0$	
C.	$u(x, t_0), 0 < t_0 < \infty$, is entire in x .	

Since $f(t)$ and $g(t)$ are analytic in the complex t -plane, $t = \sigma + i\tau$, for $\sigma > 0$ there exists for each real $t_0 > 0$ and for any $\rho < t_0/2$ a constant M such that

$$|f^{(k)}(t)| < \frac{Mk!}{\rho^k}, \quad |g^{(k)}(t)| < \frac{Mk!}{\rho^k}, \quad k = 0, 1, \dots$$

in the circle $|t - t_0| \leq t_0 - 2\rho$. Hence the above series, with any fixed real x , converges uniformly for t in that circle. By WEIERSTRASS' theorem its sum is an analytic function of t there and the series may be differentiated with respect to t there. For a fixed positive t the series is a power series convergent for all x . Denote the sum of the series by $v(x, t)$. An arbitrary point $t = \sigma + i\tau$ with $\sigma > 0$ may be included in a circle $|t - t_0| \leq t_0 - 2\rho$ by proper choice of ρ and t_0 . Hence $v(x, t)$ is analytic for $\operatorname{Re} t = \sigma > 0$ and $v(x, t_0)$, $t_0 > 0$, is entire. Differentiation of the series shows that $v(x, t) \in H$ for $-\infty < x < \infty$, $0 < t < \infty$. And $v(0, t) = f(t)$, $v_x(0, t) = g(t)$. Consequently the function $u(x, t) - v(x, t)$ satisfies the conditions of Theorem 8.1 (with $\delta = \infty$) and is consequently identically zero. This completes the proof.

We observe that the theorem contains an obvious uniqueness result: there is only one solution of the heat equation satisfying the CAUCHY conditions 3.

9. Physical interpretation of the basic transform. The relation $[f(x), d\alpha(y)]$ may be interpreted as a relation between temperature distributions in a variety of ways. Before showing this let us recall the fundamental facts about the POISSON integral representation of a temperature function,

$$(9.1) \quad u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) d\alpha(y).$$

THEOREM 9.1. - If (9.1) converges absolutely for $-\infty < x < \infty$, $0 < t < \infty$, then

- A. $u(x, t_0)$, $0 < t_0 < \infty$, is an entire function of x
- B. $u(x_0, t)$, $-\infty < x_0 < \infty$, is analytic in t for $\operatorname{Re} t > 0$
- C. All derivatives of $u(x, t)$ may be obtained by differentiation of the integrand (9.1),
- D. $u(x, t) \in H$ for $-\infty < x < \infty$, $0 < t < \infty$
- E. $\alpha(y) = \lim_{t \rightarrow 0+} \int_0^y u(r, t) dr \quad -\infty < y < \infty.$

The first four properties may be easily deduced from the theory of the LAPLACE transform if (9.1) is given one or the other of the two forms

$$u(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{xy/2t} e^{-y^2/4t} d\alpha(y)$$

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y/4t} d\alpha(x - \sqrt{y}).$$

See I. HIRSCHMAN and D. V. WIDDER [1955], Chapter VIII. For E we find

$$\int_0^y u(r, t) dr = \int_0^{\infty} [k(y-r, t) - k(r, t)] \alpha(r) dr.$$

The conclusion now follows from the above reference [1955; 189].

In the relation $[f(t), \varphi(y)]$ the function $f(t)$ may be regarded as the temperature, variable with the time t , at a fixed point of a bar which initially has its temperature assigned by the function $\varphi(y)$. As we shall see, this assignment may be done in a variety of ways, all producing the same function $f(t)$. We treat the more general STIELTJES relation $[f(t), d\alpha(y)]$ from which the other can easily be deduced.

THEOREM 9.2.

1. $[f(t), d\alpha(y)]$
2. $u(x, t) = \int_0^{\infty} k(x-y, t) d\alpha(y)$
- $\Rightarrow$
- A. $u(0, t) = f(t), u_x(0, t) = \sqrt{D} \dot{f}(t)$
- B. $\lim_{t \rightarrow 0^+} \int_0^y u(r, t) dr = \alpha(y) \quad 0 < y < \infty$
 $= 0 \quad -\infty < y \leq 0.$

The integral of hypothesis 2 converges absolutely for $-\infty < x < \infty$. This can be seen by direct comparison with the integral (1.4), using (5.3). Conclusion B follows from (9.1). The first of equations A is (1.4). By conclusion C of Theorem 9.1

$$(9.2) \quad u_x(0, t) = - \int_0^{\infty} k_y(y, t) d\alpha(y).$$

On the other hand, from (7.6)

$$(9.3) \quad \frac{1}{\sqrt{D}} f(t) = \frac{1}{2} \int_0^{\infty} l(y, t) d\alpha(y).$$

Since $l(y, t) = -2k_y(y, t)$, we have by differentiation of (9.3) that

$$(9.4) \quad \sqrt{D} f(t) = - \int_0^{\infty} k_y(y, t) d\alpha(y),$$

and this with (9.2) completes the proof.

In particular if $\alpha(y)$ is absolutely continuous, hypothesis 1 becomes $[f(t), \varphi(y)]$ and conclusion B becomes

$$\lim_{t \rightarrow 0+} u(y, t) = \varphi(y) \text{ for almost all } y > 0 \\ = 0 \quad y < 0.$$

That is, the bar is initially at temperature zero to the left of the origin, at temperature $\varphi(y)$ at a point $x = y$ to the right.

A second interpretation of the relation $[f(t), d\alpha(y)]$ follows.

THEOREM 9.3.

1. $[f(t), d\alpha(y)]$
2. $u(x, t) = \frac{1}{2} \int_0^{\infty} [k(x - y, t) + k(x + y, t)] d\alpha(y)$
- $\Rightarrow$
- A. $u(0, t) = f(t), \quad u_x(0, t) = 0 \quad 0 < t < \infty$
- B. $\lim_{t \rightarrow 0+} \int_0^y u(r, t) dr = \alpha(y)/2 \quad 0 < y < \infty$
 $= -\alpha(-y)/2 \quad -\infty < y < 0.$

The function $u(x, t)$ is now the sum of two of the transforms of Theorem 9.2 in one of which x is replaced by $-x$, so that the conclusions are immediate. In like manner we obtain the following result.

THEOREM 9.4.

1. $[f(t), d\alpha(y)]$
2. $u(x, t) = \frac{1}{2} \int_0^{\infty} [k(x - y, t) - k(x + y, t)] d\alpha(y)$
- $\Rightarrow$
- A. $u(0, t) = 0, \quad u_x(0, t) = \sqrt{D} f(t)$
- B. $\lim_{t \rightarrow 0+} \int_0^y u(r, t) dr = \alpha(y)/2 \quad 0 < y < \infty$
 $= \alpha(-y)/2 \quad -\infty < y < 0.$

10. CAUCHY's problem, by integrals. In § 8 we solved CAUCHY's problem by series. We are now able to solve it in terms of integrals if the two given functions belong to class *A*.

THEOREM 10.1

1. $[f(t), d\alpha(y)], [g(t), d\beta(y)]$
 2. $u(x, t) \in H \quad -\infty < x < \infty, \quad 0 < t < \infty$
 3. $u(0, t) = f(t), \quad u_x(0, t) = g(t) \quad 0 < t < \infty$
- $\Rightarrow$

$$(10.1) \quad u(x, t) = \frac{1}{2} \int_0^\infty [k(x-y, t) + k(x+y, t)] d\alpha(y) \\ + \frac{1}{2} \int_0^\infty [k(x-y, t) - k(x+y, t)] \beta(y) dy.$$

Denote the values of the first and second integrals (10.1) by $v(x, t)$ and $w(x, t)$, respectively. By Theorems 9.3 and 9.4.

$$v(0, t) = f(t), \quad v_x(0, t) = 0, \quad w(0, t) = 0.$$

Also by direct differentiation

$$w_x(0, t) = - \int_0^\infty k_y(y, t) \beta(y) dy = \frac{1}{2} \int_0^\infty h(y, t) \beta(y) dy.$$

But the hypothesis $[g(t), d\beta(y)]$,

$$g(t) = \int_0^\infty k(y, t) d\beta(y),$$

gives after integration by parts

$$g(t) = \frac{1}{2} \int_0^\infty h(y, t) \beta(y) dy,$$

the integrated part vanishing by Theorem 3.1. That is

$$w_x(0, t) = g(t) \quad 0 < t < \infty.$$

By Theorem 9.1 the function $v(x, t) + w(x, t)$ satisfies the same conditions as were assumed for $u(x, t)$ in hypotheses 2 and 3. Thus by Theorem 8.1 equation (10.1) is established.

By comparing Theorems 10.1 and 8.2 we see that $v(x, t)$ and $w(x, t)$ both have series developments.

COROLLARY 10.1

$$\begin{aligned} & \Rightarrow [f(t), d\alpha(y)] \\ A. \quad & \frac{1}{2} \int_0^\infty [k(x-y, t) + k(x+y, t)] d\alpha(y) = \sum_{k=0}^\infty f^{(k)}(t) \frac{x^{2k}}{(2k)!} \\ B. \quad & \frac{1}{2} \int_0^\infty [k(x-y, t) - k(x+y, t)] \alpha(y) dy = \sum_{k=0}^\infty f^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

To prove this first choose $\beta(y) = 0$ and then $\alpha(y) = 0$ in Theorem 10.1 and compare it with Theorem 8.2. Also A may be verified from B by differentiation.

As an example choose $\alpha(y) = 1$ for $y > 0$, $\alpha(0) = 0$. Then $f(t) = (4\pi t)^{-1/2}$ and A becomes

$$(10.2) \quad k(x, t) = \frac{1}{\sqrt{4\pi t}} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{x^{2n}}{(4t)^n}.$$

This can also be verified by TAYLOR's series.

11. Operational considerations. It is interesting to give operational interpretations of the three integrals of § 9. We first make the following definitions.

DEFINITION 11.1

$$(11.1) \quad \cosh x\sqrt{D} f(t) = \sum_{k=0}^\infty f^{(k)}(t) \frac{x^{2k}}{(2k)!}$$

$$(11.2) \quad \sinh x\sqrt{D} f(t) = \sum_{k=0}^\infty g^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!}, \quad g(t) = \sqrt{D} f(t).$$

$$(11.3) \quad e^{x\sqrt{D}} f(t) = \cosh x\sqrt{D} f(t) + \sinh x\sqrt{D} f(t),$$

We shall show that these operators have meanings for any function $f(t)$ of class A and indeed that the resulting functions have integral representations.

THEOREM 11.1

$$\begin{aligned} & \Rightarrow [f(t), d\alpha(y)] \\ A. \quad & \cosh x\sqrt{D} f(t) = \frac{1}{2} \int_0^\infty [k(x-y, t) + k(x+y, t)] d\alpha(y) \\ B. \quad & \sinh x\sqrt{D} f(t) = \frac{1}{2} \int_0^\infty [k(x-y, t) - k(x+y, t)] d\alpha(y) \\ C. \quad & e^{x\sqrt{D}} f(t) = \int_0^\infty k(x-y, t) d\alpha(y) \end{aligned}$$

for $-\infty < x < \infty$, $0 < t < \infty$.

The first result is a restatement of conclusion *A* of Corollary 10.1. The integral *B* defines the function $u(x, t)$ of Theorem 9.4. Consequently

$$u(0, t) = 0, \quad u_x(0, t) = \sqrt{D} f(t) = g(t).$$

By (9.4)

$$\sqrt{D} f(t) = \frac{1}{2} \int_0^\infty h(y, t) d\alpha(y),$$

and now (5.7) and (1.5) show that $g(t)$ is analytic for $\operatorname{Re} t > 0$. Hence Theorem 8.2 is applicable with $f(t) = 0$, $g(t) = \sqrt{D} f(t)$. We thus obtain the series development (11.2) of the integral *B*.

Finally equation *C* is obtained by adding equations *A* and *B*. As an example, choose $\alpha(y) = 1$ for $0 < y < \infty$, $\alpha(0) = 0$. Then $f(t) = (4\pi t)^{-1/2}$ and three integrals *A*, *B*, *C* define the functions $k(x, t)$, 0 , $k(x, t)$, respectively. Here $\sqrt{D} f(t) = 0$. For this case the series (11.1) (11.2) (11.3) can be computed directly, yielding the results predicted; compare (10.2).

12. Convolution transforms. If $x > 0$ the operator $e^{-x\sqrt{D}}$ has still another integral representation when applied to a function $f(t)$ of class *A*. The integral defines the convolution $h(x, t) * f(t)$.

THEOREM 12.1.

$$\begin{aligned} & [f(t), d\alpha(y)] \\ \Rightarrow & \\ (12.1) \quad & e^{-x\sqrt{D}} f(t) = \int_0^t h(x, t-r) f(r) dr \quad x > 0, t > 0. \end{aligned}$$

By (1.3) we have

$$\begin{aligned} \int_0^t h(x, t-r) f(r) dr &= \int_0^t h(x, t-r) dr \int_0^\infty k(y, r) d\alpha(y) = \\ &= \int_0^\infty d\alpha(y) \int_0^t h(x, t-r) k(y, r) dr = \int_0^\infty k(x+y, t) d\alpha(y). \end{aligned}$$

Here we have used equation (6.6) and FUBINI's theorem, the latter being applicable if

$$(12.2) \quad \int_0^\infty |k(x+y, t)| d\alpha(y) < \infty.$$

But the latter inequality was established for all x and t in the proof of Theorem 9.2. It should be noted that equation (12.1) is not in general true for negative x . For example if $\alpha(y) = 1$ for $y > 0$, $\alpha(0) = 0$, then the left hand side of (12.1) is $k(x, t)$ for all x , an even function of x . But the right hand side is an odd function of x .

It is natural to inquire what becomes of equation (12.1) if $h(x, t)$ is replaced by $k(x, t)$ or $l(x, t)$.

THEOREM 12.2.

$$\begin{aligned} & [f(t), d\alpha(y)] \\ \Rightarrow & e^{-x\sqrt{D}} f^{(-1/2)}(t) = 2 \int_0^t k(x, t-r) f(r) dr \quad x > 0, t > 0. \end{aligned}$$

By (1.3) we have

$$\begin{aligned} (12.3) \quad & \int_0^t k(x, t-r) f(r) dr = \int_0^t k(x, t-r) dr \int_0^\infty k(y, r) d\alpha(y) = \\ & = \int_0^\infty d\alpha(y) \int_0^t k(x, t-r) k(y, r) dr = \frac{1}{4} \int_0^\infty l(x+y, t) d\alpha(y). \end{aligned}$$

Here we have used equation (6.7) and FUBINI's theorem, the latter being applicable if

$$\int_0^\infty |l(x+y, t)| d\alpha(y) < \infty \quad t > 0.$$

But this follows from (12.2), using (5.3). The hypothesis implies $[f^{(-1/2)}(t), \alpha(y)]$. By Theorem 11.1, conclusion C, we have

$$\begin{aligned} e^{-x\sqrt{D}} f^{(-1/2)}(t) &= \int_0^\infty k(x+y, t) \alpha(y) dy \\ &= \frac{1}{2} \int_0^\infty l(x+y, t) d\alpha(y). \end{aligned}$$

The latter equation with (12.3) establishes the theorem.

THEOREM 12.3.

$$\begin{aligned} & [f(t), d\alpha(y)] \\ \Rightarrow & e^{-x\sqrt{D}} f^{(-1)}(t) = \int_0^t l(x, t-r) f(r) dr \quad x > 0, t > 0. \end{aligned}$$

Proceeding as in the previous proof, we obtain

$$\begin{aligned} \int_0^t l(x, t-r)f(r)dr &= \frac{1}{2} \int_0^t l(x, t-r)dr \int_0^\infty h(y, r)\alpha(y)dy = \\ &= \frac{1}{2} \int_0^\infty \alpha(y)dy \int_0^t l(x, t-r)h(y, r)dr = \frac{1}{2} \int_0^\infty l(x+y, t)\alpha(y)dy, \end{aligned}$$

provided

$$(12.4) \quad \int_0^\infty l(x+y, t) |\alpha(y)| dy < \infty.$$

We have here used the second of equations (6.7), and (12.4) again follows by comparison. By Theorem 7.2, applied twice, we have $[f^{(-1)}(t), \alpha^{(-1)}(y)]$, so that Theorem 11.1 yields

$$\begin{aligned} e^{-x\sqrt{D}}f^{(-1)}(t) &= \int_0^\infty k(x+y, t) \alpha^{(-1)}(y)dy \\ &= \frac{1}{2} \int_0^\infty l(x+y, t)\alpha(y)dy. \end{aligned}$$

We have integrated by parts, using Theorem 3.1 to show that the integrated part vanishes.

It is now easy to obtain the inversion of the three convolutions of the present section.

THEOREM 12.4.

$$\begin{aligned} &[f(t), d\alpha(y)] \\ \Rightarrow & \\ &\lim_{x \rightarrow 0^\pm} \int_0^t h(x, t-r)f(r)dr = \pm f(t) \\ &\lim_{x \rightarrow 0^\pm} \int_0^t k(x, t-r)f(r)dr = f^{(-1/2)}(t) \\ &\lim_{x \rightarrow 0^\pm} \int_0^t l(x, t-r)f(r)dr = f^{(-1)}(t) \end{aligned}$$

for $0 < t < \infty$.

The first of these integrals is equal, for $x > 0$, to

$$(12.5) \quad \int_0^{\infty} k(x+y, t) d\alpha(y),$$

and this integral is analytic in x for $-\infty < x < \infty$. Hence to obtain the first result, $\lim x \rightarrow 0+$, we have only to set $x=0$ in (12.5). The limit as $x \rightarrow 0-$ follows since $h(x, t)$ is odd in x . The remaining results follow in a similar way, noting that $k(x, t)$ is even and $l(x, t) - 1$ is odd in x .

13. Representation by the POISSON integral. We can now obtain a new characterization of those temperature functions which can be represented by absolutely convergent POISSON integrals.

THEOREM 13.1.

1. $u(x, t) \in H$ $-\infty < x < \infty, 0 < t < \infty$
 2. $u(0, t) = f(t)$, where $[f(t), d\alpha(y)]$
 3. $u_x(0, t) = g(t)$, where $[g(t), d\beta(y)], \beta(0+) = \beta(0) = 0$
- $\Leftrightarrow$

$$(13.1) \quad u(x, t) = \int_{-\infty}^{\infty} k(x-y, t) d\omega(y),$$

where the integral converges absolutely for $-\infty < x < \infty, 0 < t < \infty$, $\omega(y)$ is normalized, and

$$(13.2) \quad 2\omega(y) = \beta(y) + \alpha(y) \quad 0 < y < \infty$$

$$(13.3) \quad 2\omega(-y) = \beta(y) - \alpha(y) \quad 0 < y < \infty.$$

We recall that a function $\omega(y)$ is normalized on $(-\infty, \infty)$ if $\omega(0) = 0$ and

$$2\omega(y) = \omega(y+) + \omega(y-) \quad -\infty < y < \infty;$$

see D. V. WIDDER [1946; 237]. Let us prove first the necessity of the conditions, assuming the representation (13.1). The first follows from Theorem 9.1. For the second and third we have

$$\begin{aligned} u(0, t) &= \int_0^{\infty} k(y, t) d\omega(y) + \int_{-\infty}^0 k(y, t) d\omega(y) \\ &= \int_0^{\infty} k(y, t) d[\omega(y) - \omega(-y)] \end{aligned}$$

$$u_x(x, t) = -\frac{1}{2} \int_{-\infty}^{\infty} h(x-y, t) d\omega(y)$$

$$u_x(0, t) = \frac{1}{2} \int_0^{\infty} h(y, t) d[\omega(y) + \omega(-y)].$$

If we now define functions $\alpha(y)$ and $\beta(y)$ by equations (13.2) and (13.3) we will have

$$(13.4) \quad u(0, t) = \int_0^{\infty} k(y, t) d\alpha(y)$$

$$(13.5) \quad u_x(0, t) = \frac{1}{2} \int_0^{\infty} h(y, t) d\beta(y).$$

Since $\omega(y)$ is normalized, $\beta(0+) = \beta(0) = 0$. Equation (13.4) proves condition 2 and (13.5), with the help of Theorem 7.3, proves condition 3.

Conversely, assume conditions 1, 2, 3. The functions $\alpha(y)$ and $\beta(y)$ are then given and $\omega(y)$ may be defined therefrom by (13.2) and (13.3). It will be normalized by virtue of the assumption 3, with $\beta(0+) = \beta(0) = 0$. The integral transform (13.1) of this function $\omega(y)$ will define a function $v(x, t)$ which also satisfies conditions 1, 2, 3, as we have just seen. By Theorem 8.1 $v(x, t) = u(x, t)$, and the proof is complete.

As an application of this theorem let us solve CAUCHY's problem when the two given functions $f(t)$ and $g(t)$ belong to A^* .

COROLLARY 13.1.

1. $u(x, t) \in H \quad -\infty < x < \infty, 0 < t < \infty$
 2. $u(0, t) = f(t)$, where $[f(t), \varphi(y)]$
 3. $u_x(0, t) = g(t)$, where $[g(t), \psi(y)]$
- $\Rightarrow$

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t) v(y) dy,$$

where the integral converges absolutely for $-\infty < x < \infty, 0 < t < \infty$ and

$$2v(y) = \varphi(y) + \int_0^y \psi(r) dr \quad 0 < y < \infty$$

$$2v(-y) = \varphi(y) - \int_0^y \psi(r) dr \quad 0 < y < \infty.$$

This result is also a corollary of Theorem 10.1. Its converse is not true as the example $v(y) = y^{-3/4}, y > 0, \gamma(-y) = -\gamma(y)$, shows.

As a further application of the theorem let us characterize the temperature functions of Theorems 9.2, 9.3, 9.4. To simplify the statement let us introduce a name for a class of functions already employed.

DEFINITION 13.1.

$$f(t) \in B \iff [f^{(-1/2)}(t), {}^r d\alpha(y)], \alpha(0+) = \alpha(0) = 0.$$

For example, $t^{-3/4} \in B$, $\notin A$.

THEOREM 13.2. - If $u(x, t) \in H$, $-\infty < x < \infty$, $0 < t < \infty$, and $f(t) \in A$, then

$$A. \quad u(x, t) = \cosh x \sqrt{D} f(t) \iff u(0, t) \in A, u_x(0, t) = 0$$

$$B. \quad u(x, t) = \sinh x \sqrt{D} f(t) \iff u(0, t) = 0, u_x(0, t) \in B$$

$$C. \quad u(x, t) = e^{x\sqrt{D}} f(t) \iff u(0, t) \in A, u_x(0, t) \in B$$

$$\text{and } \sqrt{D} u(0, t) = u_x(0, t) \quad 0 < t < \infty.$$

These conclusions are, respectively, the cases $\omega(y) = -\omega(y)$, $\omega(y) = \omega(-y)$ and $\omega(y) = 0$ for $-\infty < y \leq 0$ in Theorem 13.1. In the last case $\omega(y)$ is in general not normalized at the origin, but none the less

$$\sqrt{D} u(0, t) = u_x(0, t) = \frac{1}{2} \int_0^\infty h(y, t) d\omega(y).$$

Note that $k(x, t)$ has both of the representations A and C , for $k(0, t) = (4\pi t)^{-1/2} \in A$ and $\sqrt{D} (4\pi t)^{-1/2} = 0$.

We record also the special case of this theorem in which $f(t) \in A^*$.

COROLLARY 13.2. - If $u(x, t) \in H$, $-\infty < x < \infty$, $0 < t < \infty$ and $f(t) \in A^*$, then

$$A. \quad u(x, t) = \cosh x \sqrt{D} f(t) \iff u(0, t) \in A^*, u_x(0, t) = 0$$

$$B. \quad u(x, t) = \sinh x \sqrt{D} f(t) \iff u(0, t) = 0, \frac{1}{\sqrt{D}} u_x(0, t) \in A^*$$

$$C. \quad u(x, t) = e^{x\sqrt{D}} f(t) \iff u(0, t) = \frac{1}{\sqrt{D}} u_x(0, t) \in A^*.$$

Let us illustrate the three classes by examples. The classical polynomial temperature functions $V_n(x, t)$ are defined by the equation

$$e^{xy+y^2t} = \sum_{n=0}^{\infty} \frac{y^n}{n!} V_n(x, t).$$

Our examples are:

$$A. \quad \cosh x \sqrt{D} t^n = \frac{(2n)!}{n!} V_{2n}(x, t) \quad n = 0, 1, \dots$$

$$B. \quad \sinh x \sqrt{D} (t^n / \sqrt{t}) = \frac{\Gamma(n + \frac{1}{2})}{(2n-1)!} V_{2n-1}(x, t) \quad n = 1, 2, \dots$$

$$C. \quad e^{-x\sqrt{D}} 1 = V_0(x, t).$$

We may verify directly the characteristic properties of the three classes by noting that $V_{2n}(x, t)$ is even, $V_{2n+1}(x, t)$ is odd, in x , and that

$$V_{2n}(0, t) = \frac{(2n)!}{n!} t^n \in A^*$$

$$\frac{\partial}{\partial x} V_{2n+1}(0, t) = (2n+1) V_{2n}(0, t)$$

$$\sqrt{D} l(0, t) = \sqrt{D} 1 = 1/\sqrt{\pi t} \in A^*$$

$$l_x(0, t) = -2k(0, t) = -1/\sqrt{\pi t}.$$

14. Positive temperature functions. It is known that temperature functions $u(x, t)$ which are non-negative for all x and for all positive t have the POISSON integral representation (13.1) with non-decreasing determining function $\omega(y)$; see D. V. WIDDER [1944; 85]. We now give a new characterization of this class of functions in terms of their behavior on a single line $x = \text{constant}$. We recall that a function $f(x)$ is completely monotonic in an interval if $(-1)^k f^{(k)}(x) \geq 0$ there; it is absolutely monotonic if $f^{(k)}(x) \geq 0$, $k = 0, 1, 2, \dots$.

THEOREM 14.1. - If $u(x, t) \in H$ for $-\infty < x < \infty$, $0 < t < \infty$ and if $u(0, t) = f(t)$, $u_x(0, t) = g(t)$, then $u(x, t) \geq 0$ there

$\Leftrightarrow$

A. $\left[f\left(\frac{1}{t}\right) \pm g^{(-1/2)}\left(\frac{1}{t}\right) \right] \frac{1}{\sqrt{t}}$ are completely monotonic for $t > 0$

B. $g^{(-1/2)}(t) = o\left(\frac{1}{\sqrt{t}}\right)$ $t \rightarrow 0+$.

We note first that $[f(t), d\alpha(y)]$ with $\alpha(y)$ non-decreasing if and only if $f(1/t)/\sqrt{t}$ is completely monotonic for $0 < t < \infty$. This follows from (1.1) and BERNSTEIN's theorem; see D. V. WIDDER [1946; 160].

Referring now to Theorem 13.1 we see that $u(x, t)$ will have the representation (13.1) with non-decreasing $\omega(y)$ if and only if the functions $\alpha(y) \pm \beta(y)$ are non decreasing, where

$$f(t) = \int_0^\infty k(y, t) d\alpha(y)$$

$$g^{(-1/2)}(t) = \int_0^\infty k(y, t) d\beta(y).$$

Condition *A* now follows from our preliminary remark. From Theorem 3.2 we see that the continuity of $\beta(y)$ at the origin, required in Theorem 13.1, is equivalent to condition *B*.

From this theorem we can obtain an easily applicable sufficient condition for the positiveness of a temperature function.

THEOREM 14.2.

1. $u(x, t) \in H$ $-\infty < x < \infty, 0 < t < \infty$
 2. $u(0, t)$ is absolutely monotonic for $0 < t < \infty$
 3. $u_x(0, t) = 0$
- $\Rightarrow$ $u(x, t) \geq 0$ $-\infty < x < \infty, 0 < t < \infty$.

In the presence of hypotheses 1 and 3 we see by Theorem 14.1 that $u(x, t) \geq 0$ for $-\infty < x < \infty, 0 < t < \infty$ if and only if $u(0, 1/t)/\sqrt{t}$ is completely monotonic for $0 < t < \infty$. Since $1/t$ and $1/\sqrt{t}$ are completely monotonic for $0 < t < \infty$ the same will be true of $u(0, 1/t)/\sqrt{t}$ under hypothesis 2; see D. V. WIDDER [1946; 143]. This proves the theorem.

A closely related result is the following.

THEOREM 14.3. - $f(t)$ is absolutely monotonic for $0 < t < \infty$

$$\Rightarrow \cosh x\sqrt{D} f(t) \in H \text{ and } \geq 0 \text{ for } -\infty < x < \infty, 0 < t < \infty.$$

The argument of the previous proof shows that $f(1/t)/\sqrt{t}$ is completely monotonic for $0 < t < \infty$, and hence by BERNSTEIN's theorem that $f(t) \in A$. By Theorem 11.1 the series defining $\cosh x\sqrt{D} f(t)$ converges and has the POISSON integral representation. By Theorem 9.1 the sum of the series belongs to H and by Theorem 14.1 is non-negative.

As an illustration we may use the first example following Corollary 13.2. The function t^n is absolutely monotonic for $0 < t < \infty$. In this case

$$\cosh x\sqrt{D} t^n = \frac{(2n)!}{n!} V_{2n}(x, t) \geq 0 \quad 0 < t < \infty.$$

We observe that the theorem shows the existence of positive temperature functions which increase arbitrarily rapidly on a line $x = \text{constant}$. For there exist absolutely monotonic functions increasing as rapidly as desired. For example, we may take $f(t) = e_n(t)$, where

$$e_1(t) = e^t, \quad e_{n+1}(t) = e_n(e^t), \quad n = 1, 2, \dots$$

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