On perfect and multiply perfect numbers.

by P. Erdös (in Haifa, Israel).

Summary. - *Denote by $P(x)$ the number of integers $n \leq x$ satisfying $\sigma(n) \equiv 0 \pmod{n}$, and by $P_2(x)$ the number of integers $n \leq x$ satisfying $\sigma(n) = 2n$. The author proves that $P(x) < x^{3/4+\varepsilon}$ and $P_2(x) < x^{(1-c)/2}$ for a certain $c > 0$.*

Denote by $\sigma(n)$ the sum of the divisors of n, $\sigma(n) = \sum_{d|n} d$. A number n is said to be perfect if $\sigma(n) = 2n$, and it is said to be multiply perfect if $\sigma(n) = kn$ for some integer k. Perfect numbers have been studied since antiquity. It is contained in the books of Euclid that every number of the form $2^{p-1}(2^p - 1)$ where both p and $2^p - 1$ are primes is perfect. Euler (1) proved that every even perfect number is of the above form. It is not known if there are infinitely many even perfect numbers since it is not known if there are infinitely many primes of the form $2^p - 1$. Recently the electronic computer of the Institute for Numerical Analysis the S.W.A.C. determined all primes of the form $2^p - 1$ for $p < 2300$. The largest prime found was $2^{2281} - 1$, which is the largest prime known at present.

It is not known if there are an odd perfect numbers. Euler (1) proved that all odd perfect numbers are of the form

$$p^{\alpha} m^2, \quad p \equiv \alpha \equiv 1 \pmod{4}, \tag{1}$$

and Sylvester (1) showed that an odd perfect number must have at least five distinct prime factors.

Multiply perfect numbers are known for various values of k but it is not known whether there are infinitely many multiply perfect numbers. Recently Kanold (2) proved that the density of multiply perfect numbers is 0. (i. e. the number of multiply perfect numbers not exceeding x is $o(x)$), and Hornfeck (3) proved that the number of perfect numbers not exceeding x is less than $x^{1/2}$.

Denote by $P(x)$ the number of multiply perfect numbers not exceeding x, and by $P_2(x)$ the number of perfect numbers not exceeding x. In the present note we are going to prove

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(1) Dickson, *History of the Theorie des Numbers*, Vol. 1, Chapter 1.
(2) « Journal für die reine und angew. Math. », 194 (1955), 218-220.
(3) « Archiv der Math. » 6 (1955), 442-443.

THEOREM 1.

$$P(x) < x^{3/4+\varepsilon} \text{ for every } \varepsilon > 0 \text{ and } x > x_0(\varepsilon).$$

THEOREM 2. - *There exists a constant* $c_1 > 0$ *so that for* $x > x_0$

$$P_2(x(< x^{(1-c_1)/2}.$$

These results are no doubt very far from being best possible. In fact it is very likely true that $P(x) = o(x^\varepsilon)$ for every $\varepsilon > 0$.

By more complicate arguments I can prove that for every constant c_2 there exists a constant c_3 so that the number of integers $n < x$ for which

$$(\sigma(n),\ n) > n^{c_2}$$

is less than x^{1-c_3}. By still more complicated arguments I can prove

THEOREM 3. - *Let* $f(x)$ *be an increasing function satisfying* $f(x) > (\log x)^{c_4}$ *for some* $c_4 > 0$. *Then the number of integers* $n < x$ *satisfying*

$$(\sigma(n),\ n) > f(x)$$

is less than $c_5 x/(f(x))^{c_6}$ *for some* $c_5 > 0$ *and* $c_6 > 0$. *The same result hold if* $\sigma(n)$ *is replaced by Euler' s* φ *function.*

We are not going to give the proof of Theorem 3. It can further be shown that Theorem 3 is best possible in the following sense: Let $f(x) = o((\log x)^\varepsilon)$ for every $\varepsilon > 0$. Then the number of integers $n < x$ satisfying

$$(\sigma(n),\ n) > f(x)$$

is greater than $x/(f(x))^{c_5}$ for every $c_5 > 0$, if x is sufficiently large.

Further I can prove the following.

THEOREM 4. - *The density of integers* n *satisfying*

$$(\sigma(n),\ n) < (\log\log n)^\alpha$$

equals $g(\alpha)$ *wkere* $g(\alpha)$, $0 < \alpha < \infty$ *is an increasing function satisfying* $g(0) = 0$, $g(\infty) = 1$. *The same result holds if* $\sigma(n)$ *is replaced by* $\varphi(n)$.

We supress the proof of Theorem 4.

Proof of Theorem 1. First we prove two Lemmas.

LEMMA 1. - $\sigma(n) < 2n \log\log n$ *for all sufficiently large* n.

Lemma 1 immediately follows from the result of LANDAU [4] according to which $\limsup_{n=\infty} \sigma(n)/n \log\log n = e^C$, (where $C = 0'577 \ldots$ is EULER' S constant).

Put $n = a_n \cdot b_n$ where

$$a_n = \prod_{\substack{p^\alpha \mid n \\ \alpha > 1}} p^\alpha, \qquad b_n = \prod_{\substack{p \mid n \\ p^2 \nmid n}} p$$

a_n is called the quadratic part of n and b_n the squarefree part of n.

(4) LANDAU, *Verteilung der Primzahlen*, Vol. 1, p. 217. LANDAU states his result for EULER' s φ function, but the result for $\sigma(n)$ follows immediately.

LEMMA 2. - *Denote by $g(x, A)$ the number of integers $n < x$ for which $a_n > A$. Then $g(x, A) < c_7 x/A^{1/2}$ where c_7 is an absolute constant independent of x and A.*

Clearly the quadratic part of n is the product of a square and a cube. Thus

$$g(x, A) < \sum_{a_n > A} \frac{x}{a_n} < \sum_{k^2 l^3 > A} \frac{x}{k^2 l^3} = x \sum_{l=1}^{\infty} \frac{1}{l^3} \sum_{k^2 > A/l^3} \frac{1}{k^2} = x \sum_{l^3 \leq A} \frac{1}{l^3} \sum_{k^2 > A/l^3} \frac{1}{k^2} +$$

$$x \sum_{l^3 > A} \frac{1}{l^3} < x \sum_{l^3 \leq A} \frac{1}{l^3} \left(\frac{l^3}{A}\right)^{1/2} + x \sum_{l^3 > A} \frac{1}{l^3} < c_7 x / A^{1/2} \text{ q. e. d.}$$

To prove Theorem 1 it will clearly be sufficient to show that

$$P(x) \quad P(x/2) < x^{3/4+\varepsilon} \text{ for } x > x_0(\varepsilon). \tag{2}$$

To prove (2) we split the multiply perfect numbers y satisfying $x/2 < y \leq x$ into two classes. In the first class are the y's with $a_y \geq x^{1/2}$. By Lemma 2 the number of the y's of the first class is less than $c_7 x^{3/4}$. For the y's of the second class we evidently have $b_y > \frac{1}{2} x^{1/2}$. Put

$$b_y = q_1 q_2 \cdots q_k, \; q_1 < q_2 < \cdots < q_k$$

where the q's are distinct primes. Define $b_y' = q_{k_1} q_{k_2} \cdots q_{k_r}$ where $q_{k_1} = q_k$ and q_{k_i} $(1 \leq i \leq r)$ is the largest q which does not divide $(q_{k_1}+1)(q_{k_2}+1)\ldots(q_{k_{i-1}}+1)$. Put $b_y = b_y' b_y''$. By our construction

$$(b_y', \sigma(b_y')) = 1, \; \sigma(b_y') \equiv 0 \pmod{b_y''} \text{ or } b_y'\sigma(b_y') \equiv 0 \pmod{b_y}. \tag{3}$$

Also since $b_y > \frac{1}{2} x^{1/2}$ we have by (3) and Lemma 1

$$\frac{1}{2} x^{1/2} < b_y \leq b_y' \sigma(b_y') < 2 b_y'^2 \operatorname{loglog} x$$

or

$$b_y' > \frac{1}{2} x^{1/4} / \operatorname{loglog} x. \tag{4}$$

Now $\sigma(y) \equiv 0 \pmod{y}$ and $\sigma(y) \equiv 0 \pmod{\sigma(b_y')}$ (since $\sigma(y) = \sigma(a_y) \cdot \sigma(b_y') \sigma(b_y'')$). Thus by (3)

$$\sigma(y) \equiv 0 \pmod{b_y' \sigma(b_y')}.$$

Now by Lemma 1 $\sigma(y) = ky < 2y \operatorname{loglog} x$. Thus

$$y B_x \equiv 0 \bmod (\sigma(y)) \equiv 0 \bmod (b_y' \sigma(b_y')) \text{ where } B_x = [\operatorname{loglog} x]\,!. \tag{5}$$

Hence by (4) and (5) if y belongs to the second class yB_x is divisible by an integer of the form $a\sigma(a)$ with $a > \frac{1}{2}x^{1/4}/\log\log x$. Thus the number of integers of the second class is less than (Σ' indicates that $a > \frac{1}{2}x^{1/4}(\log\log x)$

$$xB_x\Sigma'\frac{1}{a\sigma(a)} < xB_x\Sigma'\frac{1}{a^2} < 2B_x x^{3/4}\log\log x < x^{3/4+\varepsilon}.$$

which completes the proof of Theorem 1.

PROOF OF THEOREM 2. - The proof will be very similar to that of Theorem 1. Since by EULER's result the number of even perfect numbers not exceeding x is less than $\log x$, it suffices to consider odd perfect numbers. To prove Theorem 2 it will be sufficient to prove that

$$(6) \qquad P_2'(x) - P_2'\left(\frac{x}{2}\right) < x^{1/2-c_8},$$

where $P_2'(x)$ denotes the number of odd perfect numbers not exceeding x.

By (1) the odd perfect numbers are all of the form

$$y = p^\alpha m^2, \quad p \equiv \alpha \equiv 1 \pmod 4).$$

We now split the odd perfect numbers y satisfying $x/2 < y \leq x$ into three classes. In the first class are the y's for which $p^\alpha > x^{c_9}$. Thus if y is in the first class we have $m < x^{(1-c_9)/2}$. A simple argument shows that to each m there is at most one p^α so that $p^\alpha m^2$ is perfect ([5]). Hence the number of y's of the first class is less than $x^{(1-c_9)/2}$. For the y's of the second class we have $a_m > x^{2c_9}$. By Lemma 2 we obtain that the number of y's of the second class is less than $c_7 x^{(1-c_9)/2}$. For the y's of the third class we have $p^\alpha \leq x^{c_9}$, $a_m < x^{2c_9}$. Thus $b_m > \frac{1}{2}x^{(1-5c_9)/2}$. Put

$$b_m^2 = q_1^2 q_2^2 \dots q_k^2, \quad q_1 < q_2 < \dots < q_k$$

where the q's are distinct prime. Define $b_m'^2 = q_{k_1}^2 q_{k_2}^2 \dots q_{k_r}^2$ where $q_{k_1} = q_k$ and q_{k_i} $(1 < i \leq r)$ is the largest q which does not divide

$$(7) \qquad (q_{k_1}^2 + q_{k_1} + 1)(q_{k_2}^2 + q_{k_2} + 1) \dots (q_{k_{i-1}}^2 + q_{k_{i-1}})$$

and for which

$$(8) \qquad q_{k_j} \nmid (1 + q_{k_i} + q_{k_i}^2) \text{ for } (1 \leq j \leq i-1).$$

It follows from our construction that

$$(9) \qquad (b_m', \sigma(b_m'^2)) = 1$$

([5]) This follows immediately from the fact that $\frac{\sigma(p^\alpha)}{p^\alpha} \neq \frac{\sigma(q^\beta)}{q^\beta}$, HORNFECK's proof is also based on this idea.

and that if $q \mid b_m$, $q \neq q_{k_j}$, $1 \leq j \leq r$ then either (if (7) does not hold)

$$q \mid \sigma(b_m'^2), \tag{10}$$

or (if (8) does not hold)

$$1 + q + q^2 = \sigma(q^2) \equiv 0 \pmod{q_{k_j}} \text{ for some } 1 \leq j \leq r \text{ and } q < q_{k_j}. \tag{11}$$

Put now $b_m = b_m' b_m'' b_m'''$ where b_m'' is the product of the q's satisfying (10). Clearly Lemma 1

$$b_m'' \leq \sigma(b_m'^2) < 2b_m'^2 \log\log x. \tag{12}$$

Each prime factor of b_m''' satisfies (11). Thus for every $q \mid b_m'''$ $(1 + q + q^2, b_m') > q$. Now ($b_m' b_m'' b_m'''$ is squarefree)

$$\sigma(y) = 2y = 2p^\alpha (a_m b_m' b_m'' b_m''')^2 = \sigma(p^\alpha)\sigma(a_m^2)\sigma(b_m'^2)\sigma(b_m''^2)\sigma(b_m'''^2). \tag{13}$$

Thus for each $q_{k_j} \mid b_m'$, $q_{k_j}^2 \nmid \sigma(b_m''')$. Hence

$$(\sigma(b_m'''), b_m') \geq [\prod_{q \mid b_m'''} (1 + q + q^2, b_m')]^{1/2} > b_m'''^{1/2} \text{ [6]},$$

or

$$b_m''' < b_m'^2. \tag{14}$$

Thus from (12) and (14)

$$b_m < 2b_m'^5 \log\log x.$$

Thus since y belongs to the third class

$$b_m' > \frac{1}{4} x^{(1-5c_9)/10} / \log\log x. \tag{15}$$

Now by (13), (9) and since y is odd

$$y \equiv 0 [\mathrm{mod}\ (b_m'^2 \cdot \sigma(b_m'^2))] \text{ or } m \equiv 0 \pmod{b_m'} \text{ and } (m^2, \sigma(b_m'^2)) \geq \frac{\sigma(b_m'^2)}{p^\alpha}$$

or by (15) ($p^\alpha < x^{c_9}$)

$$m \equiv 0 \pmod{b_m'} \text{ and } (m, \sigma(b_m'^2)) \geq \left(\frac{\sigma(b_m'^2)}{p^\alpha}\right)^{1/2} > \frac{1}{4} x^{(1-10c_9)/10} / \log\log x. \tag{16}$$

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[6] To see this observe that if $q \mid b'_m$ there can be at most two prime factors q_1 and q_2 of b'''_m satisfying $\sigma(q_1^2) \equiv \sigma(q_2^2) \equiv 0 \pmod q$, also if $q \mid b'''_m$ $(\sigma(q^2), b'_m) > q$.

The number of integer $m \leq x^{1/2}$ satisfying (16) for a fixed b'_m is clearly less than (the dash indicates that $t > \frac{1}{4} x^{(1-10c_9)/10}/\log\log x$)

$$\frac{x^{1/2}}{b'_m} \sum_{t|\sigma(b_m^{2'})}' \frac{1}{t} < \frac{x^{1/2}}{b'_m} \frac{d(\sigma(b_m'^2)) \cdot 4 \log\log x}{x^{(1-10c_9)/10}} < \frac{x^{\frac{2}{5}+c_9+\varepsilon}}{b'_m} \tag{17}$$

where $d(n)$ denotes the number of divisors of n. Thus from (17) we obtain that the number of integers $m \leq x^{1/2}$ which satisfy (16) is less than

$$x^{\frac{2}{5}+c_9+\varepsilon} \sum_{b'_m < x} \frac{1}{b'_m} < x^{\frac{2}{5}+c_9+2\varepsilon}.$$

Thus the number of y, s of the third class is less than $x^{\frac{2}{5}+c_9+2\varepsilon} < x^{(1-c_9)/2}$ for sufficiently small c_9, which completes the proof of Theorem 2.

Added in proof: Denote by $Q_i(x)$ the number of odd integers $n < x$ satisfying $\sigma(n) = 2^i x$. WOLKMANN proved that $Q_i(x) = 0\ (x^{1-\frac{1}{2(i+2)}})$. (Journal für reine ung angew Math. 195 (1955), 154).