# Convergence and Evaluation of Sums of Reciprocal Powers of Eigenvalues of Certain Compact Operators (on Hilbert Space) which Are Meromorphic Functions of the Eigenvalue Parameter (*) (**).

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**Summary.** – *In this paper the trace equations*

$$\sum_i \lambda_i^{-p} = \int_0^1 k_p(x, x)\,dx$$

*arising in the Hilbert-Schmidt theory of Fredholm integral equations are extended to certain classes of compact operators $K(\lambda)$ on Hilbert space $\mathscr{H}$ which are meromorphic functions of the eigenvalue parameter $\lambda$. The operator $K(\lambda)$ is the sum of an operator valued polynomial $H(\lambda)$ plus an operator $P(\lambda)$ which is a meromorphic function of $\lambda$ and has finite dimensional range for each fixed $\lambda$. The theory is constructed so that if $\mathscr{H} = L_2[0, 1]$ and if*

$$H(\lambda) = \sum_{i=0}^{s} \lambda^i H_i$$

*where the $H_i$ are integral operators derivable from corresponding Lebesgue square integrable kernels $h_i(x, y)$, then one can systematically take advantage of various regularity conditions that some of the kernels $h_i(x, y)$ may have in improving the results.*

## I. – Introduction.

If $k(x, y)$ is a Lebesgue square integrable kernel on $[0, 1]\times[0, 1]$, then the formula

$$\sum_i \lambda_i^{-p} = \int_0^1 k_p(x, x)\,dx \tag{1.1}$$

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holds for integers $p \geq 2$, where $k_p(x, y)$ is the $p$-th iterate of $k(x, y)$, and where $\{\lambda_i\}$ is the sequence of eigenvalues of $k(x, y)$, taken according to algebraic multiplicity as zeroes of the classical Fredholm function. The formula (1.1) holds for $p = 1$ under suitable further restrictions on $k(x, y)$, as is well known (see CHANG [1], COCHRAN [3; pp. 242, 251-266], DUNFORD-SCHWARTZ [5; pp. 1116-9], HILLE-TAMARKIN [10], STINESPRING [23], SWANN [24]).

In this paper we generalize (1.1) to cover certain classes of compact operators (on a Hilbert space $\mathscr{H}$), which are meromorphic functions of the eigenvalue parameter $\lambda$. The case where the operator $H(\lambda)$ is a simple polynomial in $\lambda$ is covered in Part IV; in part V, we consider eigenvalues of $K(\lambda) = H(\lambda) + P(\lambda)$, where $P(\lambda)$ is a certain operator valued meromorphic function of $\lambda$ and has finite dimensional range for each fixed $\lambda$. Let us suppose that

$$H(\lambda) = \sum_{i=0}^{s} \lambda^i H_i \tag{1.2}$$

where the $H_i$ are certain classes of compact operators (to be defined in Part II). If it happens that $\mathscr{H} = L_2[0, 1]$ and that the operators $H_i$ are integral operators derived from square integrable kernels $h_i(x, y)$, then the theory developed will enable us *systematically* to take advantage of various regularity conditions that some of the kernels $h_i(x, y)$ may possess in order to improve our results. Such conditions are given in papers by CHANG [1], COCHRAN [3; pp. 231-248], STINESPRING [23], SWANN [24]; one such well known condition in terms of existence of partial derivatives of $h_i(x, y)$ is briefly reviewed in part II.

The results described in part II are a review of certain results in DUNFORD-SCHWARTZ [5; pp. 1088-1119]; we include them in order to make the paper as self-contained as possible, in order to facilitate referencing (since DUNFORD and SCHWARTZ do not number their equations for the most part) and in order to review briefly the rather specialized spaces of compact operators needed for this paper. The results in part III comprise a review of well-known material which is inaccessible in the literature.

In part IV we treat the eigenvalues of the operator valued polynomial $H(\lambda)$; in part V we discuss the eigenvalues of the meromorphic operator $K(\lambda)$. Finally, certain practical aspects of the uses of the formulas to be developed are treated in part VI. The major theorems in this paper are Theorems V.6, V.7, and V.8.

Our paper is a direct generalization of the work of MÜLLER [22], who considers operator-valued polynomials of degree two. The authors (see LAGINESTRA-BOYCE [13]) have previously obtained related results for certain boundary value problems for ordinary differential equations which are non-linear in the eigenvalue parameter; their methods in the latter paper were strictly classical analysis. GOODWIN [8] has also considered such problems. A few of our results apply to the case where the $H_i$ are merely compact, namely the assertions of corollaries V.5-2 and V.6-1, where we state that $[I-\lambda K(\lambda)]^{-1}$ has poles at the poles of $K(\lambda)$ and at the eigenvalues of the equation

$$\lambda K(\lambda)u = u \qquad (u \in \mathscr{H}). \tag{1.3}$$

These results are not necessarily new; they can be found in less general form in TAMARKIN [25; p. 148]. In addition, we construct Fredholm function(s) for $K(\lambda)$ in part V and investigate their properties.

Just for the record, if $K$ is any operator on $\mathscr{H}$, the eigenvalues $\lambda$ of $K$ will satisfy the equation $\lambda K u = u$ throughout this paper.

## II. – A review of the spaces $C(p)$ and of the concept of trace.

Let $A$ be any compact linear operator defined on the complex Hilbert space $\mathscr{H}$. Let $(A^*A)^{\frac{1}{2}}$ denote the nonnegative square root of $A^*A$, where $A^*$ is the adjoint of $A$. Then $(A^*A)^{\frac{1}{2}}$ is a compact self-adjoint operator on $\mathscr{H}$; its eigenvalues $\{\lambda_i[(A^*A)^{\frac{1}{2}}]\}$, written according to geometric multiplicity, are called the singular values of $A$. If $A^*A \neq 0$, i.e. if $A \neq 0$, the operator $A$ will always have at least one singular value. We write

$$\mu_i(A) = \lambda_i[(A^*A)^{\frac{1}{2}}] = \sqrt{\lambda_i[(A^*A)]} \tag{2.1}$$

and we note that (DUNFORD-SCHWARTZ [5; p. 1092])

$$\mu_i(A) = \mu_i(A^*). \tag{2.2}$$

The numbers $\mu_i(A)$ are positive, and the sequence $\{\mu_i(A)\}$ has no finite limit point.

The importance of the singular values of $A$ in this paper partially follows from the inequality (DUNFORD-SCHWARTZ [5; p. 1093])

$$\sum_i |\lambda_i(A)|^{-p} \leqslant \sum_i [\mu_i(A)]^{-p} \tag{2.3}$$

where $p$ is any positive real number. The eigenvalues of $A$ (at most denumerable, since $A$ is compact) are enumerated in the (possibly empty) sequence $\{\lambda_i(A)\}$ according to (geometric) multiplicity (which will be defined); the summations in (2.3) are taken over all elements of the sequences $\{\lambda_i(A)\}$ and $\{\mu_i(A)\}$. Convergence of the series on the right in (2.3) would obviously tell us something about the growth of the eigenvalues of $A$, provided, of course, that such eigenvalues exist. Indeed, convergence of the series $\sum_i \mu_i(A)^{-p}$ will enable us to «evaluate» the series $\sum_i [\lambda_i(A)]^{-p}$ provided $p$ is a positive integer.

Let us recall the meaning of the geometric multiplicity of the eigenvalue $\lambda = a$ of the operator $A$. Let $\mathscr{N}(A)$ denote the null-space of $A$, i.e. let

$$\mathscr{N}(A) = \{x \in \mathscr{H} : Ax = 0\}. \tag{2.4}$$

Let $\lambda = a$ be an eigenvalue of the compact operator $A$. If $A$ is compact, recall that there is a positive integer $i$ (depending on $a$) such that

$$\mathcal{N}[(I - aA)^i] = \mathcal{N}[(I - aA)^{i+1}]. \tag{2.5}$$

Furthermore, the spaces in (2.5) are finite dimensional for compact $A$ (see DUNFORD-SCHWARTZ [4; pp. 573, 579]). The space $\mathcal{N}[(I - aA)^j]$ is the $j$-th generalized eigenspace of $(I - aA)$; the geometric multiplicity of the eigenvalue $\lambda = a$ is the maximal dimension of the generalized eigenspaces of $(I - aA)$. If $A$ is a compact, self-adjoint operator, then the eigenvalue $\lambda = a$ of $A$ is real and we can take $i = 1$ in (2.5). In this case, equation (2.3) is trivial, since we may set $\mu_i(A) = |\lambda_i(A)|$.

Let $p$ be any positive real number, and let $C(p)$ denote the collection of all compact linear operators $A$ such that $\sum_i \mu_i(A)^{-p}$ converges. We write

$$|A|_p = \Big[\sum_i \mu_i(A)^{-p}\Big]^{1/p}. \tag{2.6}$$

Let $C(\infty)$ be the collection of all linear operators $A$ defined on $\mathcal{H}$ with the norm

$$|A|_\infty = \sup\{|Ax| : x \in \mathcal{H} \text{ and } |x| = 1\} < +\infty. \tag{2.7}$$

When it becomes necessary to refer to the Hilbert space $\mathcal{H}$ on which $A$ is defined, we shall frequently write $A \in C\{p; \mathcal{H}\}$ instead of $A \in C(p)$. This need to refine the notation obviously is important in discussions where several Hilbert spaces are under consideration. If $A$ is compact, then one can show (DUNFORD-SCHWARTZ [5; p. 1089]) that

$$|A|_\infty = \min_i \{[\mu_i(A)]^{-1}\}; \tag{2.8}$$

hence if $A \in C(p)$ for all sufficiently large positive $p$, we have that $|A|_p \to |A|_\infty$ as $p \to +\infty$.

If $1 \leqslant p \leqslant \infty$, then $C(p)$ is a Banach space. If $p$ satisfies $0 < p < 1$, then $C(p)$ is not a Banach space because $|\ |_p$ does not generally satisfy a triangle inequality; however $|\ |_p$ does satisfy the « unorthodox » triangle inequality

$$|A + B|_p^p \leqslant 2|A|_p^p + 2|B|_p^p \tag{2.9}$$

where the upper $p$ denotes a power; the set $C(p)$ is a linear manifold of $0 < p < 1$ and does have the property of completeness (DUNFORD-SCHWARTZ [5; p. 1095]). If $0 < p < q \leqslant \infty$, then (DUNFORD-SCHWARTZ [5; p. 1093])

$$C(p) \subseteq C(q) \tag{2.10}$$

and if $A \in C(p)$, then

$$|A|_q \leqslant |A|_p \tag{2.11}$$

if $0 < p < q \leqslant \infty$.

If $A \in C(p)$ and $B \in C(q)$, where $0 < p,\ q < \infty$, then (DUNFORD-SCHWARTZ [5; p. 1093])

$$AB \in C(r)\,, \qquad (r^{-1} = p^{-1} + q^{-1})\,. \tag{2.12}$$

Furthermore

$$|AB|_r \leqslant 2^{1/r}|A_p||B_q|\,. \tag{2.13}$$

If we restrict $p$ and $q$ so that $1 \leqslant p,\ q \leqslant \infty$ then we can improve (2.13) (DUNFORD-SCHWARTZ [5; p. 1105])

$$|AB|_r \leqslant |A|_p |B|_q\,, \qquad (r^{-1} = p^{-1} + q^{-1})\,. \tag{2.14}$$

If $A_i \in C(p_i)$, where $0 < p_i \leqslant \infty$ for each integer $i$ in $1 \leqslant i \leqslant n$, then

$$\left[\prod_{i=1}^{n} A_i\right] \in C(r), \quad \left[r^{-1} = \sum_{i=1}^{n} (p_i)^{-1}\right]. \tag{2.12$'$}$$

We shall need only the generalization of (2.14). If the $p_i$ are subjected to the additional restriction that $1 \leqslant p_i \leqslant \infty$, then

$$\left|\prod_{i=1}^{n} A_i\right|_r \leqslant \prod_{i=1}^{n} |A_i|_{p_i} \tag{2.14$'$}$$

where $r$ is defined in (2.12)$'$.

If $A \in C(1)$ and if $\{\varphi_k\}$ is any complete orthonormal basis for $\mathscr{H}$, then

$$\sum_i [\lambda_i(A)]^{-1} = \sum_\alpha \langle A\varphi_\alpha, \varphi_\alpha\rangle \tag{2.15}$$

where each of the series in (2.15) is absolutely convergent, and where all but a countable number of terms on the right side of (2.15) vanish. The limit of the series on the right side of (2.15) is independent of basis (DUNFORD-SCHWARTZ [5; pp. 1097, 1104]). We shall denote either of the quantities in (2.15) by $\tau(A)$, i.e. the trace of $A$. The operator $\tau$ is a linear functional on $C(1)$, and is continuous on $C(1)$ with respect to the $|\ |_1$ norm. Its continuity properties follow directly from the inequality

$$|\tau(A)| \leqslant |A|_1 \tag{2.16}$$

which is a restatement of inequality (2.3) (also see DUNFORD-SCHWARTZ [5; p. 1104]).

If $q$ is any positive integer then we may write $\lambda_i(A^q) = [\lambda_i(A)]^q$ by the spectral mapping theorem (DUNFORD-SCHWARTZ 4; p. 574), so if $A^q \in C(1)$, then

$$\sum_i [\lambda_i(A)]^{-q} = \sum_\alpha \langle A^q \varphi_\alpha, \varphi_\alpha \rangle . \tag{2.17}$$

Let $\mathscr{H} = L_2[0, 1]$ be the usual Hilbert space formed from the class of all complex-valued Lebesgue square-integrable functions defined on $[0, 1]$. Then (DUNFORD-SCHWARTZ [5; p. 1093]) $A \in C(2)$ if and only if there is a Lebesgue square integrable function $a(x, y)$ defined on $[0, 1] \times [0, 1]$ such that $a(x, y)$ is the kernel of $A$. (The class $C(2)$ is the class of all Hilbert-Schmidt operators on the arbitrary Hilbert space $\mathscr{H}$.) If $A \in C(1)$ then (see COCHRAN [3; p. 236]) if $\mathscr{H} = L_2[0, 1]$, we have

$$a(x, y) = \int_0^1 b(x, z) c(z, y)\, dz \tag{2.18}$$

where $b(x, y)$ and $c(x, y)$ are Lebesgue square integrable on $[0, 1] \times [0, 1]$. Hence $a(x, x)$ can be defined via (2.18) almost everywhere on $[0, 1]$, and is Lebesgue integrable on $[0, 1]$. We have (see COCHRAN [3; p. 243])

$$\tau(A) = \int_0^1 a(x, x)\, dx . \tag{2.19}$$

Mere existence of the integral in (2.19), indeed even mere continuity of $a(x, y)$ on the unit square, does not guarantee existence of $\tau(A)$ (see COCHRAN [3; p. 51]).

From what has just been said, one can easily show that the trace of $A$ and the trace of the operator whose kernel is

$$\int_0^1 c(x, z) b(z, y)\, dz \tag{2.20}$$

are equal. We shall discuss this more generally shortly.

Let $\mathscr{H} = L_2[0, 1]$, and let $A \in C(2)$. Let $a(x, y)$ be the square-integrable kernel corresponding to the operator $A$. Sufficient conditions for $A$ to be an element of $C(p)$ for $p < 2$ are given in DUNFORD-SCHWARTZ [5; p. 1117] in terms of existence of partial derivatives of $a(x, y)$, and also in terms of a Hölder condition on $a(x, y)$ (also, see CHANG [1]). Results can also be obtained if the variables $x$ and $y$ lie in a bounded region of a higher dimensional real Euclidean space (see DUNFORD-SCHWARTZ [5; p. 1119]).

As a result of the considerations of DUNFORD-SCHWARTZ [5; pp. 1116-1118]; if

$$\frac{\partial^k}{\partial y^k} a(x, y) = b(x, y) \tag{2.21}$$

is square integrable on $[0,1]\times[0,1]$, and if for some $\alpha$ in $0<\alpha<1$ and some constant $\Gamma>0$, we have

$$\left\{\int_0^1 |b(x,y)-b(x,z)|^2\,dx\right\}^{\frac{1}{2}} \leqslant \Gamma|y-z|^\alpha \tag{2.22}$$

then it can be shown that $A\in C(p)$ where $p>2/(1+2(k+\alpha))$. As a consequence of (2.2), we may replace $\partial/\partial y$ in (2.21) by $\partial/\partial x$, or assume a Hölder condition (2.22) with respect to the opposite variable. (See STINESPRING [23], SWANN [24] for further results).

We return to the general complex Hilbert space $\mathscr{H}$ in order to generalize some of the preceding. First we review a well known fact about the type of operators being studied; this will prove most useful in later considerations. Let $p$, $q$, and $r$ be positive reals satisfying

$$1/r = 1/p + 1/q \tag{2.23}$$

and suppose that $D\in C(r)$. We maintain that there exist operators $A\in C(p)$ and $B\in C(q)$ such that

$$D = AB\,. \tag{2.24}$$

In order to prove this, let $\{\mu_i(D)\}$ be the sequence of singular values of $D$, taken according to multiplicity, and let $\mu_i=\mu_i(D)$ for convenience. Let $\psi_i$ be an orthonormal sequence of eigenvectors of $(D^*D)^{\frac{1}{2}}$ satisfying

$$\mu_i\cdot[(D^*D)^{\frac{1}{2}}]\psi_i = \psi_i\,. \tag{2.25}$$

The sequence $\{\varphi_i\}$ defined by

$$\varphi_i = \mu_i D\psi_i \tag{2.26}$$

is then an orthonormal sequence of eigenvectors of $(DD^*)^{\frac{1}{2}}$ as is well known (COCHRAN [3; p. 213]). By the Hilbert theorem for compact self-adjoint operators (DUNFORD-SCHWARTZ [5; p. 905]), the vector $y$ given by

$$y = x - \sum_i \langle x, \psi_i\rangle\psi_i\,, \qquad (x\in\mathscr{H}) \tag{2.27}$$

satisfies $(D^*D)^{\frac{1}{2}}y=0$ or $Dy=0$. The series in (2.27) is taken over the whole sequence $\{\psi_i\}$. Hence by (2.26) and by the preceding we have

$$Dx = \sum_i \mu_i^{-1}\langle x, \psi_i\rangle\varphi_i\,. \tag{2.28}$$

Let the operators $U$, $V$, and $W$ be defined by

$$Uz = \sum_i \langle z, \psi_i \rangle \varphi_i$$

$$Vz = \sum_i \mu_i^{-r/p} \langle z, \psi_i \rangle \psi_i$$

$$Bz = \sum_i \mu_i^{-r/q} \langle z, \psi_i \rangle \psi_i$$

for arbitrary $z \in \mathscr{H}$. Then $U \in C(\infty)$, $V \in C(p)$ and $B \in C(q)$. If $A = UV$, then by (2.12), we have that $A \in C(p)$. Since

$$D = UVB = AB \tag{2.29}$$

by (2.23), (2.28), and by the definitions of $U$, $V$, $A$, and $B$, the equation (2.24) is verified.

Let $A \in C(p)$ and $B \in C(q)$, where $0 < p$, $q \leqslant \infty$ and where

$$1/p + 1/q \geqslant 1 \, . \tag{2.30}$$

Note that $AB \in C(1)$ and $BA \in C(1)$ by (2.12) and by the inclusion (2.10). We have (DUNFORD-SCHWARTZ [5; pp. 1098, 1104-1105])

$$\tau(AB) = \tau(BA) \, . \tag{2.31}$$

Strictly speaking, the proof in DUNFORD-SCHWARTZ (p. 1100) does not cover the case where (say) $q = \infty$, since they use the fact that $B$ can be approximated (with respect to $| \ |_q$) by operators of finite dimensional range; this is not true if $q = \infty$. However, the minor difficulty can be overcome if we factor $A$ into a product $FG$ of two operators, each in the space $C(2p) \subseteq C(2)$, where we have set $q = \infty$ in (2.30). The latter factorization is possible, of course, by (2.24). We write

$$\tau(AB) = \tau\big(F(GB)\big) \tag{2.32}$$

$$= \tau\big((GB)F\big) \tag{2.33}$$

$$= \tau\big(G(BF)\big) \tag{2.34}$$

$$= \tau\big((BF)G\big) \tag{2.35}$$

$$= \tau(BA) \, . \tag{2.36}$$

In order to obtain (2.33) and (2.35) we have used the fact that $GB$ and $BF$ are in $C(2p) \subseteq C(2)$. By induction, one can show that if

$$A_i \in C(p_i), \quad (i = 1, \dots, n) \tag{2.37}$$

where

$$0 < p_i \leqslant \infty \tag{2.38}$$

and where

$$\sum_{i=1}^{n} 1/p_i \geqslant 1 \tag{2.39}$$

then

$$\tau\Big[\prod_{i=1}^{n} A_i\Big] = \tau\Big[\prod_{i=1}^{n} \tilde{A}_i\Big] \tag{2.40}$$

where $\{\tilde{A}_i\}$ is any permutation of $\{A_i\}$.

## III. – Analytic functions and product spaces-A review.

We continue to assume that $\mathscr{H}$ is a complex Hilbert space.

Let $p$ be a *fixed* element in $1 \leqslant p \leqslant \infty$, so that $|\ |_p$ is a true norm. Let $\varepsilon$ be a fixed element in $0 < \varepsilon \leqslant +\infty$, and let $A(\lambda)$ be a linear operator defined on all of $\mathscr{H}$ for each $\lambda$ in $|\lambda| < \varepsilon$. We say the mapping (1) $\lambda \to A(\lambda)$ is analytic with respect to $|\ |_p$ for each $\lambda$ in $|\lambda| < \varepsilon$ if

(i) $A(\lambda) \in C\{p; \mathscr{H}\}$ for each $\lambda$ in $|\lambda| < \varepsilon$;

(ii) the mapping $\lambda \to A(\lambda)$ is continuous with respect to $|\ |_p$ for $\lambda$ in $|\lambda| < \varepsilon$;

(iii) the limit (with respect to $|\ |_p$)

$$\lim_{\Delta\lambda \to 0} \frac{A(\lambda + \Delta\lambda) - A(\lambda)}{\Delta\lambda} \tag{3.1}$$

exists for each $\lambda$ in $|\lambda| < \varepsilon$, where $|\lambda + \Delta\lambda| < \varepsilon$.

The mapping $\lambda \to A(\lambda)$ is analytic with respect to $|\ |_p$ in $|\lambda| < \varepsilon$ iff (see DUNFORD-SCHWARTZ [4; p. 228])

$$A(\lambda) = \sum_{i=1}^{\infty} \lambda^i A_i \tag{3.2}$$

where $A_i \in C(p)$ and where the series in (3.2) is absolutely and uniformly convergent with respect to $|\ |_p$ in $|\lambda| \leqslant \varepsilon'$ for each $\varepsilon'$ in $0 < \varepsilon' < \varepsilon$. The proofs involved here are similar to the proofs in the case where the functions are complex-valued.

If $1 \leqslant p < q \leqslant \infty$, then by the inclusion (2.10), by the inequality (2.11), and by the definition of analyticity, the reader can show that $A(\lambda)$ is analytic with respect

(1) In the future, we will frequently say (somewhat inexactly) that $A(\lambda)$ is an analytic function or continuous function with respect to $|\ |_p$.

to $|\ |_q$ in $|\lambda|<\varepsilon$. The inequality (2.11) can be used to show that

$$\lim_{n\to\infty}\left|A(\lambda)-\sum_{i=0}^{n}\lambda^i A_i\right|_p\to 0 \tag{3.3}$$

uniformly in $|\lambda|\leqslant\varepsilon'<\varepsilon$. The absolute convergence of the series (3.2) with respect to $|\ |_p$ implies its absolute convergence with respect to $|\ |_q$ by (2.11). Obviously, although $A(\lambda)$ is analytic here in two different senses, the Maclaurin coefficients $A_i$ are *norm invariant*.

If $A(\lambda)$ is analytic with respect to $|\ |_p$ for each $\lambda$ in $|\lambda|<\varepsilon$, then, of course, all of the derivatives of $A(\lambda)$ (with respect to $|\ |_p$) exist, and are analytic with respect to $|\ |_p$ in $|\lambda|<\varepsilon$.

If $A(\lambda)$ is the function previously described, then it is convenient to define the operators $\boldsymbol{m}_k$ for $k=0,1,\ldots$ by the equations (see DUNFORD-SCHWARTZ [4; pp. 228-229])

$$\boldsymbol{m}_k(A(\lambda))=A_k \tag{3.4}$$

$$=(1/k!)(d^kA/d\lambda^k)(0) \tag{3.5}$$

$$=(1/2\pi i)\int_\Gamma\frac{A(\lambda)}{\lambda^{k+1}}\,d\lambda \tag{3.6}$$

where $\Gamma$ is the positively oriented contour

$$\lambda=\varepsilon'\exp[i\theta]\,,\qquad 0\leqslant\theta\leqslant 2\pi\,. \tag{3.7}$$

The derivative in (3.5) and the integral in (3.6) are defined via the usual limiting processes with respect to $|\ |_p$. The norm invariance of the Maclaurin coefficients $A_k$ yields the norm invariance of the derivative and integral, considered as limits, in (3.5) and (3.6). If $A_k\in C(1)$ for some fixed $k$, then we define the operator $\tau_k$ by

$$\tau_k(A(\lambda))=\tau\{\boldsymbol{m}_k(A(\lambda))\} \tag{3.8}$$

$$=\tau(A_k)\,. \tag{3.9}$$

The notation $\tau_k$ will be used without reference to the Hilbert space in question, since frequently several spaces will be discussed in the same proof. However, no confusion will arise due to this.

If $A(\lambda)$ is analytic in $|\lambda|<\varepsilon$ with respect to the $|\ |_1$ norm, then we may write

$$\tau_k(A(\lambda))=\boldsymbol{m}_k\tau(A(\lambda)) \tag{3.10}$$

since $\tau$ is a continuous operator on $C(1)$.

Let $p$ be a *fixed* element in $1\leqslant p\leqslant\infty$, and let $b_i(\lambda)\in C\{p;\mathscr{H}\}$ for each non-negative integer $i$ and each $\lambda$ in $|\lambda|<\varepsilon$. Suppose, in addition, that the functions

$b_i(\lambda)$ are analytic in $|\lambda|<\varepsilon$ with respect to $|\ |_p$ and that

$$B(\lambda)=\sum_{i=0}^{\infty} b_i(\lambda) \tag{3.11}$$

uniformly in $|\lambda|\leqslant\varepsilon'$ for each $\varepsilon'$ in $0<\varepsilon'<\varepsilon$. Then $B(\lambda)$ is analytic with respect to $|\ |_p$ in $|\lambda|<\varepsilon$, and the formula

$$(d^j/d\lambda^j)B(\lambda)=\sum_{i=1}^{\infty}(d^j/d\lambda^j)b_i(\lambda) \tag{3.12}$$

holds for $|\lambda|<\varepsilon$, since $d^j/d\lambda^j$ may be expressed as an integral operator by using the Cauchy formula (for *fixed* $\lambda$)

$$\frac{1}{j!}\left(\frac{d^j}{d\lambda^j}B(\lambda)\right)=\frac{1}{2\pi i}\int_{\Gamma(\lambda)}\frac{B(\mu)}{(\mu-\lambda)^{j+1}}\,d\mu \tag{3.13}$$

where $\Gamma(\lambda)$ is the positively oriented contour

$$\mu=\lambda+(\varepsilon'-|\lambda|)\exp[i\theta]\,,\qquad (0\leqslant\theta\leqslant 2\pi) \tag{3.14}$$

and where $\lambda$ satisfies the condition $0<|\lambda|<\varepsilon'<\varepsilon$. The derivatives in (3.12) continue to be norm invariant, and the series in (3.12) converges uniformly with respect to $|\ |_p$ on the set $|\lambda|\leqslant\varepsilon'<\varepsilon$ as a direct result of (3.13) and of the uniform convergence of the series in (3.12).

We may write (3.12) as

$$\boldsymbol{m}_k(B(\lambda))=\sum_{i=0}^{\infty}\boldsymbol{m}_k(b_i(\lambda))\,. \tag{3.15}$$

Furthermore, if $p=1$ in the preceding, then

$$\tau(B(\lambda))=\sum_{i=0}^{\infty}\tau(b_i(\lambda))\,. \tag{3.16}$$

Since $\tau$ is continuous on $C(1)$, equation (3.16) holds uniformly in $|\lambda|\leqslant\varepsilon'$. We may also write

$$\tau_k(B(\lambda))=\sum_{i=0}^{\infty}\tau_k(b_i(\lambda)) \tag{3.17}$$

if $p=1$ in the preceding, since we may apply $\boldsymbol{m}_k$ to both sides of (3.16).

COMMENT. - Strictly speaking, we should denote the mapping (= function) $\lambda\to A(\lambda)$ by $A$ rather than by $A(\lambda)$ and we should use the notations $\boldsymbol{m}_k(A)$ and $\tau_k(A)$.

However, the notations $\boldsymbol{m}_k[A(\lambda)]$ and $\tau_k[A(\lambda)]$ are more convenient in this paper. In this connection, we shall use boldface typing whenever an expression of the type

$$\sum_{j=0}^{\infty} \lambda^j \boldsymbol{m}_j[A(\boldsymbol{\lambda})]$$

or

$$\sum_{j=0}^{\infty} \lambda^j \tau_j[A(\boldsymbol{\lambda})]$$

appears, in order to « distinguish » between the two « variables ».

Note that we have preferred not to talk about analyticity with respect to $|\ |_p$ if $0<p<1$, since $|\ |_p$ here does not satisfy a triangle inequality.

LEMMA III.1. – Let $p$ and $q$ satisfy $1 \leqslant p,\ q \leqslant \infty$ and let $0<\varepsilon\leqslant +\infty$. Suppose $D(\lambda)$ and $E(\lambda)$ are operator valued functions such that $D(\lambda)\in C\{p, \mathscr{H}\}$ and $E(\lambda)\in C\{q, \mathscr{H}\}$ for each complex $\lambda$ in $|\lambda|<\varepsilon$. Furthermore suppose that $D(\lambda)$ is analytic with respect to $|\ |_p$ for each $\lambda$ in $|\lambda|<\varepsilon$, and that $E(\lambda)$ is analytic with respect to $|\ |_q$ for each $\lambda$ in $|\lambda|<\varepsilon$. Then $D(\lambda)E(\lambda)$ is analytic with respect to $|\ |_{\tilde r}$ for each $\lambda$ in $|\lambda|<\varepsilon$, where

$$\tilde r = \max\{1, (p^{-1}+q^{-1})^{-1}\}. \tag{3.18}$$

We have

$$D(\lambda)E(\lambda) = \sum_{i=0}^{\infty} \lambda^i \sum_{j=0}^{i} [\boldsymbol{m}_j(D(\boldsymbol{\lambda}))][\boldsymbol{m}_{i-j}(E(\boldsymbol{\lambda}))] \tag{3.19}$$

for each $\lambda$ in $|\lambda|<\varepsilon$; the series in (3.19) converges uniformly and absolutely with respect to $|\ |_{\tilde r}$ for each $\lambda$ in $|\lambda|\leqslant\varepsilon'$, where $\varepsilon'$ is any number in $0<\varepsilon'<\varepsilon$.

PROOF. – One can show from the definition of the derivative, from (2.14), and from the inequality

$$|A|_{\tilde r} \leqslant |A|_r, \tag{3.20}$$

where $r$ is defined in (2.14), that $D(\lambda)E(\lambda)$ is differentiable with respect to $|\ |_{\tilde r}$, and that

$$\big(D(\lambda)E(\lambda)\big)' = D'(\lambda)E(\lambda) + D(\lambda)E'(\lambda). \tag{3.21}$$

Furthermore the continuity of $D(\lambda)E(\lambda)$ with respect to $|\ |_{\tilde r}$ follows from the continuity of $D(\lambda)$ with respect to $|\ |_p$, from the continuity of $E(\lambda)$ with respect to $|\ |_q$, from (3.20), and from (2.14). Hence $D(\lambda)E(\lambda)$ is analytic with respect to $|\ |_{\tilde r}$ in $|\lambda|<\varepsilon$. The series

$$D(\lambda)E(\lambda) = \sum_{i=0}^{\infty} \lambda^i \boldsymbol{m}_i[D(\boldsymbol{\lambda})E(\boldsymbol{\lambda})] \tag{3.22}$$

converges uniformly and absolutely with respect to $| \, |_{\tau}$ in $|\lambda| < \varepsilon'$, where $0 < \varepsilon' < \varepsilon$, by DUNFORD-SCHWARTZ [4; p. 229]. The proof that the coefficients of $\lambda^i$ in (3.19) and (3.22) are equal may be carried out by an appropriate generalization of (3.21). ∎

COROLLARY III.1. – If $f(\lambda)$ is a complex valued analytic function in $|\lambda| < \varepsilon$, and if $D(\lambda)$ satisfies the assumptions of lemma III.1, then $f(\lambda)D(\lambda)$ is analytic in $|\lambda| < \varepsilon$ with respect to $| \, |_p$. The Maclaurin series for $f(\lambda)$ $D(\lambda)$ about $\lambda = 0$ may be computed by the usual Cauchy rule for products of Taylor series.

PROOF. – Let $E(\lambda) = f(\lambda)I$ and let $q = \infty$ in lemma III.1.

LEMMA III.2. – Let $0 < \varepsilon \leqslant +\infty$, and let $D(\lambda)$ be an operator valued function such that $D(\lambda) \in C\{1; \mathscr{H}\}$ for each $\lambda$ in $|\lambda| < \varepsilon$. If $D(\lambda)$ is analytic with respect to $| \, |_1$ for each $\lambda$ in $|\lambda| < \varepsilon$, then the function $\tau(D(\lambda))$ is analytic (in the usual sense, i.e. with respect to the norm on the set of complex numbers) for each $\lambda$ in $|\lambda| < \varepsilon$. Furthermore the equation

$$(d/d\lambda)\{\tau[D(\lambda)]\} = \tau\{(d/d\lambda)[D(\lambda)]\} \tag{3.23}$$

holds for each $\lambda$ in $|\lambda| < \varepsilon$.

PROOF. – The assertion (3.23) follows directly from the inequality (see (2.16))

$$\left| \frac{\tau[D(\lambda + \Delta\lambda)] - \tau[D(\lambda)]}{\Delta\lambda} - \tau\{(d/d\lambda)D(\lambda)\} \right| \leqslant \left| \frac{D(\lambda + \Delta\lambda) - D(\lambda)}{\Delta\lambda} - (d/d\lambda)D(\lambda) \right|_1$$

and from the fact that the function $D(\lambda)$ is differentiable (with respect to $| \, |_1$). The continuity of the function $\tau[D(\lambda)]$ follows similarly from (2.16) and from the continuity of $D(\lambda)$ with respect to $| \, |_1$. Hence the lemma is proved. ∎

PRODUCT SPACES. – The following is presented for review only, and is not believed to be new.

Let $n$ be a positive integer. We define $\mathscr{H}^{[n]}$ to be the Hilbert space of all $n \times 1$ column vectors

$$\boldsymbol{u} = (u_1, \ldots, u_n)^T$$

where $T$ indicates a transpose, and where $u_i \in \mathscr{H}$ for each $i$ in $1 \leqslant i \leqslant n$. The (standard) inner product on $\mathscr{H}^{[n]}$ is given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{n} \langle u_i, v_i \rangle \tag{3.24}$$

where

$$\boldsymbol{v} = (v_1, \ldots, v_n)^T, \qquad (v_i \in \mathscr{H}).$$

The operations of addition and scalar multiplication on $\mathcal{H}^{[n]}$ are defined in the usual way.

Let $a$ and $b$ be *fixed* integers such that $1 \leqslant a, b \leqslant n$. Let $A$ be any operator on $\mathcal{H}$, and let $\mathcal{A}$ be the $n \times n$ matrix with $A$ as the element in the $a$-th row, $b$-th column, and with the zero operator (on $\mathcal{H}$) elsewhere. Note that $\mathcal{A}$ is a compact operator on $\mathcal{H}^{[n]}$ iff $A$ is a compact operator on $\mathcal{H}$. If $\mathcal{A}$ or $A$ is compact, then we may write, as the reader may show,

$$\mu_k(\mathcal{A}) = \mu_k(A) \tag{3.25}$$

where $\mu_k(\mathcal{A})$ and $\mu_k(A)$ are the (non-decreasing) sequences of singular values of $\mathcal{A}$ and $A$, taken according to multiplicity. Hence

$$|\mathcal{A}|_p = |A|_p \tag{3.26}$$

for $p$ in $0 < p \leqslant \infty$ and for compact $\mathcal{A}$ or $A$, where both sides of (3.26) may possibly equal $+\infty$. If either $\mathcal{A}$ or $A$ is not compact, but has finite « sup » norm (see (2.7)), then both have finite « sup » norm, and (3.26) is valid with $p = \infty$ as the reader may prove. Since the space $C(p)$ is a linear manifold, a matrix operator on $\mathcal{H}^{[n]}$ is in $C\{p, \mathcal{H}^{[n]}\}$ if each of its elements (which are operators on $\mathcal{H}$) is in $C\{p; \mathcal{H}\}$.

A converse to the last statement holds. Let $\mathcal{M}$ be an $n \times n$ matrix of operators on $\mathcal{H}$. Let $M_{ij}$ be the element in the $i$-th row, $j$-th column of $\mathcal{M}$, where $1 \leqslant i, j \leqslant n$. We claim that $\mathcal{M} \in C\{p; \mathcal{H}^{[n]}\}$ implies $M_{ij} \in C\{p; \mathcal{H}\}$ for each $i$ and $j$ satisfying $1 \leqslant i, j \leqslant n$. Let $a$ and $b$ be *fixed* integers such that $1 \leqslant a, b \leqslant n$. Let $\mathcal{F}_{ab}$ be the $n \times n$ matrix with the identity operator on $\mathcal{H}$ as the element in its $a$-th row, $b$-th column, and with zero elsewhere. Then by (3.26), it suffices to show that the matrix operator $(M_{ab}\mathcal{F}_{ab})$ is in $C\{p; \mathcal{H}^{[n]}\}$. (The matrix $M_{ab}\mathcal{F}_{ab}$ has $M_{ab}$ as the element in its $a$-th row, $b$-th column, and zero elsewhere; the multiplication here is, of course, similar to the multiplication of a matrix by a scalar.) But

$$M_{ab}\mathcal{F}_{ab} = \mathcal{F}_{aa}\mathcal{M}\mathcal{F}_{bb} \tag{3.27}$$

and

$$|\mathcal{F}_{aa}|_\infty = |\mathcal{F}_{bb}|_\infty = 1 .$$

Use of (2.12)′ and use of the assumption that $\mathcal{M} \in C\{p; \mathcal{H}^{[n]}\}$ will then yield the result that $M_{ab}\mathcal{F}_{ab} \in C\{p; \mathcal{H}^{[n]}\}$ and hence that $M_{ab} \in C\{p; \mathcal{H}\}$. If $0 < p < \infty$ we have, a fortiori, the compactness of $M_{ab}$. If $p = \infty$ and if $\mathcal{M}$ is compact, then by (3.27), the matrix operator $M_{ab}\mathcal{F}_{ab}$ must be compact; hence, by previous comments, $M_{ab}$ must be compact. We have:

LEMMA III.3. – Let $M_{ij}$ be an operator on $\mathcal{H}$ for each integer $i$ and each integer $j$ satisfying $1 \leqslant i, j \leqslant n$. Let $\mathcal{M}$ be the $n \times n$ matrix with $M_{ij}$ in its $i$-th row,

$j$-th column for each pair of integers $i$ and $j$ in $1 \leqslant i,\ j \leqslant n$. Then $\mathcal{M} \in C\{p; \mathcal{H}^{[n]}\}$ if and only if $M_{ij} \in C\{p; \mathcal{H}\}$ for each integer $i$ and each integer $j$ in $1 \leqslant i,\ j \leqslant n$, where $p$ is some fixed element in $0 < p \leqslant \infty$. If $p = 1$ in the preceding, then

$$\tau(\mathcal{M}) = \sum_{i=1}^{n} \tau(M_{ii}) . \tag{3.28}$$

PROOF. – Clearly only the last assertion in the lemma needs to be proved. Let $\{\varphi_\alpha\}$ be a complete orthonormal set in the space $\mathcal{H}$. Let $\Phi_{i,\alpha}$ be the $n \times 1$ column vector with $\varphi_\alpha$ in its $i$-th row (for *fixed* $i$) and with the zero vector in $\mathcal{H}$ elsewhere. The set $\Phi_{i,\alpha}$ is a complete orthonormal set in $\mathcal{H}^{[n]}$; use of the analogue of (2.15) will easily yield (3.28). ∎

We wish to consider the case where the matrix $\mathcal{M}$, previously described, is dependent on a parameter $\lambda$. As before, we let $n$ denote a fixed positive integer, and we let $\varepsilon$ denote a fixed element in $0 < \varepsilon \leqslant +\infty$. For each pair of integers $i$ and $j$ satisfying $1 \leqslant i,\ j \leqslant n$, and each complex $\lambda$ in $|\lambda| < \varepsilon$, we assume that the function $M_{ij}(\lambda)$ is defined and has values in $C\{\infty, \mathcal{H}\}$. Let $\mathcal{M}(\lambda)$ be the $n \times n$ matrix with $M_{ij}(\lambda)$ in its $i$-th row, $j$-th column, for each pair of integers $i$ and $j$ in $1 \leqslant i,\ j \leqslant n$.

LEMMA III.4. – Suppose the matrix valued function $\mathcal{M}(\lambda)$ is analytic with respect to $|\ |_p$ in $|\lambda| < \varepsilon$, where $0 < \varepsilon \leqslant +\infty$ and $1 \leqslant p \leqslant \infty$. Let the Maclaurin series for $\mathcal{M}(\lambda)$ be given by

$$\mathcal{M}(\lambda) = \sum_{h=0}^{\infty} \lambda^h \mathcal{M}^{\{h\}} . \tag{3.29}$$

Then $M_{ij}(\lambda)$ is also analytic with respect to $|\ |_p$ in $|\lambda| < \varepsilon$ for each pair of integers $i$ and $j$ satisfying $1 \leqslant i,\ j \leqslant n$; if the Maclaurin series for $M_{ij}(\lambda)$ is given by

$$M_{ij}(\lambda) = \sum_{h=0}^{\infty} \lambda^h M_{ij}^{\{h\}} \tag{3.30}$$

then $M_{ij}^{\{h\}}$ is the element in the $i$-th row, $j$-th column of the matrix $\mathcal{M}^{\{h\}}$ in (3.29). The series in (3.29) and (3.30) are uniformly and absolutely convergent with respect to $|\ |_p$ in $|\lambda| \leqslant \varepsilon'$ for each $\varepsilon'$ in $0 < \varepsilon' < \varepsilon$.

PROOF. – Let $a$ and $b$ be fixed integers in $1 \leqslant a,\ b \leqslant n$. As before, let $\mathcal{F}_{ab}$ denote the $n \times n$ matrix with the identity operator on $\mathcal{H}$ as the element in the $a$-th row, $b$-th column, and with the zero operator (on $\mathcal{H}$) elsewhere. Let $M_{ab}^{\{h\}}$ denote the element in the $a$-th row, $b$-th column of the matrix $\mathcal{M}^{\{h\}}$ in (3.29). First we show convergence with respect to $|\ |_p$ of the series $\sum_{h=0}^{\infty} \lambda^h M_{ab}^{\{h\}}$; then we shall show that the latter series is precisely the element in the $a$-th row, $b$-th column of the matrix $\mathcal{M}(\lambda)$.

Once these facts have been proved, the equality of $M_{ab}(\lambda)$ and of the latter series would follow by the very definition of $M_{ab}(\lambda)$.

In order to prove convergence of the series $\sum_{h=0}^{\infty} \lambda^h M_{ab}^{\{h\}}$, we first note the relation

(3.31) $$\left[\sum_{h=k}^{l} \lambda^h M_{ab}^{\{h\}}\right] \mathscr{F}_{ab} = \mathscr{F}_{aa}\left[\sum_{h=k}^{l} \lambda^h \mathscr{M}^{\{h\}}\right] \mathscr{F}_{bb}$$

which is obtained by replacing $M_{ab}$ in (3.27) by $\sum_{h=k}^{l} \lambda^h M_{ab}^{\{h\}}$ and by replacing $\mathscr{M}$ in (3.27) by $\sum_{h=k}^{l} \lambda^h \mathscr{M}^{\{h\}}$. We claim that

(3.32) $$\left|\sum_{h=k}^{l} \lambda^h M_{ab}^{\{h\}}\right|_p = \left|\left[\sum_{h=k}^{l} \lambda^h M_{ab}^{\{h\}}\right] \mathscr{F}_{ab}\right|_p$$

(3.33) $$\leqslant \left|\sum_{h=k}^{l} \lambda^h \mathscr{M}^{\{h\}}\right|_p .$$

The equality (3.32) follows from (3.26) with $\sum_{h=k}^{l} \lambda^h M_{ab}^{\{h\}}$ replacing $A$ and $\sum_{h=k}^{l} \lambda^h M_{ab}^{\{h\}} \mathscr{F}_{ab}$ replacing $\mathscr{A}$. The assertion (3.33) follows from (3.31), from the fact that $|\mathscr{F}_{aa}|_\infty = |\mathscr{F}_{bb}|_\infty = 1$, and from (2.14)'. Since the series $\sum_{h=0}^{\infty} \lambda^h \mathscr{M}^{\{h\}}$ is (uniformly) convergent with respect to $|\ |_p$ in $|\lambda| \leqslant \varepsilon' < \varepsilon$, the relations (3.32)-(3.33) show that the sequence $\left\{\sum_{h=0}^{l} \lambda^h M_{ab}^{\{h\}}\right\}$ must be uniformly Cauchy convergent in $|\lambda| \leqslant \varepsilon' < \varepsilon$. Since $C\{p;\mathscr{H}\}$ is complete, the latter sequence must converge uniformly with respect to $|\ |_p$ to its limit $\sum_{h=0}^{\infty} \lambda^h M_{ab}^{\{h\}}$ in $C\{p;\mathscr{H}\}$.

In order to show that $\sum_{h=0}^{\infty} \lambda^h M_{ab}^{\{h\}}$ is precisely the element in the $a$-th row $b$-th column of the matrix $\mathscr{M}(\lambda)$, we set $k = 0$ and $l = \infty$ in (3.31). The reader may show that this is permissible. The ensuing relation, along with (3.29), proves that $\sum_{h=0}^{\infty} \lambda^h M_{ab}^{\{h\}}$ is the element in the $a$-th row, $b$-th column of $\mathscr{M}(\lambda)$; by definition of $M_{ab}(\lambda)$, the equality (3.30) must follow. ■

For convenience in referencing only, we state the following lemma, which is a consequence of previous considerations. Recall the definition of the operator $\boldsymbol{m}_j$ in (3.4).

LEMMA III.5. – Let $\mathscr{M}(\lambda)$ be an $n\times n$ matrix of operators $M_{ij}(\lambda)$ on $\mathscr{H}$. Let $\mathscr{M}(\lambda)$ be analytic with respect to $|\ |_\infty$ about $\lambda = 0$. Then the functions $M_{ij}(\lambda)$ are also analytic with respect to $|\ |_\infty$ about $\lambda = 0$. Suppose

(3.35) $$\boldsymbol{m}_k(\mathscr{M}(\lambda)) \in C\{p;\mathscr{H}^{[n]}\}$$

for some fixed integer $k \geqslant 0$ and some fixed $p$ in $0 < p \leqslant \infty$. Then

$$m_k(M_{ij}(\lambda)) \in C\{p, \mathscr{H}\} \tag{3.36}$$

for each $i$, $j$ in $1 \leqslant i$, $j \leqslant n$. If $p = 1$ in the preceding, then

$$\tau_k(\mathscr{M}(\lambda)) = \sum_{i=1}^{n} \tau_k(M_{ii}(\lambda)) . \tag{3.37}$$

## IV. – The eigenvalues of operator polynomials.

We assume that $H(\lambda)$ is given by

$$H(\lambda) = \sum_{i=0}^{s} \lambda^i H_i \tag{4.1}$$

where the $H_i$ are compact linear operators on the complex Hilbert space $\mathscr{H}$. Under additional assumptions, we will obtain results concerning the convergence of the sums $\sum_i \lambda_i^{-p}$ and its evaluation for integral values of $p$, where $\{\lambda_i\}$ is the sequence of eigenvalues of the equation

$$\lambda H(\lambda) u = u \qquad (u \in \mathscr{H}) \tag{4.2}$$

taken according to multiplicity in some sense yet to be defined. It is not clear either that this multiplicity is finite, or even that the eigenvalues of (4.2) are denumerable. LANCASTER [14] and KELDYSH [12] have obtained results in this area; we will shortly discuss a geometric definition of multiplicity given by KELDYSH [12]. Our approach will ultimately yield a definition of multiplicity which is equivalent to that of KELDYSH. Since the results on operators independent of $\lambda$ are extensive, we will transform (4.2) into a finite sequence of «equivalent» matrix operator systems, of which the last system will involve an operator independent of $\lambda$. We can then tentatively define the multiplicity of an eigenvalue in the usual manner described in part II for operators independent of $\lambda$; this definition can then be related to the (more natural) one of Keldysh, and to the Fredholm function(s) to be constructed. Also, by utilizing the properties of the system whose operator is independent of $\lambda$, we can initially set up a formula for the evaluation of the sums $\sum_i \lambda_i^{-p}$ via formula (2.17), and then translate the results back in terms of the operator $H(\lambda)$. Finally, certain bounds will be obtained on $[I - \lambda H(\lambda)]^{-1}$ and on the Fredholm function; these will assist us later on when we consider operators which are meromorphic functions of $\lambda$. First, however, we will briefly discuss the results of KELDYSH [12].

Under the assumption that the $H_i$ are merely compact, KELDYSH [12] states, without proof, the proper construction of a chain corresponding to the eigenvalue $\lambda = a$ of (4.2). A chain is an appropriately chosen finite sequence $(y_0, \ldots, y_m)$ such that

$$y_k = \sum_{j=0}^{k} (1/j!) \frac{d^j}{d\lambda^j} (\lambda H(\lambda)) \Big|_{\lambda=a} y_{k-j} \tag{4.3}$$

for $k = 0, \ldots, m$. The integer $m$ here is assumed to be maximal. If $H(\lambda)$ is independent of $\lambda$, the vector $y_k$ is simply an element of the generalized eigenspace $\mathcal{N}\{(I - aH)^{k+1}\}$ described by (2.4). The $y_k$ are not assumed to be linearly independent; if $\varphi \in \mathscr{H}$ and $|\varphi| = 1$, then the operator $H(\lambda)$ defined by the equation

$$H(\lambda)v = 2\varphi\langle v, \varphi\rangle - \lambda\varphi\langle v, \varphi\rangle \tag{4.4}$$

has the property that $\lambda = 1$ is an eigenvalue of (4.2), and $m = 1$ with $y_0 = y_1 = \varphi$.

If the length of the chain $(y_0, \ldots, y_m)$ is defined to be $(m + 1)$, then the Keldysh multiplicity $M(a)$ of the eigenvalue $\lambda = a$ is the sum of the lengths of all chains generated by an appropriate basis for the eigenspace. Keldysh implies that $M(a) < + \infty$. Let $R(\lambda)$ denote the resolvent of $H(\lambda)$, i.e. let $R(\lambda)$ satisfy

$$\mathrm{I} + \lambda R(\lambda) = [\mathrm{I} - \lambda H(\lambda)]^{-1} \tag{4.5}$$

(Keldysh calls $\lambda R(\lambda)$ the resolvent.) Keldysh asserts that if $\lambda = a$ is an eigenvalue of (4.2), then $\lambda = a$ is a pole (2) of $R(\lambda)$. Let $\mathscr{P}(a)R(\lambda)$ denote the principal (i.e. singular) part of $R(\lambda)$ near $\lambda = a$; then Keldysh states that

$$\tau\mathscr{P}(a)(\lambda H(\lambda))'(\lambda R(\lambda)) = \frac{M(a)}{a - \lambda}. \tag{4.6}$$

Since there are no proofs in the paper of Keldysh, we will develop our theory independently of his; howèver, it will be interesting to see how the (algebraic) multiplicity of the Fredholm function(s) (to be constructed) at the eigenvalue $\lambda = a$ (considered as a zero of the Fredholm function(s)) is related to the geometric definition of multiplicity given by Keldysh.

For review and for future reference, we state a few well known results. Let $z_0$ be some arbitrary complex number. Let us suppose that $[I - z_0 H(z_0)]^{-1}$ exists i.e. that $[I - z_0 H(z_0)]$ is a one-to-one operator or equivalently, that $\lambda = z_0$ is not an eigenvalue of (4.2). Then we maintain that $\varepsilon > 0$ exists such that if $|\lambda - z_0| < \varepsilon$, then $[I - \lambda H(\lambda)]^{-1}$ exists, is defined on all of $\mathscr{H}$, and the mapping

$$\lambda \to [I - \lambda H(\lambda)]^{-1} \tag{4.7}$$

(2) This will be proved independently in theorem IV.3.

is analytic with respect to $|\ |_\infty$. The latter implies, of course, that $[I-\lambda H(\lambda)]^{-1}$ is an element of $C(\infty)$. First of all, we maintain that $\lambda=z_0$ is not an eigenvalue of (4.2) iff $[I-z_0H(z_0)]^{-1}$ exists, is defined on all of $\mathscr{H}$, and is an element of $C(\infty)$. These facts follow from the compactness of $H(z_0)$, and from standard theorems concerning the operator $[I-\lambda H(z_0)]^{-1}$ (see DUNFORD-SCHWARTZ [4; p. 579]). From the results stated in DUNFORD-SCHWARTZ [4; p. 584], we have that

$$(4.8)\qquad [I-\lambda H(\lambda)]^{-1}=[I-z_0H(z_0)]^{-1}\sum_{j=0}^{\infty}\{[\lambda H(\lambda)-z_0H(z_0)][I-z_0H(z_0)]^{-1}\}^j$$

for each complex $\lambda$ near the point $z_0$. More specifically, if $0<\eta<1$, then $\varepsilon=\varepsilon(\eta)$ exists such that the condition $|\lambda-z_0|<\varepsilon$ implies that

$$(4.9)\qquad |\lambda H(\lambda)-z_0H(z_0)|_\infty<\eta|I-z_0H(z_0)^{-1}|_\infty^{-1}.$$

Clearly (4.9) is a consequence of the analyticity (and hence the continuity) of the mapping $\lambda\to H(\lambda)$ with respect to $|\ |_\infty$. Using (2.14)$'$ with $p_i=r=\infty$, we can verify the absolute and uniform convergence of the series in (4.8) with respect to $|\ |_\infty$, provided $|\lambda-z_0|<\varepsilon$. The results previously stated concerning the operator $[I-\lambda H(\lambda)]^{-1}$ follow immediately. The reader should recall the definition of the trace operator $\tau$, and the operators $\boldsymbol{m}_k$ and $\tau_k$ given in (3.4) and (3.8).

We would like to say a few words about the work we are about to do. One of our goals, of course, is to provide a formula (see (4.102)) for the evaluation of sums of reciprocal powers of the eigenvalues of (4.2). We will transform (4.2) into a successive (finite) sequence of « equivalent » matrix equations; each equation will involve an operator polynomial whose degree is one less than the degree of its predecessor. The first matrix equation « equivalent » to (4.2) is given by (4.22) (see (4.12) and (4.15)). In the same way we constructed $J(\lambda)$ from $H(\lambda)$, we will continue the process of matrix construction in theorem IV.3 until we arrive at an equation (« equivalent » to (4.1)) whose operator is independent of $\lambda$. Then we can apply (2.17), and translate the results in terms of $H(\lambda)$ in order to obtain (4.102). Equation (4.41) will play a key role for the latter purpose, as we will see. Equations (4.41) and (4.102) are to be proved for integers $k\geqslant k_0-1$ (see (4.11) and (4.16)). Note the « invariance » of $k_0$ by comparing (4.16) with (4.29) (see (4.24) through (4.28) for the definition of the numbers $\beta_i$).

Several things are important in this lemma: we must prove existence and equality of both sides of (4.41) for integers $k\geqslant k_0-1$, and we must prove the analyticity of (4.42) with respect to $|\ |_1$, provided $k\geqslant k_0-1$ and provided $\lambda$ is not an eigenvalue of (4.2). The trace of (4.42) is related to the logarithmic derivatives of the Fredholm functions to be constructed in theorem IV.3 (see (4.97)), and so the analyticity properties of (4.42) are essential. In order to prove existence of the traces in (4.41) (see e.g. (4.38) in (v)), and in order to prove analyticity of (4.42) with respect to $|\ |_1$, we shall assume (4.17) and certain analyticity properties of (4.19).

These assumptions are to be considered as induction assumptions; their analogues will be easy to prove for the last operator (i.e. the operator which is independent of $\lambda$) in the sequence of operators to be constructed. We show here that if (4.17) holds, and if (4.19) has certain analyticity properties, then their analogues (see (iii)) hold for the predecessor $H(\lambda)$ of $J(\lambda)$. With the aid of (iv), we can then relate (4.30) with (4.38) and prove (4.41). With the aid of (iv), we can also relate the analyticity properties of (4.31) with the analyticity of (4.42).

The inequality (4.44) will help us later on to establish a bound on $[I-\lambda H(\lambda)]^{-1}$, which will be necessary when we add a certain type of meromorphic operator to $H(\lambda)$ in part V in order to generalize our results. In part VI, we will show how (4.102) can be used to obtain more explicit expressions for the sums $\sum_i \lambda_i^{-(k+1)}$ in terms of the operators $H_i$.

The conclusions (i), (ii), and (viii) of lemma IV.1 do not depend on acceptance of (4.17) and acceptance of the analyticity of (4.19). We remark that (4.17) and the analyticity of (4.19), along with (iii), can be proved in a direct way which is, perhaps, more computationally involved than the methods we have chosen here.

LEMMA IV.1. – Let $H(\lambda)$ be given by (4.1), where $s$ is an integer such that $s\geqslant 1$, where $\lambda$ is a complex parameter and where the $H_i$ are linear operators on the complex Hilbert space $\mathscr{H}$. Let us assume that

$$H_i \in C\{\alpha_i;\mathscr{H}\} \tag{4.11}$$

for each integer $i$ in $0\leqslant i\leqslant s$, where $\alpha_i$ is a positive real for each integer $i$ in $0\leqslant i\leqslant s$. There are linear operators $A_s$ and $B_s$ on $\mathscr{H}$ such that

$$H_s = A_s B_s \tag{4.12}$$

where

$$A_s \in C\{((s+1)/s)\alpha_s\} \tag{4.13}$$

and

$$B_s \in C\{(s+1)\alpha_s\}. \tag{4.14}$$

We define the $2\times 2$ matrix $J(\lambda)$ by

$$J(\lambda) = \left(\begin{array}{c|c} \sum_{i=0}^{s-1} \lambda^i H_i & \lambda^{s-1} A_s \\ \hline B_s & 0 \end{array}\right) \tag{4.15}$$

and we regard $J(\lambda)$ as an operator on the Hilbert space $\mathscr{H}^{[2]} = \mathscr{H}\times\mathscr{H}$. Let

$$k_0 = \max\{(i+1)\alpha_i : 0\leqslant i\leqslant s\} \tag{4.16}$$

and let us suppose that

$$\boldsymbol{m}_k\{[\mathscr{I}-\lambda J(\lambda)]^{-1}\}\in C\{k_0/k;\mathscr{H}^{[2]}\} \tag{4.17}$$

for each non-negative (3) integer $k$, where $\mathscr{I}$ is the identity operator on $\mathscr{H}^{[2]}$. Let $\gamma(k)$ be defined by

$$\gamma(k)=\max\{k_0/k, 1\} \tag{4.18}$$

and let us further suppose that

$$[\mathscr{I}-\lambda J(\lambda)]^{-1}-\sum_{j=0}^{k-1}\lambda^j\boldsymbol{m}_j\{[\mathscr{I}-\boldsymbol{\lambda}J(\boldsymbol{\lambda})]^{-1}\} \tag{4.19}$$

is an analytic function with respect to $|\ |_{\gamma(k)}$ for each positive integer $k$, and for each $\lambda$ which is not an eigenvalue of the equation

$$\lambda J(\lambda)\vec{u}=\vec{u}\qquad(\vec{u}\in\mathscr{H}^{[2]})\,.$$

The following results are then valid.

(i) Let

$$\vec{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix} \tag{4.20}$$

where $u_i\in\mathscr{H}$ for $i=1,2$ We have

$$\lambda H(\lambda)u=u \tag{4.21}$$

if and only if

$$\lambda J(\lambda)\vec{u}=\vec{u} \tag{4.22}$$

provided

$$u_1=u\,,\qquad u_2=\lambda B_s u\,. \tag{4.23}$$

Hence $\lambda$ is an eigenvalue of (4.21) iff $\lambda$ is an eigenvalue of (4.22).

(ii) We may write

$$J(\lambda)=\sum_{i=0}^{s-1}\lambda^i J_i \tag{4.24}$$

where the $J_i$ are independent of $\lambda$. If

$$\beta_{s-1}=(1/s)\max\{s\alpha_{s-1},(s+1)\alpha_s\} \tag{4.25}$$

$$\beta_0\ \ =\max\{\alpha_0,(s+1)\alpha_s\} \tag{4.26}$$

(3) We shall assume that $k_0/k=\infty$ if $k=0$. It is clear that $m_0\{[\mathscr{I}-\lambda J(\lambda)]^{-1}\}=\mathscr{I}$ so (4.17) is correct if $k=0$.

and

(4.27) $$\beta_i = \alpha_i\,, \qquad 1 \leqslant i \leqslant s-2$$

then

(4.28) $$J_i \in C\{\beta_i; \mathscr{H}^{[2]}\}$$

for each $i$ in $0 \leqslant i \leqslant s-1$. Furthermore

(4.29) $$k_0 = \max\{(i+1)\beta_i \colon 0 \leqslant i \leqslant s-1\}\,.$$

(iii) We have

(4.30) $$\boldsymbol{m}_k\{[I-\lambda H(\lambda)]^{-1}\} \in C\{k_0/k; \mathscr{H}\}$$

if $k$ is a non-negative integer and if (4.17) is accepted. Furthermore if the assumption about the analyticity of (4.19) is also accepted, then

(4.31) $$[I-\lambda H(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j\{[I-\boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\}$$

is an analytic function with respect (4) to $|\ |_{\gamma(k)}$ at all points $\lambda$ which are not eigenvalues of (4.2), where $\gamma(k)$ is given by (4.18).

(iv) Let $i$ be a fixed integer satisfying $0 \leqslant i \leqslant s$. Let $A$ and $B$ be operators on $\mathscr{H}$ such that

(4.32) $$A \in C\{p; \mathscr{H}\}$$

and

(4.33) $$B \in C\{q; \mathscr{H}\}$$

where $0 < p$, $q \leqslant \infty$ and where

(4.34) $$1/p + 1/q \geqslant 1/\alpha_i\,.$$

Then if $k$ is any integer such that $k \geqslant k_0 - 1$, we have that

(4.35) $$\boldsymbol{m}_k\{\lambda^i AB[I-\lambda H(\lambda)]^{-1}\} \in C\{1, \mathscr{H}\}$$

(4.36) $$\boldsymbol{m}_k\{\lambda^i B[I-\lambda H(\lambda)]^{-1}A\} \in C\{1, \mathscr{H}\}$$

and

(4.37) $$\tau_k\{\lambda^i AB[I-\lambda H(\lambda)]^{-1}\} = \tau_k\{\lambda^i B[I-\lambda H(\lambda)]^{-1}A\}$$

provided (4.17) is accepted.

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(4) In particular, if $k \geqslant k_0 - 1$, then (4.31) is analytic (where defined) with respect to $|\ |_1$.

If (4.17) is accepted, then the previous results continue to be valid if the space $\mathscr{H}$ is replaced by $\mathscr{H}^{[2]}$, if $H(\lambda)$ and $I$ are replaced by $J(\lambda)$ and $\mathscr{I}$ respectively, if $\alpha_i$ in (4.34) is replaced by $\beta_i$, and if $i$ is now restricted so that $0 \leqslant i \leqslant s-1$.

(v) If $k$ is an integer such that $k \geqslant k_0 - 1$, and if (4.17) is accepted, then we have

$$\boldsymbol{m}_k\{[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1}\} \in C\{1, \mathscr{H}\} \tag{4.38}$$

$$\boldsymbol{m}_k\{[I-\lambda H(\lambda)]^{-1}[\lambda H(\lambda)]'\} \in C\{1, \mathscr{H}\} \tag{4.39}$$

and

$$\tau_k\{[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1}\} = \tau_k\{[I-\lambda H(\lambda)]^{-1}[\lambda H(\lambda)]'\} \tag{4.40}$$

where the derivatives in (4.38)-(4.40) are taken with respect to (say) $|\ |_\infty$.

If (4.17) is accepted, then the previous results continue to be valid if the space $\mathscr{H}$ in (4.38) and (4.39) is replaced by $\mathscr{H}^{[2]}$, if $H(\lambda)$ and $I$ are replaced by $J(\lambda)$ and $\mathscr{I}$, and if $k$ continues to be restricted so that $k \geqslant k_0 - 1$.

(vi) We have that

$$\tau_k\{[\lambda J(\lambda)]'[\mathscr{I}-\lambda J(\lambda)]^{-1}\} = \tau_k\{[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1}\} \tag{4.41}$$

provided $k$ is an integer such that $k \geqslant k_0 - 1$, and provided (4.17) is accepted.

(vii) The functions ([5])

$$[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j\{[\boldsymbol{\lambda} H(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\} \tag{4.42}$$

and

$$[\lambda J(\lambda)]'[\mathscr{I}-\lambda J(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j\{[\boldsymbol{\lambda} J(\boldsymbol{\lambda})]'[\mathscr{I}-\boldsymbol{\lambda} J(\boldsymbol{\lambda})]^{-1}\} \tag{4.43}$$

are analytic with respect to $|\ |_1$, provided $k$ is an integer such that $k \geqslant k_0 - 1$, provided $\lambda$ is not an eigenvalue of (4.21) (or equivalently of (4.22)), and provided (4.17) and the assumption about (4.19) are accepted.

If the factors $[\lambda H(\lambda)]'$ and $[I-\lambda H(\lambda)]^{-1}$ in (4.42) are commuted, the conclusions of this part remain valid. A similar statement holds for (4.43).

(viii) The inequality

$$|[I-\lambda H(\lambda)]^{-1}|_\infty \leqslant |[\mathscr{I}-\lambda J(\lambda)]^{-1}|_\infty \tag{4.44}$$

([5]) If we may take $k=0$ in (4.42) and (4.43), the series in (4.42) and (4.43) are assumed to vanish.

holds at all points $\lambda$ which are not eigenvalues of (4.21) (or equivalently, of (4.22)).

PROOF. – Assertion (i) is easy to show; hence the verification is left to the reader.

Note that the factorization (4.12) (which is not unique) is possible because of (4.11) and because of the discussion concerning equation (2.24).

For the proof of (4.28), note that

$$J_i = \begin{pmatrix} H_i & A_s \delta_{i,s-1} \\ B_s \delta_{i,0} & 0 \end{pmatrix} \tag{4.45}$$

for $0 \leqslant i \leqslant s-1$, where $\delta_{ij}$ is the Kronecker delta. Use of (4.11), (4.13), (4.14), (2.10) and of lemma III.3, yields the result (4.28). Note that the special case where $i = 0 = s-1$ must be handled separately, although the basic formulas (4.25) and (4.26) are still consistent. The complication here is that the matrix $J_i$ will have three non-zero components if $i = 0 = s-1$, whereas $J_i$ will have one or two non-zero components if $i > 0$ or $s > 1$.

The rest of the assertions in (ii) can be proved by the reader.

In order not to break up the continuity of the proof, we will assume (for the time being) that

$$[\mathscr{I} - \lambda J(\lambda)]^{-1} = \begin{pmatrix} [I - \lambda H(\lambda)]^{-1} & \lambda^s [I - \lambda H(\lambda)]^{-1} A_s \\ \lambda B_s [I - \lambda H(\lambda)]^{-1} & I + \lambda^{s+1} B_s [I - \lambda H(\lambda)]^{-1} A_s \end{pmatrix}. \tag{4.46}$$

Assertion (4.30) follows from (4.17) and from lemma III.5 since (by (4.46)) the operator $[I - \lambda H(\lambda)]^{-1}$ is the element in the first row, first column, of the matrix $[\mathscr{I} - \lambda J(\lambda)]^{-1}$. Similarly, the assertion about the analyticity of (4.31) is a consequence of the analyticity of (4.19) and of lemma III.4, since, by (4.46) and lemma III.4, the operator (4.31) is the element in the first row, first column of the matrix operator (4.19).

We shall prove (iv) only in the case where $A$ and $B$ are operators on $\mathscr{H}$, and not on $\mathscr{H}^{[2]}$. We shall prove only (4.35), since the proof of (4.36) is essentially identical. The only case of interest in the proof of (4.35) is the case where $k \geqslant i$, since if $k < i$, we have that

$$\boldsymbol{m}_k\{\lambda^i AB[I - \lambda H(\lambda)]^{-1}\} = 0 \tag{4.47}$$

by the analyticity of $[I - \lambda H(\lambda)]^{-1}$ with respect to $|\ |_\infty$ about $\lambda = 0$ (see (4.8)). We have that

$$\boldsymbol{m}_k\{\lambda^i AB[I - \lambda H(\lambda)]^{-1}\} = AB\boldsymbol{m}_k\{\lambda^i[I - \lambda H(\lambda)]^{-1}\} \tag{4.48}$$

since $A$ and $B$ are independent of $\lambda$. By (4.30), we see that

$$\boldsymbol{m}_k\{\lambda^i[I - \lambda H(\lambda)]^{-1}\} = \boldsymbol{m}_{k-i}\{[I - \lambda H(\lambda)]^{-1}\} \in C\{k_0/(k-i); \mathscr{H}\} \tag{4.49}$$

provided $k \geqslant i$. Hence

$$\boldsymbol{m}_k\{\lambda^i AB[I-\lambda H(\lambda)]^{-1}\} \in C\{t; \mathscr{H}\}, \tag{4.50}$$

where by (4.32), (4.33), (4.49), and (2.12)′

$$1/t = 1/p + 1/q + (k-i)/k_0. \tag{4.51}$$

By (4.16), we see that

$$1/\alpha_i \geqslant (i+1)/k_0. \tag{4.52}$$

Hence by (4.34), (4.51), and (4.52), if $k \geqslant k_0 - 1$, and $k \geqslant i$, the inequalities

$$1/t \geqslant 1/\alpha_i + (k-i)/k_0 \tag{4.53}$$

$$\geqslant (k+1)/k_0 \tag{4.54}$$

$$\geqslant 1 \tag{4.55}$$

hold. Thus by the preceding and by the inclusion (2.10), the operator on the right (and hence the left) side of (4.48) is an element of $C\{1, \mathscr{H}\}$ and does indeed have a trace. The same conclusions, of course apply to the operator $\boldsymbol{m}_k\{\lambda^i B[I-\lambda H(\lambda)]^{-1}A\}$ in (4.36).

In order to prove (4.37), we apply the trace operator $\tau$ to both sides of (4.48), and note that if $k \geqslant i$, then

$$\tau_k\{\lambda^i AB[I-\lambda H(\lambda)]^{-1}\} = \tau\Big\{AB\boldsymbol{m}_k\{\lambda^i[I-\lambda H(\lambda)]^{-1}\}\Big\} \tag{4.56}$$

$$= \tau\Big\{B\Big\{\boldsymbol{m}_k[\lambda^i[I-\lambda H(\lambda)]^{-1}]\Big\}A\Big\} \tag{4.57}$$

$$= \tau_k\{\lambda^i B[I-\lambda H(\lambda)]^{-1}A\}. \tag{4.58}$$

Equation (4.56) follows from (4.48) and (3.8). Equation (4.57) follows from (2.40); the requirement (2.39) for use of (2.40) follows from (4.51)-(4.55). Equation (4.58) is proved in a manner similar to the proof of (4.56). Hence (4.37) follows.

The other assertions in (iv) are similarly proved because of the « invariance » of $k_0$ (see (4.16) and (4.29)).

In order to prove assertion (4.38), it suffices to show that

$$\boldsymbol{m}_k\{\lambda^i H_i[I-\lambda H(\lambda)]^{-1}\} \in C\{1; \mathscr{H}\} \tag{4.59}$$

for each integer $i$ in $0 \leqslant i \leqslant s$ and each integer $k \geqslant k_0 - 1$, since $\boldsymbol{m}_k$ is a linear operator. The latter follows immediately by setting

$$A = H_i \qquad B = I \tag{4.60}$$

in (4.35). By making the identification (4.60) in (4.36) and (4.37), it is thus clear that (4.39) and (4.40) also follow. Similarly, the remaining assertions in (v) can be demonstrated; the symmetrical relations (4.16) and (4.29) come into play here.

In order to prove (4.41), note that the elements in the first row, first column and second row, second column of the matrix $[\lambda J(\lambda)]'[I-\lambda J(\lambda)]^{-1}$ are (see (4.15) and (4.46))

$$\sum_{i=0}^{s-1}(i+1)\lambda^i H_i[I-\lambda H(\lambda)]^{-1}+s\lambda^s H_s[I-\lambda H(\lambda)]^{-1} \tag{4.61}$$

and

$$\lambda^s B_s[I-\lambda H(\lambda)]^{-1}A_s \tag{4.62}$$

respectively. By (v), we have that

$$\boldsymbol{m}_k\{[\lambda J(\lambda)]'[\mathscr{I}-\lambda J(\lambda)]^{-1}\}\in C\{1;\mathscr{H}^{[2]}\} \tag{4.63}$$

provided $k\geqslant k_0-1$. Hence by (3.37), if $k\geqslant k_0-1$, we have

$$\tau_k\{[\lambda J(\lambda)]'[\mathscr{I}-\lambda J(\lambda)]^{-1}\}= \tag{4.64}$$
$$\tau_k\Big\{\sum_{i=0}^{s-1}(i+1)\lambda^i H_i[I-\lambda H(\lambda)]^{-1}+s\lambda^s H_s[I-\lambda H(\lambda)]^{-1}\Big\}+\tau_k\{\lambda^s B_s[I-\lambda H(\lambda)]^{-1}A_s\}.$$

By (4.12), (4.13), (4.14), and by the results of assertion (iv) with

$$A=A_s \qquad B=B_s \tag{4.65}$$

we have, if $k\geqslant k_0-1$,

$$\tau_k\{\lambda^s B_s[I-\lambda H(\lambda)]^{-1}A_s\}=\tau_k\{\lambda^s H_s[I-\lambda H(\lambda)]^{-1}\}. \tag{4.66}$$

Hence (4.41) follows from (4.64) and (4.66). Note that the assertion (4.59) guarantees existence of each term in (4.64) and (4.66) of the form

$$\tau_k\{(i+1)\lambda^i H_i[I-\lambda H(\lambda)]^{-1}\} \tag{4.67}$$

provided $k\geqslant k_0-1$.

We will prove the analyticity of (4.42) with respect to $|\;|_1$, provided $k\geqslant k_0-1$ and provided $\lambda$ is not an eigenvalue of (4.2). It suffices to prove analyticity (with respect to $|\;|_1$) of

$$\lambda^i H_i[I-\lambda H(\lambda)]^{-1}\sum_{j=0}^{k-1}\lambda^j \boldsymbol{m}_j\{\boldsymbol{\lambda}^i H_i[I-\boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\} \tag{4.68}$$

for each integer $i$ in $0 \leqslant i \leqslant s$, subject of course, to the restrictions just imposed on $\lambda$ and $k$. We may write (4.68) as the sum of

$$\lambda^i H_i[I - \lambda H(\lambda)]^{-1} - \sum_{j=0}^{k+i-1} \lambda^j \boldsymbol{m}_j\{\boldsymbol{\lambda}^i H_i[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\} \tag{4.69}$$

and of

$$\sum_{j=k}^{k+i-1} \lambda^j \boldsymbol{m}_j\{\boldsymbol{\lambda}^i H_i[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\}. \tag{4.70}$$

Now (4.70) is a polynomial in $\lambda$, since the coefficient of $\lambda^j$ in (4.70) is an operator independent of $\lambda$. By (4.59), it is clear that (4.70) is an entire function of $\lambda$ with respect to $|\;|_1$, provided $k \geqslant k_0 - 1$. It suffices to show, therefore, that (4.69) is an analytic function with respect to $|\;|_1$, provided $k \geqslant k_0 - 1$ and $\lambda$ is not an eigenvalue of (4.2). We note that

$$\boldsymbol{m}_j\{\lambda^i H_i[I - \lambda H(\lambda)]^{-1}\} = 0 \tag{4.71}$$

for each integer $j$ in $0 \leqslant j \leqslant i-1$, by the analyticity of $[I - \lambda H(\lambda)]^{-1}$ with respect to $|\;|_\infty$ about $\lambda = 0$ (see (4.8)). We may rewrite (4.69) as

$$\lambda^i H_i[I - \lambda H(\lambda)]^{-1} - \sum_{j=i}^{k+i-1} \lambda^j \boldsymbol{m}_{j-i}\{H_i[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\}. \tag{4.72}$$

Setting $h = j - i$ in (4.72), and eliminating $j$, we obtain equivalence of (4.69) and of

$$\lambda^i H_i\{[I - \lambda H(\lambda)]^{-1} - \sum_{h=0}^{k-1} \lambda^h \boldsymbol{m}_h[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\}. \tag{4.73}$$

From (4.16), (4.18), and (2.10), we see that

$$H_i \in C\{\alpha_i\} \subseteq C\{k_0\} \subseteq C\{\gamma(1)\} \tag{4.74}$$

so that $\lambda^i H_i$ is an entire function with respect to $|\;|_{\gamma(1)}$. Since (4.31) is analytic with respect to $|\;|_{\gamma(k)}$, an easy application of lemma III.1 suffices to prove analyticity of (4.73) with respect to $|\;|_1$, since (see (3.18) and (4.18))

$$\max\left\{[(\gamma(1))^{-1} + (\gamma(k))^{-1}]^{-1}, 1\right\} = 1 \tag{4.75}$$

provided $k \geqslant k_0 - 1$.

In a similar manner, the other assertions in (vii) can be proved.

In order to prove (4.44) we first have to define the matrix $\mathscr{F}_{11}$ by

$$\mathscr{F}_{11} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.76}$$

We maintain that

$$|[I-\lambda H(\lambda)]^{-1}|_\infty = |[I-\lambda H(\lambda)]^{-1}\mathscr{F}_{11}|_\infty \tag{4.77}$$

$$= |\mathscr{F}_{11}[\mathscr{I}-\lambda J(\lambda)]^{-1}\mathscr{F}_{11}|_\infty \tag{4.78}$$

$$\leqslant |[\mathscr{I}-\lambda J(\lambda)]^{-1}|_\infty . \tag{4.79}$$

The $2\times 2$ matrix $[I-\lambda H(\lambda)]^{-1}\mathscr{F}_{11}$ has $[I-\lambda H(\lambda)]^{-1}$ in its first row, first column, and the zero operator (on $\mathscr{H}$) elsewhere. Hence (4.77) follows from (3.26); equation (4.78) follows from (4.46) and from (3.27) with $a=b=1$, $\mathscr{M}=[\mathscr{I}-\lambda J(\lambda)]^{-1}$ and $M_{11}=[I-\lambda H(\lambda)]^{-1}$. Inequality (4.79) follows from (2.14)$'$, since $|\mathscr{F}_{11}|_\infty=1$.

It remains only to prove (4.46) in order to complete the proof. Let

$$\vec{w}=\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and

$$\vec{f}=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where $w_i\in\mathscr{H}$ and $f_i\in\mathscr{H}$ for $i=1,2$. The « solution » to the equation

$$[\mathscr{I}-\lambda J(\lambda)]\vec{w}=\vec{f} \tag{4.80}$$

is

$$\vec{w}=[\mathscr{I}-\lambda J(\lambda)]^{-1}\vec{f}\,. \tag{4.81}$$

Rewriting (4.80) in « scalar » terms, we obtain

$$w_1-\lambda\sum_{i=0}^{s-1}\lambda^i H_i w_1-\lambda^s A_s w_2=f_1 \tag{4.82}$$

and

$$w_2-\lambda B_s w_1=f_2\,. \tag{4.83}$$

Solving (4.83) for $w_2$ and substituting the result into (4.82), we obtain

$$[I-\lambda H(\lambda)]w_1=f_1+\lambda^s A_s f_2 \tag{4.84}$$

or

$$w_1=[I-\lambda H(\lambda)]^{-1}f_1+\lambda^s[I-\lambda H(\lambda)]^{-1}A_s f_2\,. \tag{4.85}$$

Substituting (4.85) into (4.83), we have

$$w_2=\lambda B_s[I-\lambda H(\lambda)]^{-1}f_1+\{I+\lambda^{s+1}B_s[I-\lambda H(\lambda)]^{-1}A_s\}f_2\,. \tag{4.86}$$

Comparing (4.81) with (4.85) and (4.86), we easily obtain (4.46). ■

COROLLARY IV.1. – If $[\mathscr{I} - \lambda J(\lambda)]^{-1}$ has a pole at $\lambda = a$ of multiplicity $\nu$, (i.e. $(\lambda - a)^{\nu}[\mathscr{I} - \lambda J(\lambda)]^{-1}$ has a removable singularity at $\lambda = a$ while $(\lambda - a)^{\nu-1} \cdot [\mathscr{I} - \lambda J(\lambda)]^{-1}$ does not), where $\nu$ is a positive integer, then so does $[I - \lambda H(\lambda)]^{-1}$. Convergence of the Laurent series involved here is assumed to be with respect to $|\ |_{\infty}$.

PROOF. – The proof follows easily from (4.46), from the fact that $(\lambda - a)^{\nu} \cdot [\mathscr{I} - \lambda J(\lambda)]^{-1}$ has a removable singularity at $\lambda = a$, and from lemma III.4. ■

COMMENT. – The reader is again reminded that the conclusions (iii) through (vii) of lemma IV.1 have been proved under the assumption (4.17), and under the assumption about the analyticity of (4.19). We will prove these assumptions in theorem IV.3 by induction.

LEMMA IV.2. – Let $\mathscr{H}$ be any complex Hilbert space, and let $N$ be a compact linear operator on $\mathscr{H}$. Let $\beta > 0$, and let

$$N \in C\{\beta, \mathscr{H}\}. \tag{4.87}$$

If $I$ is the identity operator on $\mathscr{H}$, then

$$\boldsymbol{m}_k\{[I - \lambda N]^{-1}\} \in C\{\beta/k; \mathscr{H}\} \tag{4.88}$$

for each non-negative integer $k$. Let

$$\varrho(k) = \max\{\beta/k; 1\}. \tag{4.89}$$

Then

$$[I - \lambda N]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j\{[I - \lambda N]^{-1}\} \tag{4.90}$$

is analytic with respect to $|\ |_{\varrho(k)}$ about all points $\lambda$ which are not eigenvalues of the equation

$$\lambda N u = u \qquad (u \in \mathscr{H}). \tag{4.91}$$

PROOF. – Since $N$ is compact, the (distinct) eigenvalues of (4.91) may be written in a (possibly finite or empty) sequence $\{\omega_i\}$, which may have only $\infty$ as a limit point.

If $|\lambda|$ is sufficiently small, we have

$$[I - \lambda N]^{-1} = \sum_{j=0}^{\infty} \lambda^j N^j, \tag{4.92}$$

where convergence of the series in (4.92) is uniform with respect to $|\ |_\infty$ near $\lambda = 0$. The assertion (4.88) follows trivially from (4.87) and from (4.92). We obtain from (4.92) the equality of (4.90) and of

$$\lambda^k N^k [I - \lambda N]^{-1} \tag{4.93}$$

provided $|\lambda|$ is small. Since the sequence $\{\omega_i\}$ is composed of isolated points (i.e. the sequence cannot have a finite limit point), we may appeal to analytic continuation to establish the equality of (4.90) and of (4.93) for all complex $\lambda$ which are not elements of the sequence $\{\omega_i\}$.

If $z_0$ is not an element of $\{\omega_i\}$, then $[I - \lambda N]^{-1}$ is analytic with respect to $|.|_\infty$ for all $\lambda$ near $z_0$, by the analogue of (4.8). Furthermore $N^k \in C(\beta/k)$, so that $\lambda^k N^k$ is an entire function with respect to $|\ |_{\varrho(k)}$. Hence an easy application of lemma III.1 guarantees analyticity of (4.93) (and hence of (4.90)) with respect to $|\ |_{\varrho(k)}$, provided, of course, that $\lambda$ is not an element of the sequence $\{\omega_i\}$. ∎

DEFINITION. – Let $\{\zeta_i\}$ be a (possibly finite) sequence of complex numbers having no finite limit point. We shall say that $\{\zeta_i\}$ is ordered if $\{\zeta_i\}$ satisfies (*a*) and (*b*) below:

*a*) $\{|\zeta_i|\}$ is a non-decreasing sequence.

*b*) If

$$|\zeta_i| = |\zeta_{i+1}|$$

for some positive integer $i$, then

$$0 \leqslant \arg \zeta_i \leqslant \arg \zeta_{i+1} < 2\pi .$$

THEOREM IV.3. – Let $H(\lambda)$ be given by (4.1) where $s$ is a non-negative integer, where $\lambda$ is a complex parameter, and where the $H_i$ are operators on the complex Hilbert space $\mathscr{H}$. Let

$$H_i \in C\{\alpha_i; \mathscr{H}\} \tag{4.96}$$

for each integer $i$ in $0 \leqslant i \leqslant s$, where $\alpha_i$ is a positive real for each integer $i$ in $0 \leqslant i \leqslant s$. Let $k_0$ be given by (4.16). The following conclusions are valid.

(i) The (distinct) eigenvalues of (4.2) form a denumerable (or possibly empty) set of isolated points.

(ii) The function (4.42) is analytic with respect to $|\ |_1$, provided $k$ is an integer such that $k \geqslant k_0 - 1$ and provided $\lambda$ is not an eigenvalue of (4.2).

(iii) For each integer $k \geqslant k_0 - 1$, there exists a unique entire function $\Delta(\lambda; k)$ satisfying $\Delta(0, k) = 1$ such that

$$-\Delta'(\lambda; k)/\Delta(\lambda; k) = \tau\Big\{[\lambda H(\lambda)]'[I - \lambda H(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j[[\boldsymbol{\lambda} H(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}]\Big\} \tag{4.97}$$

holds for each integer $k \geqslant k_0 - 1$ and for each $\lambda$ which is not an eigenvalue of (4.2). (The derivative of $\lambda H(\lambda)$ in (4.97) is taken with respect to (say) $|\ |_\infty$.)

(iv) The (possibly finite or empty) *ordered* sequence of zeroes of $\Delta(\lambda; k)$, taken according to multiplicity, is independent of $k$, where $k$ is an integer satisfying $k \geqslant k_0 - 1$. This sequence will be denoted by $\{\lambda_i\}$ in the sequel. The sequence $\{\lambda_i\}$, if infinite, can have no finite limit point; furthermore, $\lambda = a$ is an eigenvalue of (4.2) if and only if $a = \lambda_i$ for some positive integer $i$.

(v) The equation

$$\frac{-\Delta'(\lambda; k)}{\Delta(\lambda; k)} = \sum_{j=k}^{\infty} \lambda^j \tau_j\{[\boldsymbol{\lambda} H(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\} \tag{4.98}$$

holds for each integer $k \geqslant k_0 - 1$ and for each complex $\lambda$ such that

(4.99) $|\lambda| < \infty$ if there are no eigenvalues of equation (4.2)

(4.100) $|\lambda| < |\lambda_1|$ if $\{\lambda_i\}$ is not void,

where $|\lambda_1| > 0$ in (4.100). The series in (4.98) converges uniformly and absolutely on all compact subsets of the applicable set described in (4.99)-(4.100).

(vi) We have

$$\sum_i |\lambda_i|^{-k_0} < +\infty \tag{4.101}$$

and

$$\sum_i \lambda_i^{-(k+1)} = \tau_k\{[\lambda H(\lambda)]'[I - \lambda H(\lambda)]^{-1}\} \tag{4.102}$$

$$= \tau_k\{[I - \lambda H(\lambda)]^{-1}[\lambda H(\lambda)]'\}, \tag{4.103}$$

where $k$ is an integer such that $k \geqslant k_0 - 1$ in (4.102)-(4.103), and where the summations in (4.101) and (4.102) are taken over the whole sequence $\{\lambda_i\}$; the left side of (4.102) is understood to be zero if there are no eigenvalues of equation (4.2).

(vii) The function $[I - \lambda H(\lambda)]^{-1}$ is analytic with respect to $|\ |_\infty$ at all points $\lambda$ not in the sequence $\{\lambda_i\}$. The function $[I - \lambda H(\lambda)]^{-1}$ has poles at the points $\lambda = \lambda_i$. The function $\Delta(\lambda; k)[I - \lambda H(\lambda)]^{-1}$ has removable singularities at the points $\lambda = \lambda_i$, and hence its appropriate extension will be an entire function with respect to $|\ |_\infty$.

(viii) Let $k_1$ be the smallest integer such that $k_1 \geqslant k_0 - 1$. There exists a constant $\Gamma > 0$ such that

$$|\Delta(\lambda; k_1)| \leqslant \exp[\Gamma |\lambda|^{k_0}] \tag{4.104}$$

$$|\Delta(\lambda; k_1)[I - \lambda H(\lambda)]^{-1}|_\infty \leqslant \exp[\Gamma |\lambda|^{k_0}]. \tag{4.105}$$

(ix) The multiplicity of the eigenvalue $\lambda = a$ in the sequence $\{\lambda_i\}$ is precisely the Keldysh multiplicity $M(a)$ of the eigenvalue $\lambda = a$.

COMMENT. – The order of the proofs of the various assertions of this theorem will not be the same as the order in which the assertions have been stated. The reason for this is that we wish to avoid reference in the statement of the theorem to the auxiliary systems $(4.119)_q$ to be constructed in the proof.

If $\mathscr{H} = L_2[0, 1]$ and if $0 < \alpha_i \leqslant 2$ for each integer $i$ in $0 \leqslant i \leqslant s$, then it turns out that the functions $\Delta(\lambda; k)$ constructed in this theorem are closely related to the classical Fredholm determinant of the square-integrable kernel $h(x, y, \lambda)$ of $H(\lambda)$, where we assume that $h(x, x, \lambda)$ is Lebesgue integrable on $[0, 1]$ so that the classical Fredholm determinant exists. The reason for this close relation is that the classical Fredholm determinant satisfies an equation closely related to (4.97) (see FREDHOLM [6; p. 380]). We shall not, however, pursue the matter further, since the actual relation is not useful for our purposes here.

PROOF. – The Hilbert space $\mathscr{H}^{\{p\}}$ are defined as follows;

$$\mathscr{H}^{\{s\}} = \mathscr{H}, \tag{4.106}$$

$$\mathscr{H}^{\{p-1\}} = \mathscr{H}^{\{p\}} \times \mathscr{H}^{\{p\}}, \tag{4.107}$$

where $p$ is an integer such that $1 \leqslant p \leqslant s$ in (4.107). The superscripts $\{s\}$, $\{p-1\}$, and $\{p\}$ are indices here; the space $\mathscr{H}^{\{2\}}$, for example, is not generally the same as $\mathscr{H}^{[2]} = \mathscr{H} \times \mathscr{H}$. We define the numbers $\beta_i^{\{p\}}$ for integers $i$ and $p$ such that $0 \leqslant i \leqslant p \leqslant s$ by the equations

$$\beta_0^{\{p\}} = \max\{(j+1)\alpha_j : j = 0 \text{ or } p+1 \leqslant j \leqslant s\} \tag{4.108}$$

$$\beta_p^{\{p\}} = \frac{1}{p+1} \max\{(j+1)\alpha_j : p \leqslant j \leqslant s\} \tag{4.109}$$

$$\beta_i^{\{p\}} = \alpha_i, \quad 1 \leqslant i \leqslant p-1 \leqslant s-1. \tag{4.110}$$

Let

$$J^{\{s\}}(\lambda) \quad = H(\lambda) \tag{4.111}$$

$$J^{\{s-1\}}(\lambda) = J(\lambda) \tag{4.112}$$

where $J(\lambda)$ is given by the matrix (4.15). We assume, roughly speaking, that the matrix operator $J^{\{p-1\}}(\lambda)$ is defined inductively from $J^{\{p\}}(\lambda)$ in the same manner that $J^{\{s-1\}}(\lambda)$ was defined from $J^{\{s\}}(\lambda)$. More precisely, let $p$ be any *fixed* integer such that $1 \leqslant p \leqslant s-1$, and let us assume that for each integer $q$ in $1 \leqslant p \leqslant q \leqslant s$, there exist operators $J^{\{q\}}(\lambda)$ on $\mathscr{H}^{\{q\}}$ with the following properties:

$(a)_p$ We have for each integer $q$ in $1 \leqslant p \leqslant q \leqslant s$

$$J^{\{q\}}(\lambda) = \sum_{i=0}^{q} \lambda^i J_i^{\{q\}} \tag{4.113}$$

where $J_i^{\{q\}}$ is independent of $\lambda$; furthermore, we have (see (4.108)-(4.110))

$$J_i^{\{q\}} \in C\{\beta_i^{\{q\}}; \mathscr{H}^{\{q\}}\} \tag{4.114}$$

for each pair of integers $i$ and $q$ such that $1 \leqslant p \leqslant q \leqslant s$ and $0 \leqslant i \leqslant q$.

$(b)_p$ For each integer $q$ in $1 \leqslant p < q \leqslant s$, the operator $J_q^{\{q\}}$ may be factored so that

$$J_q^{\{q\}} = A_q^{\{q\}} B_q^{\{q\}} \tag{4.115}$$

where

$$A_q^{\{q\}} \in C\{((q+1)/q)\beta_q^{\{q\}}; \mathscr{H}^{\{q\}}\} \tag{4.116}$$

and where

$$B_q^{\{q\}} \in C\{(q+1)\beta_q^{\{q\}}; \mathscr{H}^{\{q\}}\} \tag{4.117}$$

where $\beta_q^{\{q\}}$ is given by (4.109).

$(c)_p$ We assume that $J^{\{q\}}(\lambda)$ is defined by (4.111) for $q = s$ and that

$$J^{\{q\}}(\lambda) = \begin{pmatrix} \sum_{i=0}^{q} \lambda^i J_i^{\{q+1\}} & \lambda^q A_{q+1}^{\{q+1\}} \\ B_{q+1}^{\{q+1\}} & 0 \end{pmatrix} \tag{4.118}$$

for integers $q$ satisfying $1 \leqslant p \leqslant q \leqslant s-1$.

$(d)_p$ The complex number $\lambda$ is either an eigenvalue of

$$\lambda J^{\{q\}}(\lambda) u^{\{q\}} = u^{\{q\}} \qquad (u^{\{q\}} \in \mathscr{H}^{\{q\}}) \tag{4.119$_q$}$$

for each integer $q$ in $1 \leqslant p \leqslant q \leqslant s$ or for no integer $q$ in $1 \leqslant p \leqslant q \leqslant s$.

Note that $(a)_p$, $(b)_p$, $(c)_p$, $(d)_p$ hold for $p = s-1$ by lemma IV.1. This enables us to start the inductions. Having defined $J^{\{s\}}(\lambda)$, $J^{\{s-1\}}(\lambda), \ldots, J^{\{p\}}(\lambda)$, we proceed to define $J^{\{p-1\}}(\lambda)$. Since (see (4.114))

$$J_p^{\{p\}} \in C\{\beta_p^{\{p\}}; \mathscr{H}^{\{p\}}\} \tag{4.120}$$

we factor $J_p^{\{p\}}$ by writing

$$J_p^{\{p\}} = A_p^{\{p\}} B_p^{\{p\}} \tag{4.121}$$

with

$$A_p^{\{p\}} \in C\{((p+1)/p)\beta_p^{\{p\}}; \mathscr{H}^{\{p\}}\} \tag{4.122}$$

$$B_p^{\{p\}} \in C\{(p+1)\beta_p^{\{p\}}; \mathscr{H}^{\{p\}}\}. \tag{4.123}$$

This is possible by (2.24). We define $J^{\{p-1\}}(\lambda)$ by setting $q=p-1$ in (4.118). Note that $J^{\{p-1\}}(\lambda)$ assumes the form (4.113) with $q=p-1$, and with

$$J_i^{\{p-1\}} = \begin{pmatrix} J_i^{\{p\}} & A_p^{\{p\}}\delta_{i,p-1} \\ B_p^{\{p\}}\delta_{i,0} & 0 \end{pmatrix} \tag{4.124}$$

where $\delta_{ij}$ is the Kronecker delta.

For $i=0$ and for each integer $p$ in $2 \leqslant p \leqslant s-1$, the result

$$J_i^{\{p-1\}} \in C\{\beta_i^{\{p-1\}}; \mathscr{H}^{\{p-1\}}\} \tag{4.125}$$

follows from (4.114) with $i=0$ and $q=p$, from (4.123), from (4.124) with $i=0$, from (2.10), from lemma III.3, and from the relation

$$\beta_0^{\{p-1\}} = \max\{\beta_0^{\{p\}}\,(p+1)\beta_p^{\{p\}}\}. \tag{4.126}$$

The restriction $2 \leqslant p \leqslant s-1$ enables us to set the entry in the first row, second column of $J_0^{\{p-1\}}$ equal to the zero operator (on $\mathscr{H}^{\{p\}}$).

For $i=p-1$ and $2 \leqslant p \leqslant s-1$, the matrix $J_i^{\{p-1\}}$ again has (at most) two non-zero elements; in this case, the relation (4.125) follows in similar fashion if we note that

$$\beta_{p-1}^{\{p-1\}} = \max\left\{\beta_{p-1}^{\{p\}}, \frac{p+1}{p}\beta_p^{\{p\}}\right\}. \tag{4.127}$$

The case where $i=0=p-1$ in (4.125) needs separate treatment, since the matrix $J_0^{\{0\}}$ will have three non-zero components; however, the basic formulas (4.126) and (4.127) are valid and consistent if $i=0=p-1$.

The proof of (4.125) for integers $p$ and $i$ such that $3 \leqslant p \leqslant s-1$, and $1 \leqslant i \leqslant p-2$ is easy, and is left to the reader. This exhausts all possibilities.

Finally, the fact that $\lambda$ is an eigenvalue of $(4.119)_{p-1}$ iff $\lambda$ is an eigenvalue of $(4.119)_p$ is similar to assertion (i) of lemma IV.1, and is left to the reader.

Note that $J^{\{0\}}(\lambda)$ is independent of $\lambda$; for brevity, we shall sometimes write $J^{\{0\}}$ instead of $J^{\{0\}}(\lambda)$. Furthermore $\lambda$ is an eigenvalue of $(4.119)_q$ for all $q$ in $0 \leqslant q \leqslant s$ or for no value of $q$ in $0 \leqslant q \leqslant s$. Hence equation (4.2) is «equivalent» to $(4.119)_q$ with $q=0$ in the sense described. Also, we may see that

$$J^{\{0\}} = J_0^{\{0\}} \in C(\beta_0^{\{0\}}) = C(k_0) \tag{4.128}$$

where $k_0$ is defined by (4.16). A fortiori, we have the compactness of $J^{\{0\}}$. Hence the (distinct) eigenvalues of $(4.119)_q$ with $q=0$ or of (4.2) must form a denumerable (or possibly empty) set. We denote the sequence of eigenvalues of $(4.119)_q$ with $q=0$ by $\{\lambda_i\}$, where each eigenvalue $\lambda=a$ of $(4.119)_q$ with $q=0$ appears in the sequence $\{\lambda_i\}$ according to its geometric multiplicity, which is the maximum dimension of the null spaces

$$\mathcal{N}\{[I^{\{0\}}-aJ^{\{0\}}]^j\} \qquad (j=1,2,\ldots). \tag{4.129}$$

The operator $I^{\{p\}}(0\leqslant p\leqslant s)$ here denotes the identity operator on $\mathcal{H}^{\{p\}}$. We assume that the sequence $\{\lambda_i\}$ satisfies the ordering previously described. Of course, we must show that the sequence $\{\lambda_i\}$ so defined is precisely the sequence described in the statement of this theorem. The result that the eigenvalues of (4.2) form a denumerable set is not new; it has been proven in the case where the $H_i$ are merely compact (see KELDYSH [12]). Clearly the sequence $\{\lambda_i\}$ has no finite limit point, since $J^{\{0\}}$ is compact. It is also clear that $|\lambda_i|>0$ for each applicable $i$.

By (4.128) and (2.3), we have the inequality (4.101). For integers $k\geqslant k_0-1$, we define $\Delta(\lambda;k)\equiv 1$ if no eigenvalues of (4.2) exist; if eigenvalues of (4.2) do exist, then we define

$$\Delta(\lambda;k)=\prod_i(1-\lambda/\lambda_i)\exp\left\{\sum_{j=1}^{k}(1/j)(\lambda/\lambda_i)^j\right\}, \tag{4.130}$$

where the product in (4.130) is taken over the entire sequence $\{\lambda_i\}$. The product so defined converges uniformly on bounded subsets of the complex plane as a direct result of (4.101), as we shall show. (Obviously, the only case of interest here is if the sequence $\{\lambda_i\}$ is infinite). Let

$$b(\sigma;k)=\left[(1+\sigma)\exp\sum_{j=1}^{k}((-1)^j/j)\,\sigma^j\right]-1. \tag{4.131}$$

Now there exists $\tilde{\Gamma}>0$ such that if $k\geqslant k_0-1$, then

$$|b(\sigma;k)|\leqslant\tilde{\Gamma}|\sigma|^{k+1}\leqslant\tilde{\Gamma}|\sigma|^{k_0} \tag{4.132}$$

for all complex $\sigma$ in $|\sigma|\leqslant 1$; here $\tilde{\Gamma}$ depends only on $k$ (see DUNFORD-SCHWARTZ [5; p. 1107, inequality no. 4]). If we let

$$a_i(\lambda;k)=b(-\lambda/\lambda_i;k) \tag{4.133}$$

then we can use (4.101), (4.132), (4.133) to apply theorem 7.3, LEVINSON-REDHEFFER [15; p. 385], to obtain uniform convergence of the product

$$\prod_i[1+a_i(\lambda;k)]=\Delta(\lambda;k) \tag{4.134}$$

on bounded subsets of the complex plane; furthermore, by the theorem quoted in Levinson-Redheffer, the product in (4.134) or (4.130) is zero if and only if at least one of its factors is zero. Hence the product in (4.130) defines an entire function $\Delta(\lambda; k)$ whose zeroes, taken according to algebraic multiplicity, form precisely the sequence $\{\lambda_i\}$. Thus it is clear that assertions (i), (iv), and inequality (4.101) of assertion(vi) are proved by the preceding considerations.

In order to prove (4.102), we note that if $k \geqslant k_0 - 1$, then

$$\sum_i \lambda_i^{-(k+1)} = \tau\{(J^{\{0\}})^{k+1}\} \tag{4.135}$$

$$= \tau_k\{[\lambda J^{\{0\}}]'[I^{\{0\}} - \lambda J^{\{0\}}]^{-1}\} \tag{4.136}$$

where the left side of (4.135) is zero if (4.2) has no eigenvalues. Equation (4.135) follows from (2.17) and from the fact that (see (4.128))

$$[J^{\{0\}}]^{k+1} \in C\{k_0/(k+1); \mathscr{H}^{\{0\}}\}. \tag{4.137}$$

Equation (4.136) is obtained by left multiplying the relation

$$[I^{\{0\}} - \lambda J^{\{0\}}]^{-1} = \sum_{j=0}^{\infty} \lambda^j [J^{\{0\}}]^j \tag{4.138}$$

through by $[\lambda J^{\{0\}}]'$. It is clear that the right side of (4.138) is convergent with respect to $|\ |_\infty$, provided $\lambda$ is small.

Noting (4.135)-(4.136), we see that the proof of (4.102) can be established if we can prove that the relation

$$\tau_k\{[\lambda J^{\{q-1\}}(\lambda)]'[I^{\{q-1\}} - \lambda J^{\{q-1\}}(\lambda)]^{-1}\} = \tau_k\{[\lambda J^{\{q\}}(\lambda)]'[I^{\{q\}} - \lambda J^{\{q\}}(\lambda)]^{-1}\} \tag*{(4.139)$_q$}$$

holds for all integers $q$ such that $1 \leqslant q \leqslant s$ and all integers $k \geqslant k_0 - 1$. The derivatives in $(4.139)_q$ are taken with respect to (say) $|\ |_\infty$.

By (4.137) and by (4.138), we have that

$$\boldsymbol{m}_k\{[I^{\{p\}} - \lambda J^{\{p\}}(\lambda)]^{-1}\} \in C\{k_0/k; \mathscr{H}^{\{p\}}\} \tag*{(4.140)$_p$}$$

for $p = 0$ and for all non-negative integers $k$. Also, by (4.128) and by lemma IV.2, the function

$$[I^{\{p\}} - \lambda J^{\{p\}}(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j m_j\{[I^{\{p\}} - \boldsymbol{\lambda} J^{\{p\}}(\boldsymbol{\lambda})]^{-1}\} \tag*{(4.141)$_p$}$$

is analytic with respect to $|\ |_{\gamma(k)}$ (see (4.18)) provided $p = 0$ in (4.141) and provided $\lambda$ is not an eigenvalue of (4.2). Note that if $k_0$ is given by (4.16), then the relation

$$k_0 = \max\{(i+1)\beta_i^{\{p\}}: 0 \leqslant i \leqslant p\} \tag{4.142}$$

holds for each fixed integer $p$ in $0 \leqslant p \leqslant s$. Hence if we replace $H(\lambda)$ and $J(\lambda)$ in lemma IV.1 by $J^{\{1\}}(\lambda)$ and $J^{\{0\}}(\lambda)$ respectively and if we make other necessary notational changes in lemma IV.1, then by the validity of (4.114) for integers $i$ and $q$ in $0 \leqslant i \leqslant q \leqslant 1$, by (4.118) with $q = 0$, by the validity of (4.142) for $p = 0$ and $p = 1$, by the validity of $(4.140)_p$ for $p = 0$, and by the analyticity of $(4.141)_p$ with respect to $| \; |_{\gamma(k)}$, provided $p = 0$ in $(4.141)_p$, the conclusions $(1)_p$, $(2)_p$, and $(3)_p$ listed below are valid for $p = 1$.

$(1)_p$ The assertion $(4.140)_p$ is valid for all nonnegative integers $k$.

$(2)_p$ The function $(4.141)_p$ is analytic with respect to $| \; |_{\gamma(k)}$ for each non-negative integer $k$ and each $\lambda$ which is not an element of the sequence $\{\lambda_i\}$.

$(3)_p$ The relation $(4.139)_q$ holds for each integer $q$ in $1 \leqslant q \leqslant p$ and each integer $k$ such that $k \geqslant k_0 - 1$.

By induction, and by appropriate use of lemma IV.1, we can show that the statements $(1)_p$, $(2)_p$, and $(3)_p$ are valid for each integer $p$ in $0 \leqslant p \leqslant s$. In particular, $(1)_{s-1}$ is equivalent to (4.17) by (4.112), and $(2)_{s-1}$ yields the analyticity of (4.19) (where defined) with respect to $| \; |_{\gamma(k)}$. Hence (4.17) and the analyticity of (4.19) can be unconditionally verified (i.e. they do not have to be assumed). Assertion (vii) of lemma IV.1 implies the validity of assertion (ii) in the statement of this theorem, while (4.135), (4.136), and $(3)_s$ imply equation (4.102). Finally (4.112) and $(1)_{s-1}$ imply (4.103) by the results of assertion (v) of lemma IV.1. At this point, assertions (i), (ii), (iv), and (vi) in the statement of this theorem are established.

In order to prove (4.98), we note that (see LEVINSON-REDHEFFER [10; p. 392]) if $k \geqslant k_0 - 1$, then

$$\frac{\Delta'(\lambda;k)}{\Delta(\lambda;k)} = \sum_i \frac{a_i'(\lambda;k)}{1 + a_i(\lambda;k)} \tag{4.143}$$

where the summation in (4.143) is taken over all $i$ for which $a_i(\lambda; k)$ is defined (see 4.133)). Convergence of the series in (4.143) is uniform for $\lambda$ on compact subsets of the complex plane which do not contain any points of the sequence $\{\lambda_i\}$. In particular, convergence of the series in (4.143) is uniform for $\lambda$ on compact subsets of the set described in (4.100) (here, we assume that there do exist eigenvalues of (4.2)). Since

$$\frac{a_i'(\lambda;k)}{1 + a_i(\lambda;k)} = -\sum_{j=k}^{\infty} \lambda^j/\lambda_i^{j+1} \tag{4.144}$$

for $\lambda$ satisfying (4.100), we have for integers $k \geqslant k_0 - 1$

$$\boldsymbol{m}_j\left[\frac{a_i'(\lambda;k)}{1 + a_i(\lambda;k)}\right] = \begin{cases} -\lambda_i^{-(j+1)} & \text{if } j \geqslant k \\ 0 & \text{if } j < k \end{cases} \tag{4.145}$$

Since the operator $\boldsymbol{m}_j$ is an integral operator, and since the series in (4.143) converges uniformly for (say) $|\lambda| \leqslant (\frac{1}{2})|\lambda_1|$, we have

$$\boldsymbol{m}_j \frac{\Delta'(\lambda;\, k)}{\Delta(\lambda;\, k)} = \begin{cases} -\sum\limits_i \lambda_i^{-(j+1)} & \text{if } j \geqslant k \\ 0 & \text{if } j < k \end{cases} \tag{4.146}$$

where the summation in (4.146) is over the whole sequence $\{\lambda_i\}$. Hence

$$\frac{-\Delta'(\lambda;k)}{\Delta(\lambda;\, k)} = \sum_{j=k}^{\infty} \lambda^j \sum_i \lambda_i^{-(j+1)}\,. \tag{4.147}$$

Equation (4.147) follows immediately from (4.146) and is valid for $\lambda$ satisfying (4.100). Hence (4.98) follows from (4.102) and from (4.147), at least in the case where (4.2) has eigenvalues. If (4.2) has no eigenvalues, then (4.98) follows from (4.102) and from the fact that $\Delta(\lambda;\, k) \equiv 1$.

We continue to assume that equation (4.2) has eigenvalues. If we use the result (ii) of this theorem, along with (3.8) and the fact that $\tau$ is a continuous linear functional on $C(1)$ (see (2.16)), then (4.97) follows (*) immediately from (4.98), provided $\lambda$ satisfies the restriction (4.100). Since the sequence $\{\lambda_i\}$ may have only $\lambda = \infty$ as a (possible) limit point, we may use analytic continuation arguments to establish (4.97) for all $\lambda$ not in the sequence $\{\lambda_i\}$. The analyticity of the right side of (4.97) follows from lemma III.2 provided $\lambda \neq \lambda_i$ for any $i$. Hence assertions (i) through (vi) inclusive are verified.

In (vii), we note that the fact that $[I - \lambda H(\lambda)]^{-1}$ has a pole at the points $\lambda = \lambda_i$ follows from the fact that $[I^{(0)} - \lambda J^{(0)}]^{-1}$ has a pole at the points $\lambda = \lambda_i$ (see DUNFORD-SCHWARTZ [4; p. 579]) and from appropriate use of Corollary IV.1.

In order to prove (4.104), we shall verify the inequality (see (4.131))

$$|b(\sigma;\, k_1) + 1| \leqslant \exp\{\hat{\Gamma}|\sigma|^{k_0}\} \tag{4.148}$$

where $k_1$ is the integer described in (viii), where $\hat{\Gamma}$ is a positive constant which depends only on $k_0$, and where $\sigma$ is any complex number. The inequality (4.148) is a restatement of inequality (2), page 1107 in DUNFORD-SCHWARTZ [5]. Their quantities $\lambda$, $\Gamma$, $p$, and $k$ are our $\sigma$, $\hat{\Gamma}$, $k_0$, and $(1 + k_1)$ respectively. Their condition (in our notation) $1 + k_1 \geqslant k_0 \geqslant k_1$ is necessary for the validity of (4.148),and follows easily from the definition of $k_1$. If we set $\sigma = -\lambda/\lambda_i$ in (4.148), and use (4.133) and (4.134), the inequality (4.104) follows immediately, with

$$\Gamma \geqslant \hat{\Gamma} \sum_i |\lambda_i|^{-k_0}. \tag{4.149}$$

Now the function

$$\Delta(\lambda;\, k)[I^{(p)} - \lambda J^{(p)}(\lambda)]^{-1} \tag{4.150}$$

(*) The MacLaurin coefficients of (4.42) are norm invariant w.r.t $|\ |_1$ and $|\ |_\infty$.

is clearly analytic with respect to $|\ |_\infty$, provided $\lambda$ is not an element of the sequence $\{\lambda_i\}$, and has (at worst) poles at the points $\lambda = \lambda_i$.

In order to establish (4.105), let us assume for the time being that there exists a positive constant $\Gamma'$ (which as we shall see, depends only on $k_0$, and not on $J^{\{0\}}$ or on the operators $H_i$) such that

$$|\Delta(\lambda; k_1)[I^{\{0\}} - \lambda J^{\{0\}}]^{-1}|_\infty \leqslant \exp\{\Gamma' |J^{\{0\}}|_{k_0}^{k_0} |\lambda^{k_0}|\}. \tag{4.151}$$

We assume, of course, that $\lambda$ is not an element of the sequence $\{\lambda_i\}$ in (4.151). The superscript on the right side of (4.151) refers to a power; the subscript refers to the norm $|\ |_{k_0}$ on the space $C\{k_0; \mathscr{H}^{\{0\}}\}$. If (4.151) is temporarily accepted, and if $\Gamma > 0$ satisfies the inequalities (4.149) and

$$\Gamma \geqslant \Gamma' |J^{\{0\}}|_{k_0}^{k_0} \geqslant \Gamma' \sum_i |\lambda_i|^{-k_0} \tag{4.152}$$

then we have

$$|\Delta(\lambda; k_1)\,[I^{\{p\}} - \lambda J^{\{p\}}]^{-1}|_\infty \leqslant \exp[\Gamma |\lambda|^{k_0}] \tag{4.153}$$

for $p = 0$, and for $\lambda$ not in the sequence $\{\lambda_i\}$. If we replace $H(\lambda)$ and $J(\lambda)$ in inequality (4.44) of lemma IV.1 by $J^{\{p\}}(\lambda)$ and $J^{\{p-1\}}(\lambda)$ respectively, and make other appropriate notational substitutions, then we have

$$|[I^{\{p\}} - \lambda J^{\{p\}}]^{-1}|_\infty \leqslant |[I^{\{p-1\}} - \lambda J^{\{p-1\}}(\lambda)]^{-1}|_\infty \tag{4.154}$$

for integers $p$ in $1 \leqslant p \leqslant s$ and for complex $\lambda$ not an element of the sequence $\{\lambda_i\}$. Hence (4.153) is valid for all $p$ in $0 \leqslant p \leqslant s$; the inequality (4.105) follows immediately from (4.153) with $p = s$.

The inequality (4.151) follows from corollary 25, page 1112, from theorem 26, p. 1113, and from lemma 22(*f*), p. 1106 in DUNFORD-SCHWARTZ [5]. Their quantities $\Gamma$, $k$, and $p$ are equal to our $\Gamma'$, $(1 + k_1)$, and $k_0$ respectively; their operator $T$ is replaced by our operator $(-\lambda J^{\{0\}})$, their function $\det_k(I + T)$ is to be replaced by our function $\Delta(\lambda; k_1)$. Also their eigenvalues $\{\lambda_i\}$ in the definition of their function $\det_k(I + T)$ on page 1106 would have to be replaced by our quantity $(-\lambda/\lambda_i)$, since the analogue of their equation $Tu = \lambda_i u$ would be $-\lambda J^{\{0\}} u = -(\lambda/\lambda_i) u$ in our notation. Hence (4.151) and therefore (4.105) are established.

We have already noted that (4.150) has (at worst) a pole at the points $\lambda = \lambda_i$. If (4.150) had a pole at $\lambda = \lambda_i$, then the left side of (4.153) would tend to $\infty$ as $\lambda \to \lambda_i$. Hence (4.150) must have only a removable singularity at $\lambda = \lambda_i$.

In order to prove assertion (ix), let $M(a)$ be the Keldysh multiplicity of the eigenvalue $\lambda = a$ of (4.2), and let $m(a)$ be the algebraic multiplicity of $\lambda = a$ as a zero of the functions $\Delta(\lambda; k)$. Let $\mathscr{P}(a) g(\lambda)$ denote the principal (or singular) part of any complex or operator-valued function $g(\lambda)$ about $\lambda = a$. We freely use the fact that $\mathscr{P}(a)g(\lambda)$ is zero if $g(\lambda)$ is analytic at $\lambda = a$ with respect to an appropriate norm. As before, $R(\lambda)$ will denote the resolvent of $H(\lambda)$ (see (4.5)). By (4.6), we have that

$$\frac{M(a)}{a-\lambda} = \tau\mathscr{P}(a)\{[\lambda H(\lambda)]'[\lambda R(\lambda)]\} \tag{4.155}$$

$$= \tau\mathscr{P}(a)\{[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1}\} \tag{4.156}$$

$$= \tau\mathscr{P}(a)\Big\{[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1} - \sum_{j=0}^{k-1}\lambda^j \boldsymbol{m}_j[[\boldsymbol{\lambda} H(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}]\Big\} \tag{4.157}$$

$$= \mathscr{P}(a)\left[-\frac{\Delta'(\lambda;k)}{\Delta(\lambda;k)}\right] \tag{4.158}$$

$$= m(a)/(a-\lambda)\,. \tag{4.159}$$

Equation (4.158) follows from (4.97) and from the fact that $\tau\mathscr{P}(a)g(\lambda) = \mathscr{P}(a)\tau g(\lambda)$ if $g(\lambda)$ is an operator valued function analytic with respect to $|\ |_1$ in a deleted neighborhood about $\lambda = a$. The equation (4.159) is easily proved using Laurent series, so $M(a) = m(a)$, and assertion (ix) is proved.

We remark that KELDYSH [7] states that $R(\lambda)$ has a pole at $\lambda = \lambda_i$, and that he characterizes the coefficients of the negative powers of $(\lambda - \lambda_i)$ in the Laurent expansion of $(\lambda R(\lambda))$ about $\lambda = \lambda_i$ as being operators whose range is contained in the space generated by the set of vectors $y_j$ described in (4.3). ■

COROLLARY IV.3. – Let $H(\lambda)$ be given by (4.1), with $s \geqslant 1$, and let $s'$ be any integer satisfying $0 \leqslant s' \leqslant s-1$. Let $\alpha_i$ be a positive real for each integer $i$ in $0 \leqslant i \leqslant s'$, and let (4.96) hold for each integer in $0 \leqslant i \leqslant s'$. Furthermore, for each integer $i$ in $s'+1 \leqslant i \leqslant s$, let $H_i \in C\{\infty;\mathscr{H}\}$ be an operator of finite dimensional range. Then $H_i \in C\{p;\mathscr{H}\}$ for each $p > 0$ and each integer $i$ in $s'+1 \leqslant i \leqslant s$; hence, the number $k_0$ in (4.16) is equal to

$$\max\{(i+1)\alpha_i \colon 0 \leqslant i \leqslant s'\}\,. \tag{4.160}$$

COMMENT. – Loosely speaking, under the assumptions of this corollary, the higher order terms in $H(\lambda)$ do not substantially affect the convergence of the series $\sum \lambda_i^{-p}$ and our ability to evaluate the latter series for integral $p$.

The conclusion that $H_i \in C\{p;\mathscr{H}\}$ for each $p > 0$ and each $i$ in $s'+1 \leqslant i \leqslant s$ comes from the fact that $H_i^* H_i$ and hence $(H_i^* H_i)^{\frac{1}{2}}$ are also operators with finite dimensional range; hence the singular values of $H_i$, i.e. the eigenvalues of $(H_i^* H_i)^{\frac{1}{2}}$, are finite in number. This is well-known.

The reader can make various modifications of this corollary.

EXAMPLE. – Let the complex Hilbert space $\mathscr{H}$ be infinite dimensional, and let

$$\{\varphi_{i,j} \colon i = 0, \ldots, s \text{ and } j = 1, 2, \ldots\} \tag{4.161}$$

be any infinite orthonormal set in $\mathscr{H}$. Let $b_j$ be any positive sequence such that

(4.162)
$$\sum_{j=1}^{\infty} b_j^{-\varepsilon}$$

converges for real $\varepsilon$ if and only if $\varepsilon \geqslant 1$. Let $\alpha_1, \dots, \alpha_s$ be any collection of positive reals, and let the operator $H_i$ be defined by

(4.163)
$$H_i u = \sum_{j=1}^{\infty} b_j^{-(\alpha_i^{-1})} \varphi_{i,j} \langle u, \varphi_{i,j} \rangle .$$

The eigenvalues and singular values of $H_i$ are given by the sequence $\{b_1^{(\alpha_i^{-1})}, b_2^{(\alpha_i^{-1})}, \dots\}$. Hence $H_i \in C\{\alpha_i; \mathscr{H}\}$ for each $i$ in $0 \leqslant i \leqslant s$. The positive eigenvalues of (4.2) are given by the multiple sequence $\{b_j^{[(i+1)\alpha_i]^{-1}}\}$, where $i$ and $j$ vary as in (4.161); this follows since (4.161) is an orthonormal set. Hence $\sum_i \lambda_i^{-q}$ converges absolutely for real $q$ iff $q \geqslant k_0$, where $k_0$ is given by (4.16).

COMMENTS. – Let $D \in C\{\infty; \mathscr{H}\}$ and suppose that $[I-D]^{-1} \in C\{\infty; \mathscr{H}\}$. Let us further suppose that $s$, $H(\lambda)$, and $H_i$ satisfy the assumptions of theorem IV.3. We assume that $H_i$ is not the zero operator for some value of $i$ in $0 \leqslant i \leqslant s$. Let

(4.164)
$$\tilde{H}(\lambda) = D + \lambda H(\lambda) .$$

We wish to consider briefly the eigenvalues of the equation

(4.165)
$$\tilde{H}(\lambda) u = u \qquad (u \in \mathscr{H}) .$$

It turns out that equation (4.165) has eigenvalues which share all of the properties that the eigenvalues $\{\lambda_i\}$ of (4.2) enjoyed. We write (4.165) as

(4.166)
$$\lambda (I-D)^{-1} H(\lambda) u = u$$

with

(4.167)
$$(I-D)^{-1} H(\lambda) = \sum_{i=0}^{s} \lambda^i (I-D)^{-1} H_i$$

since

(4.168)
$$(I-D)^{-1} H_i \in C\{\alpha_i; \mathscr{H}\} .$$

We may apply theorem IV.3 to the operator $(I-D)^{-1}H(\lambda)$ instead of to $H(\lambda)$. For each integer $k \geqslant k_0 - 1$, where $k_0$ is given by (4.16), there exists an entire function $\tilde{\Delta}(\lambda; k)$ satisfying the analogue of (4.97) obtained by replacing $H(\lambda)$ in (4.97) by $(I-D)^{-1}H(\lambda)$. However this analogue can be written in the more «natural» form

(4.169)
$$-\frac{\tilde{\Delta}'(\lambda; k)}{\tilde{\Delta}(\lambda; k)} = \tau \left\{ [\tilde{H}(\lambda)]' [I - \tilde{H}(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j [[\tilde{H}(\boldsymbol{\lambda})]' [I - \tilde{H}(\boldsymbol{\lambda})]^{-1}] \right\}$$

if we observe that

$$(4.170)\qquad [\lambda(I-D)^{-1}H(\lambda)]'[I-\lambda(I-D)^{-1}H(\lambda)]^{-1} = (I-D)^{-1}[\tilde{H}(\lambda)]'[I-\tilde{H}(\lambda)]^{-1}(I-D),$$

where the derivatives in (4.170) are with respect to $|\ |_\infty$. We point out that the expression whose trace is being taken in (4.169) is indeed an analytic function with respect to $|\ |_1$, except at the eigenvalues of (4.165), of course. The justification of the latter statement is this: if we replace $H(\lambda)$ in (4.42) by $(I-D)^{-1}H(\lambda)$, the ensuing expression is analytic (where defined) with respect to $|\ |_1$, provided $k\geqslant k_0-1$. By (4.170), we can show analyticity of the expression whose trace is being taken in (4.169) with respect to $|\ |_1$, provided $\lambda$ is not an eigenvalue of (4.165) and provided $k\geqslant k_0-1$.

The ordered sequence of zeroes of $\tilde{\Delta}(\lambda; k)$, taken according to algebraic multiplicity, continues to be independent of $k$, and is denoted by $\{\tilde{\lambda}_i\}$. We have that

$$(4.171)\qquad \sum_i |\tilde{\lambda}_i|^{-k_0} < +\infty$$

and

$$(4.172)\qquad \sum_i \tilde{\lambda}_i^{-(k+1)} = \tau_k\{[\tilde{H}(\lambda)]'[I-\tilde{H}(\lambda)]^{-1}\},$$

where the latter is valid for integers $k\geqslant k_0-1$. While (4.172) is, perhaps, more appealing in form than the equation obtained by replacing $\lambda_i$ by $\tilde{\lambda}_i$ and $H(\lambda)$ by $(I-D)^{-1}H(\lambda)$ in (4.102), it turns out that (4.172) is somewhat less practical for actual computational purposes. We will pursue evaluation of the right side of (4.102) in part V.

## V. – The eigenvalues of certain operators which are meromorphic functions of the eigenvalue parameter.

The symbols $\mathscr{H}$, $C(p)$, $H(\lambda)$, $H_i$, $s$, $\lambda_i$, $\alpha_i$, $\Delta(\lambda; k)$, $\boldsymbol{m}_k$, $\tau$, $\tau_k$, $k_0$, and $k_1$ of the previous section will continue to have the same meaning. We shall continue to suppose that $H(\lambda)$ satisfies the assumptions of theorem IV.3.

Let $f(\lambda)$ be an entire complex-valued function such that $f(0)\neq 0$. Let $l$ be a positive integer; for each integer $j$ in $1\leqslant j\leqslant l$, and for each complex $\lambda$, let $x_j(\lambda)$ and $y_j(\lambda)$ be elements of $\mathscr{H}$. We assume that the mappings $\lambda\to x_j(\lambda)$ and $\lambda\to y_j(\lambda)$ are entire functions with respect to the norm $|\ |$ of the Hilbert space $\mathscr{H}$. Let the operator $K(\lambda)$ be defined for each fixed complex $\lambda$ such that $f(\lambda)\neq 0$ by the equation

$$(5.1)\qquad K(\lambda)u = H(\lambda)u + \frac{1}{f(\lambda)}\sum_{j=1}^{l} x_j(\lambda)\langle u, y_j(\bar{\lambda})\rangle$$

where $u \in \mathscr{H}$. The reason for writing $\langle u, y_j(\bar{\lambda})\rangle$ is that the mapping $\lambda \to \langle u, y_j(\bar{\lambda})\rangle$ is an analytic (complex-valued) function of $\lambda$ for each fixed $u \in \mathscr{H}$, while the mapping $\lambda \to \langle u, y_j(\lambda)\rangle$ is not analytic for each fixed $u \in \mathscr{H}$. We will study the eigenvalues of the equation

$$\lambda K(\lambda) u = u \tag{5.2}$$

in this section; one difficulty will, of course, be the fact that $K(\lambda)$ has possible poles at the zeroes of $f(\lambda)$. Later on, we will have to place more restrictions on $x_j(\lambda)$, $y_j(\lambda)$, and $f(\lambda)$.

Let the operator $P(\lambda)$ be defined by

$$P(\lambda) u = K(\lambda) u - H(\lambda) u \tag{5.3}$$

$$= (1/f(\lambda)) \sum_{j=1}^{l} x_j(\lambda) \langle u, y_j(\bar{\lambda})\rangle . \tag{5.4}$$

We will have to show that $P(\lambda)$ is an operator-valued function analytic with respect to $|\ |_1$ about all points $\lambda$ such that $f(\lambda) \neq 0$. For this purpose, we first perform an auxiliary computation. Let the operator $\Lambda$ be defined on $\mathscr{H}$ by the equation

$$\Lambda u = x \langle u, y\rangle \tag{5.5}$$

where $x$ and $y$ are *fixed* elements of $\mathscr{H}$. Note that if $\Lambda^*$ is the adjoint of $\Lambda$, then

$$\Lambda^* v = y \langle v, x\rangle \tag{5.6}$$

for each $v \in \mathscr{H}$. Hence

$$\Lambda^* \Lambda u = y \langle u, y\rangle \langle x, x\rangle . \tag{5.7}$$

Hence if $\{\mu_i(\Lambda)\}$ are the singular values of $\Lambda$, we have from (5.7)

$$|\Lambda|_p = \left[\sum_i \mu_i(\Lambda)^{-p}\right]^{1/p} \tag{5.8}$$

$$= [\mu_1(\Lambda)]^{-1} \tag{5.9}$$

$$= |x||y| . \tag{5.10}$$

The analyticity of $P(\lambda)$ with respect to $|\ |_1$ about points $\lambda$ such that $f(\lambda) \neq 0$ follows directly from the next lemma, and from corollary III.1.

LEMMA V.1. – Let $a$ be a fixed complex number, and let $\varepsilon > 0$. Let $x(\lambda) \in \mathscr{H}$ for each complex $\lambda$ in the set

$$\{\bar{\lambda}\colon |\bar{\lambda} - a| < \varepsilon\} \tag{5.11}$$

and let $y(\lambda) \in \mathscr{H}$ for each complex $\lambda$ in the set

$$\{\hat{\lambda} \colon |\hat{\lambda} - \bar{a}| < \varepsilon\}. \tag{5.12}$$

Furthermore, let $x(\lambda)$ be analytic with respect to the norm $|\ |$ of the Hilbert space $\mathscr{H}$ for each $\lambda$ in the set (5.11), and let $y(\lambda)$ be analytic with respect to the norm $|\ |$ for each $\lambda$ in (5.12). For each $\lambda$ in (5.11), let the operator $A(\lambda)$ be defined by

$$A(\lambda) u = x(\lambda) \langle u, y(\bar{\lambda}) \rangle \tag{5.13}$$

where $u \in \mathscr{H}$. Then $A(\lambda)$ is analytic with respect to $|\ |_1$ for each $\lambda$ in (5.11).

PROOF. – Let $v_0$ be a fixed element of $\mathscr{H}$ such that

$$|v_0| = 1 \tag{5.14}$$

and let the operators $D(\lambda)$ and $E(\lambda)$ be defined by the equations

$$D(\lambda) u = x(\lambda) \langle u, v_0 \rangle \tag{5.15}$$

$$E(\lambda) u = v_0 \langle u, y(\bar{\lambda}) \rangle \tag{5.16}$$

for each $u \in \mathscr{H}$ and for each $\lambda$ in the set (5.11). Note that

$$D(\lambda) E(\lambda) = A(\lambda). \tag{5.17}$$

It suffices by lemma III.1 to show that $D(\lambda)$ and $E(\lambda)$ are each analytic with respect to $|\ |_1$ for $\lambda$ in the set (5.11). We shall do this for the function $E(\lambda)$; indeed, we shall prove that

$$E'(\lambda) u = v_0 \langle u, y'(\bar{\lambda}) \rangle \tag{5.18}$$

for each $\lambda$ in (5.11). For now, let the operator $F(\lambda)$ be defined by the equation

$$F(\lambda) u = v_0 \langle u, y'(\bar{\lambda}) \rangle \tag{5.19}$$

for $\lambda$ in (5.11) and $u \in \mathscr{H}$. Let $\lambda$ be in (5.11), and let $\Delta\lambda$ be a non-zero complex number such that $(\lambda + \Delta\lambda)$ is in (5.11). We set

$$\varrho = \bar{\lambda}, \quad \Delta\varrho = \overline{\Delta\lambda}. \tag{5.20}$$

Clearly $\varrho$ and $(\varrho + \Delta\varrho)$ are in (5.12). We have

$$[E(\lambda + \Delta\lambda) - E(\lambda)] u = v_0 \langle u, [y(\varrho + \Delta\varrho) - y(\varrho)] \rangle \tag{5.21}$$

so that by (5.8)-(5.10) and by (5.14), the relation

$$|E(\lambda+\Delta\lambda)-E(\lambda)|_1=|y(\varrho+\Delta\varrho)-y(\varrho)| \tag{5.22}$$

holds. We have assumed that if $\varrho$ is in the set (5.12), then $y(\varrho)$ is analytic (and hence continuous) with respect to the norm $|\ |$ of $\mathscr{H}$. Thus the continuity of $E(\lambda)$ with respect to $|\ |_1$ follows from (5.22), provided that $\lambda$ is in (5.11).

The proof of the relation

$$\left|\frac{E(\lambda+\Delta\lambda)-E(\lambda)}{\Delta\lambda}-F(\lambda)\right|_1=\left|\frac{y(\varrho+\Delta\varrho)-y(\varphi)}{\Delta\varrho}-y'(\varrho)\right| \tag{5.23}$$

is similar to that of (5.22); by arguments of the previous type, the relation (5.23) proves the analyticity of $E(\lambda)$ with respect to $|\ |_1$ and the relation (5.18). ∎

The numbers $\lambda_i$ and $k_0$ in the next lemma are the same numbers defined in part IV.

LEMMA V.2. – We shall *always* assume here that $\lambda$ is an element of the open set

$$\{\hat{\lambda}: f(\hat{\lambda})\neq 0\}-\bigcup_i\{\lambda_i\} \tag{5.24}$$

where the union in (5.24) is taken over all elements of the sequence $\{\lambda_i\}$.

Let the operator $G(\lambda)$ be defined by

$$G(\lambda)=[I-\lambda H(\lambda)]^{-1}P(\lambda) \tag{5.25}$$

for each $\lambda$ in the set (5.24). The function $G(\lambda)$ is analytic with respect to $|\ |_1$ for each $\lambda$ in (5.24).

Suppose *either* $[I-bG(b)]^{-1}$ or $[I-bK(b)]^{-1}$ exists for some fixed $b$ in (5.24). Then there exists $\varepsilon>0$ such that the disc $\{\lambda: |\lambda-b|<\varepsilon\}$ is a subset of the set (5.24) with the following properties

(i) The inverses $[I-\lambda K(\lambda)]^{-1}$ and $[I-\lambda G(\lambda)]^{-1}$ both exist and are both defined on all of $\mathscr{H}$, provided $\lambda$ satisfies $|\lambda-b|<\varepsilon$.

(ii) The inverses $[I-\lambda K(\lambda)]^{-1}$ and $[I-\lambda G(\lambda)]^{-1}$ are analytic functions of $\lambda$ with respect to $|\ |_\infty$, provided $|\lambda-b|<\varepsilon$.

For each $\lambda$ in (5.24) such that either $[I-\lambda K(\lambda)]^{-1}$ or $[I-\lambda G(\lambda)]^{-1}$ exists, we have the relation

$$\begin{aligned}[\lambda K(\lambda)]'[I-\lambda K(\lambda)]^{-1}&=[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1}+\\&+[I-\lambda H(\lambda)][\lambda G(\lambda)]'[I-\lambda G(\lambda)]^{-1}[I-\lambda H(\lambda)]^{-1}.\end{aligned} \tag{5.26}$$

The function

$$(5.27)_k \qquad [\lambda K(\lambda)]'[I-\lambda K(\lambda)]^{-1}-\sum_{j=0}^{k-1}\lambda^j \boldsymbol{m}_j\{[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}\}$$

is analytic with respect to $|\ |_1$ on the open set of all points $\lambda$ in (5.24) such that $[I-\lambda K(\lambda)]^{-1}$ exists, provided $k$ is an integer such that $k \geqslant k_0-1$.

COMMENTS. – It is clear that the trace of (5.27) can be expected to play a role similar to the quantity on the right side of (4.97). In effect, we will eventually « solve » a differential equation similar to (4.97) and obtain a function $\mathscr{D}(\lambda; k)$ which is very roughly analogous to the function $\Delta(\lambda; k)$ of the previous section. The equality (5.26) will play an important role in the construction of the function $\mathscr{D}(\lambda; k)$, and will be important in the establishment of certain inequalities which will yield an analogue of (4.102).

Note that the operator $G(\lambda)$ has a finite dimensional range for each fixed $\lambda$. This fact along with equation (5.26) will aid us greatly, since at this point, we know much about $H(\lambda)$ and $[I-\lambda H(\lambda)]^{-1}$.

We have an inherent disadvantage in the construction of $G(\lambda)$ in that $G(\lambda)$ is not defined at points $\lambda=\lambda_i$; consequently we cannot (in *this* lemma) prove that (5.27) is analytic with respect to $|\ |_1$ about the points $\lambda=\lambda_i$. We will easily overcome this in lemma V.3, however.

PROOF. – We shall continue to assume that $\lambda$ is an element of the set (5.24). The analyticity of $G(\lambda)$ with respect to $|\ |_1$ follows immediately from lemma III.1 and from the analyticity of $[I-\lambda H(\lambda)]^{-1}$ with respect to $|\ |_\infty$ and the analyticity of $P(\lambda)$ with respect to $|\ |_1$. Another approach, which yields some useful minor results, will be given later.

For each $\lambda$ in (5.24), we can show that

$$(5.28) \qquad [I-\lambda H(\lambda)][I-\lambda G(\lambda)]=[I-\lambda K(\lambda)].$$

For each $\lambda$ in (5.24), the existence of $[I-\lambda H(\lambda)]^{-1}$ is assured. Hence we see from (5.28) that $[I-\lambda G(\lambda)]^{-1}$ exists if and only if $[I-\lambda K(\lambda)]^{-1}$ exists, provided $\lambda$ is in (5.24). The assertions (i) and (ii) immediately follow from the discussion concerning (4.8), since the latter is valid if $G(\lambda)$ or $K(\lambda)$ replaces $H(\lambda)$. Clearly, if either $[I-\lambda G(\lambda)]^{-1}$ or $[I-\lambda K(\lambda)]^{-1}$ exist for some $\lambda$ in (5.24), then we have

$$(5.29) \qquad [I-\lambda K(\lambda)]^{-1}=[I-\lambda G(\lambda)]^{-1}[I-\lambda H(\lambda)]^{-1}.$$

In order to prove (5.26), we note that

$$(5.30) \qquad P(\lambda)=[I-\lambda H(\lambda)]G(\lambda)$$

so that we have

$$[\lambda P(\lambda)]' = -[\lambda H(\lambda)]'[\lambda G(\lambda)] + [I - \lambda H(\lambda)][\lambda G(\lambda)]'. \tag{5.31}$$

If we add $[\lambda H(\lambda)]'$ to both sides of (5.31), we obtain from (5.3)

$$[\lambda K(\lambda)]' = [\lambda H(\lambda)]'[I - \lambda G(\lambda)] + [I - \lambda H(\lambda)][\lambda G(\lambda)]'. \tag{5.32}$$

Equation (5.26) then follows from (5.29) and (5.32).

In order to prove the analyticity of $(5.27)_k$ with respect to $|\ |_1$, we first note that the functions

$$[\lambda G(\lambda)]'[I - \lambda G(\lambda)]^{-1} \tag{5.33}$$

and

$$[I - \lambda H(\lambda)][\lambda G(\lambda)]'[I - \lambda G(\lambda)]^{-1}[I - \lambda H(\lambda)]^{-1} \tag{5.34}$$

are analytic (where defined) with respect to $|\ |_1$ by lemma III.1. It is thus clear from (5.26), from the analyticity of (4.42) with respect to $|\ |_1$ (provided $k \geqslant k_0 - 1$ and provided $\lambda \neq \lambda_i$ for any $i$), and from the preceding, that

$$[\lambda K(\lambda)]'[I - \lambda K(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j \{[\boldsymbol{\lambda} H(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\} \tag{5.35}$$

is analytic with respect to $|\ |_1$ for $\lambda$ in (5.24) such that $[I - \lambda K(\lambda)]^{-1}$ exists, provided $k \geqslant k_0 - 1$. From the analyticity of (5.34) with respect to $|\ |_1$, it is clear that

$$\sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j \{[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})][\boldsymbol{\lambda} G(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} G(\boldsymbol{\lambda})]^{-1}[I - \boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}\} \tag{5.36}$$

is an operator-valued polynomial which is an entire function with respect to $|\ |_1$. If we subtract (5.36) from (5.35), we obtain precisely the expression $(5.27)_k$; this follows from (5.26) and from the linearity of $\boldsymbol{m}_j$. Hence $(5.27)_k$ is analytic (where defined) with respect to $|\ |_1$, provided $\lambda$ is in (5.24) and provided $k \geqslant k_0 - 1$.

An alternate proof of the analyticity of $G(\lambda)$ with respect to $|\ |_1$ can be given here. We have

$$G(\lambda)\, u = \frac{1}{f(\lambda)} \sum_{j=0}^{l} z_j(\lambda) \langle u, y_j(\bar{\lambda}) \rangle \tag{5.37}$$

where

$$z_j(\lambda) = [I - \lambda H(\lambda)]^{-1} x_j(\lambda). \tag{5.38}$$

Now $[I - \lambda H(\lambda)]^{-1}$ is analytic with respect to $|\ |_\infty$ for $\lambda$ in (5.24). This fact, along with the analyticity of $x_j(\lambda)$ with respect to the norm $|\ |$ of the Hilbert space $\mathscr{H}$ guarantees that $z_j(\lambda)$ is also analytic (where defined) with respect to the norm $|\ |$ of $\mathscr{H}$.

The proof of this is similar to that of lemma III.1, and is done by proving an analogue of (3.21); the inequality $|Ax| \leqslant |A|_\infty |x|$ where $x \in \mathscr{H}$ and $A \in C\{\infty; \mathscr{H}\}$ would replace (2.14) in the proof. Since $z_j(\lambda)$ is analytic (where defined) with respect to the norm $|\ |$ of $\mathscr{H}$, we have analyticity of $G(\lambda)$ with respect to $|\ |_1$ by lemma V.1 and by corollary III.1. ■

LEMMA V.3. – Let $\lambda_m$ be an element of the sequence $\{\lambda_i\}$, where $m$ henceforth denotes a fixed positive integer. There exists a compact linear operator $Q^{\{m\}}$ on $\mathscr{H}$ of finite dimensional range such that if

$$H^{\{m\}}(\lambda) = H(\lambda) + Q^{\{m\}} \tag{5.39}$$

then the equation

$$\lambda H^{\{m\}}(\lambda) v = v \tag{5.40}$$

has no eigenvalues in the region $|\lambda - \lambda_m| < \varepsilon_m$ where $\varepsilon_m$ is a small positive number. We have

$$H^{\{m\}}(\lambda) = \sum_{i=0}^{s} \lambda^i H_i^{\{m\}} \tag{5.41}$$

with

$$H_i^{\{m\}} = H_i + \delta_{i,0} Q^{\{m\}} \in C\{\alpha_i, \mathscr{H}\} \tag{5.42}$$

where $\delta_{ij}$ is the Kronecker delta. Hence for each integer $k \geqslant k_0 - 1$, we may define an entire function $\Delta^{\{m\}}(\lambda; k)$ by replacing $H(\lambda)$ in (4.97) by $H^{\{m\}}(\lambda)$. Let $\{\lambda_i^{\{m\}}\}$ be the (possibly void or finite) ordered sequence of zeroes of $\Delta^{\{m\}}(\lambda; k)$, taken according to algebraic multiplicity. We shall *always* assume that $\lambda$ is an element of the open set

$$\{\lambda : f(\lambda) \neq 0\} - \bigcup_i \{\lambda_i^{\{m\}}\}, \tag*{$(5.43)_m$}$$

where the union in $(5.43)_m$ is taken over the whole sequence $\{\lambda_i^{\{m\}}\}$ for *fixed m*.

Let the operators $P^{\{m\}}(\lambda)$ and $G^{\{m\}}(\lambda)$ be defined by the equations

$$P^{\{m\}}(\lambda) = P(\lambda) - Q^{\{m\}} = K(\lambda) - H^{\{m\}}(\lambda) \tag{5.44}$$

$$G^{\{m\}}(\lambda) = [I - \lambda H^{\{m\}}(\lambda)]^{-1} P^{\{m\}}(\lambda) \tag{5.45}$$

for each $\lambda$ in the set $(5.43)_m$. The operator $P^{\{m\}}(\lambda)$ is analytic with respect to $|\ |_1$ about each point $\lambda$ such that $f(\lambda) \neq 0$; the operator $G^{\{m\}}(\lambda)$ is analytic with respect to $|\ |_1$ for each $\lambda$ in the open set $(5.43)_m$.

Suppose either $[I - bK(b)]^{-1}$ or $[I - bG^{\{m\}}(b)]^{-1}$ exists for some fixed $b$ in $(5.43)_m$. Then there exists $\varepsilon > 0$ such that the disc $\{\lambda : |\lambda - b| < \varepsilon\}$ is a subset of the set $(5.43)_m$ with the following properties:

(i) The inverses $[I-\lambda K(\lambda)]^{-1}$ and $[I-\lambda G^{\{m\}}(\lambda)]^{-1}$ both exist and are both defined on all of $\mathscr{H}$, provided $\lambda$ satisfies $|\lambda-b|<\varepsilon$;

(ii) The inverses $[I-\lambda K(\lambda)]^{-1}$ and $[I-\lambda G^{\{m\}}(\lambda)]^{-1}$ are analytic functions of $\lambda$ with respect to $|\ |_\infty$, provided $\lambda$ satisfies $|\lambda-b|<\varepsilon$.

For each $\lambda$ in $(5.43)_m$ such that either $[I-\lambda K(\lambda)]^{-1}$ or $[I-\lambda G^{\{m\}}(\lambda)]^{-1}$ exists, we have the relation

$$(5.46)\qquad [\lambda K(\lambda)]'[I-\lambda K(\lambda)]^{-1}=[\lambda H^{\{m\}}(\lambda)]'[I-\lambda H^{\{m\}}(\lambda)]^{-1} + [I-\lambda H^{\{m\}}(\lambda)][\lambda G^{\{m\}}(\lambda)]'[I-\lambda G^{\{m\}}(\lambda)]^{-1}[I-\lambda H^{\{m\}}(\lambda)]^{-1}.$$

The function $(5.27)_k$ is analytic with respect to $|\ |_1$ on the open set of all points $\lambda$ in $(5.43)_m$ such that $[I-\lambda K(\lambda)]^{-1}$ exists, provided $k$ is an integer such that $k\geqslant k_0-1$.

PROOF. – In order to prove existence of $Q^{\{m\}}$, we let $\xi_1,\dots,\xi_q$ be an orthonormal basis for the space $\mathscr{N}\{[I-\lambda_m H(\lambda_m)]\}$, which is finite dimensional, since $H(\lambda_m)$ is compact. It goes without saying that the latter space, and its dimension $q$, are dependent on $\lambda_m$. Note that $\bar\lambda_m$ is an eigenvalue of the equation

$$\bar\lambda_m[H(\lambda_m)]^*w=w\qquad(w\in\mathscr{H});$$

also note that $\mathscr{N}\{[I-\lambda_m H(\lambda_m)]\}$ and $\mathscr{N}\{[I-\bar\lambda_m(H(\lambda_m))^*]\}$ have the same dimension (see exercise 35, p. 584, in DUNFORD-SCHWARTZ [4]). Hence we may let $\eta_1,\dots,\eta_q$ denote any orthonormal basis for the latter space.

Let the operator $Q^{\{m\}}$ be defined by

$$(5.47)\qquad Q^{\{m\}}u=\sum_{i=1}^{q}\eta_i\langle u,\xi_i\rangle .$$

Let us suppose that $v\in\mathscr{H}$ satisfies (5.40) with $\lambda=\lambda_m$. Then we have

$$(5.48)\qquad [I-\lambda_m H(\lambda_m)]v=\lambda_m Q^{\{m\}}v .$$

If we take the inner product of both sides of (5.48) with $\eta_j$, and if we note that $\lambda_m\neq0$, then we see that

$$(5.49)\qquad \langle Q^{\{m\}}v,\eta_j\rangle=0,\quad (j=1,\dots,q).$$

But (5.47) and (5.49) yield

$$0=\langle Q^{\{m\}}v,\eta_j\rangle=\langle v,\xi_j\rangle\qquad(j=1,\dots,q)$$

since $\eta_i,\dots,\eta_q$ is an orthonormal set. Hence $Q^{\{m\}}v=0$ and hence $v$ satisfies

$$[I-\lambda_m H(\lambda_m)]v=\vec{0}.$$

Since $v \in \mathcal{N}\{[I - \lambda_m H(\lambda_m)]\}$ and since $\xi_1, \ldots, \xi_q$ is an orthonormal basis for the latter space, we may write

$$v = \sum_{j=1}^{q} \langle v, \xi_j \rangle \xi_j = \vec{0}$$

and so $\lambda = \lambda_m$ is not an eigenvalue of (5.40).

We note (see corollary IV.3) that $Q^{\{m\}} \in C\{p; \mathcal{H}\}$ for each real $p > 0$; hence (5.42) holds for $i = 0$ since $C(\alpha_0)$ is a linear manifold. Since (5.42) holds for each integer $i$ in $0 \leqslant i \leqslant s$, we may conclude by assertion (ii) of theorem IV.3 that

$$[\lambda H^{\{m\}}(\lambda)]'[I - \lambda H^{\{m\}}(\lambda)]^{-1} - \sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j\{[\boldsymbol{\lambda} H^{\{m\}}(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} H^{\{m\}}(\boldsymbol{\lambda})]^{-1}\} \tag{5.50}$$

is analytic with respect to $|\ |_1$, provided $k \geqslant k_0 - 1$ and provided that $\lambda$ is not an eigenvalue of (5.40). Hence the functions $\Delta^{\{m\}}(\lambda; k)$ described in the statement of the lemma may be constructed; the ordered sequence of zeroes of $\Delta^{\{m\}}(\lambda; k)$, where each zero is taken according to algebraic multiplicity, is independent of $k$ for $k \geqslant k_0 - 1$.

Note that we may write

$$P^{\{m\}}(\lambda) u = \big(1/f(\lambda)\big) \sum_{j=1}^{l\{m\}} x_j^{\{m\}}(\lambda) \langle u, y_j^{\{m\}}(\bar{\lambda}) \rangle \tag{5.51}$$

where, since $q$ depends on $m$, we write

$$l^{\{m\}} = l + q \tag{5.52}$$

and where

$$\left\{ \begin{aligned} x_j^{\{m\}}(\lambda) &= \begin{cases} x_j(\lambda) & j = 1, \ldots, l \\ -f(\lambda)\eta_{j-l} & j = l+1, \ldots, l+q \end{cases} \\ y_j^{\{m\}}(\lambda) &= \begin{cases} y_j(\lambda) & j = 1, \ldots, l \\ \xi_{j-l} & j = l+1, \ldots, l+q \end{cases} \end{aligned} \right. \tag{5.53}$$

Hence $P^{\{m\}}(\lambda)$ is analytic with respect to $|\ |_1$ about each point $\lambda$ such that $f(\lambda) \neq 0$ by the same arguments that we used to prove the analyticity of $P(\lambda)$. It is easy to see that $x_j^{\{m\}}(\lambda)$ and $y_j^{\{m\}}(\lambda)$ are entire functions with respect to the norm of the space $\mathcal{H}$. The rest of the proof follows as before. ■

In the next lemma we state a few well-known facts for future reference.

LEMMA V.4. – We will *always* assume henceforth that $\lambda$ is an element of the set (5.24). Let $d(\lambda)$ be the $l \times l$ determinant with (see (5.1) and (5.38))

$$\delta_{ij} - \frac{\lambda}{f(\lambda)} \langle z_j(\lambda), y_i(\bar{\lambda}) \rangle \tag{5.54}$$

in its $i$-th row, $j$-th column, where $\delta_{ij}$ is the Kronecker delta. Then

(i) Let $a$ be an element of the set (5.24). The following assertions are equivalent:

(1) $d(a) = 0$;

(2) $\lambda = a$ is an eigenvalue of (5.2);

(3) $\lambda = a$ is an eigenvalue of the equation (5.55);

$$\lambda G(\lambda) w = w \qquad (w \in \mathscr{H}). \tag{5.55}$$

If $b$ is an element of the set (5.24) then the following are equivalent:

(4) $d(b) \neq 0$;

(5) the inverse $[I - bK(b)]^{-1}$ exists and is defined on all of $\mathscr{H}$.

(6) the inverse $[I - bG(b)]^{-1}$ exists and is defined on all of $\mathscr{H}$.

(ii) Let $D_j(\lambda)$ be the transformation (from $\mathscr{H}$ onto the set of complex numbers) such that $D_j(\lambda)u$ is the determinant obtained by replacing the $j$-th column of $d(\lambda)$ by

$$(\langle u, y_1(\bar{\lambda})\rangle, \ldots, \langle u, y_l(\bar{\lambda})\rangle)^T \tag{5.56}$$

where $T$ indicates a transpose. The equation

$$[I - \lambda G(\lambda)]^{-1} u = u + \frac{\lambda}{f(\lambda)} \sum_{j=1}^{l} z_j(\lambda) \frac{D_j(\lambda) u}{d(\lambda)} \tag{5.57}$$

holds at all points $\lambda$ in the set (5.24) such that $d(\lambda) \neq 0$.

(iii) The equation

$$-\frac{d'(\lambda)}{d(\lambda)} = \tau\{[\lambda G(\lambda)]'[I - \lambda G(\lambda)]^{-1}\} \tag{5.58}$$

holds at all points $\lambda$ in the set (5.24) such that $d(\lambda) \neq 0$.

PROOF. – The proofs of (i) and (ii) for the most part involve standard results, and will not be done here. Note that the results of Lemma V.2 and equation (5.28) are useful in the proof of (i).

In order to establish (5.58), it will be helpful to generalize slightly the functions $d(\lambda)$ and $D_j(\lambda)$. (We shall always assume that $\lambda$ is an element of the set (5.24) in the following.) Let $d(\sigma; \lambda)$ be the $l \times l$ determinant with

$$\delta_{ij} - \frac{\sigma}{f(\lambda)} \langle z_j(\lambda), y_i(\bar{\lambda})\rangle \tag{5.59}$$

in its $i$-th row, $j$-th column. Of course, we have that $d(\lambda; \lambda) = d(\lambda)$. Clearly, the function $d(\sigma; \lambda)$ is continuous (where defined) in $\sigma$ and $\lambda$ together, and is analytic (where defined) in $\lambda$ for fixed $\sigma$ and vice versa (the reader may verify that $\langle z_j(\lambda), y_i(\bar{\lambda})\rangle$ is analytic (where defined) with respect to the norm on the set of complex numbers; here the analyticity of $z_j(\lambda)$ and $y_i(\lambda)$, where defined, are needed). Let the transformation $D_j(\sigma, \lambda)$ (from $\mathscr{H}$ onto the set of complex numbers) be defined so that $D_j(\sigma, \lambda)u$ is the $l\times l$ determinant in which the $j$-th column of $d(\sigma; \lambda)$ has been replaced by (5.56). The solution of the equation

$$(I-\sigma G(\lambda))w = h \qquad (h\in\mathscr{H}) \tag{5.60}$$

is

$$w = \frac{\sigma}{f(\lambda)}\sum_{j=1}^{l} z_j(\lambda)\frac{D_j(\sigma; \lambda)h}{d(\sigma; \lambda)} + h \tag{5.61}$$

provided, of course that $d(\sigma; \lambda)\neq 0$ for fixed $\sigma$ and $\lambda$. Let $V(\sigma; \lambda)$ be the resolvent defined by the equation

$$I + \sigma V(\sigma; \lambda) = [I-\sigma G(\lambda)]^{-1} \tag{5.62}$$

and let the operator $E_j(\sigma; \lambda)$ (on $\mathscr{H}$ to $\mathscr{H}$) be defined by

$$E_j(\sigma; \lambda)u = z_j(\lambda)D_j(\sigma; \lambda)u\,. \tag{5.63}$$

From (5.61)-(5.63) we note that

$$V(\sigma; \lambda) = \frac{1}{d(\sigma; \lambda)f(\lambda)}\sum_{j=1}^{l} E_j(\sigma; \lambda)\,. \tag{5.64}$$

If $z_j(\lambda)\neq 0$ for some *fixed* $j$ and $\lambda$, and if we extend the set $\{z_j(\lambda)/|z_j(\lambda)|\}$ (where $j$ and $\lambda$ are *fixed*) into an orthonormal basis for $\mathscr{H}$, then we see that

$$\tau[E_j(\sigma; \lambda)] = \frac{\langle E_j(\sigma; \lambda)z_j(\lambda), z_j(\lambda)\rangle}{|z_j(\lambda)|^2} = D_j(\sigma; \lambda)z_j(\lambda) \tag{5.65}$$

from (5.63) and from the definition of trace (see the right side of (2.15)). If $z_j(\lambda) = \vec{0}$ for some *fixed* $j$ and $\lambda$, the equation

$$\tau[E_j(\sigma; \lambda)] = D_j(\sigma; \lambda)z_j(\lambda) \tag{5.66}$$

still holds, as both sides are zero. From the usual « rule » (using columns) of taking the derivative of a determinant, we obtain

$$-\frac{\partial}{\partial\sigma}d(\sigma; \lambda) = \frac{1}{f(\lambda)}\sum_{j=1}^{l} D_j(\sigma; \lambda)z_j(\lambda)\,. \tag{5.67}$$

From (5.64), (5.66), and (5.67), we obtain

$$-\frac{1}{d(\sigma;\lambda)}\frac{\partial}{\partial\sigma}d(\sigma;\lambda)=\tau[V(\sigma;\lambda)] \tag{5.68}$$

which holds at all points $(\sigma,\lambda)$ such that $d(\sigma,\lambda)\neq 0$. Note that

$$V(\sigma;\lambda)=\sum_{j=0}^{\infty}\sigma^j[G(\lambda)]^{j+1} \tag{5.69}$$

where the Neumann series (5.69) for $V(\sigma;\lambda)$ converges uniformly and absolutely with respect to $|\ |_1$ for $|\lambda|+|\sigma|<\varepsilon$, where $\varepsilon$ is a small positive number (recall that $G(\lambda)$ is in the space $C\{1;\mathscr{H}\}$ and is analytic where defined with respect to $|\ |_1$). We claim, therefore, that

$$\frac{\partial}{\partial\lambda}\int_0^\sigma\tau\{V(\hat\sigma;\lambda)\}\,d\hat\sigma=\sum_{j=0}^{\infty}\frac{\sigma^{j+1}}{j+1}\frac{d}{d\lambda}\tau\{[G(\lambda)]^{j+1}\} \tag{5.70}$$

where the path of integration is inside the region $|\hat\sigma|+|\lambda|<\varepsilon$. In order to obtain (5.70), we first note that $\tau$ and the summation in (5.69) may be interchanged, since by (2.16) the operator $\tau$ is a continuous linear function on $C(1)$. As a result of (2.16) and of the uniform convergence of the series (5.69) with respect to $|\ |_1$, the series obtained by interchanging $\tau$ with the summation in the series in (5.69) continues to converge uniformly (with respect to the norm on the set of complex numbers) for $|\lambda|+|\sigma|<\varepsilon$, and the latter series with $\hat\sigma$ replacing $\sigma$ may integrated in term by term fashion with respect to $\hat\sigma$ to obtain

$$\int_0^\sigma\tau[V(\hat\sigma;\lambda)]\,d\hat\sigma=\sum_{j=0}^{\infty}\frac{\sigma^{j+1}}{j+1}\tau\{[G(\lambda)]^{j+1}\}. \tag{5.71}$$

The series in (5.71) continues to be uniformly convergent in the region $|\sigma|+|\lambda|<\varepsilon$, and hence uniformly convergent with respect to $\lambda$ in $|\lambda|<\varepsilon-|\sigma|$ for small *fixed* $\sigma$. By lemma III.2, each term in the series in (5.71) is an analytic, complex-valued function of $\lambda$ for each fixed $\sigma$, and each $\lambda$ in (5.24). Hence the series in (5.71) may be differentiated with respect to $\lambda$ in term-by-term fashion, since, by Cauchy's formula, $\partial/\partial\lambda$ can be expressed as an integral operator. Therefore (5.70) is justified.

By lemma III.2, we may interchange $\tau$ and $d/d\lambda$ in (5.70) so

$$\frac{d}{d\lambda}\tau\{[G(\lambda)]^{j+1}\}=\tau\left\{\frac{d}{d\lambda}[G(\lambda)]^{j+1}\right\} \tag{5.72}$$

$$=\tau\sum_{i=0}^{j}[G(\lambda)]^iG'(\lambda)[G(\lambda)]^{j-i} \tag{5.73}$$

$$=(j+1)\tau\{G'(\lambda)[G(\lambda)]^j\} \tag{5.74}$$

where (5.73) follows from (3.21) and from the fact that $G(\lambda) \in C(1)$ and $G'(\lambda) \in C(1)$. Clearly (5.74) follows from (2.40). By (5.69) and (5.70) and by (5.72)-(5.74), we have

$$\frac{\partial}{\partial\lambda}\int_0^\sigma \tau\{V(\hat\sigma;\lambda)\}\,d\hat\sigma = \tau\{\sigma G'(\lambda) + \sigma^2 G'(\lambda)\,V(\sigma;\lambda)\} \tag{5.75}$$

provided $|\hat\sigma| + |\lambda| < \varepsilon$ along a suitable path connecting $\hat\sigma = 0$ with $\hat\sigma = \sigma$. In order to obtain (5.75), one must prove the validity of an interchange of $\tau$ with the summation of a certain series; the justification is again obtained by noting that the series in (5.69) converges with respect to $|\;|_1$ for $|\sigma| + |\lambda| < \varepsilon$, and by using the fact that $\tau$ is a bounded linear functional on the space $C(1)$.

Integrating (5.68), we obtain for $|\hat\sigma| + |\lambda| < \varepsilon$

$$d(\sigma;\lambda) = \exp\left\{-\int_0^\sigma \tau[V(\hat\sigma;\lambda)]\,d\hat\sigma\right\} \tag{5.76}$$

since $d(0,\lambda) = 1$. From (5.75) and (5.76) we obtain

$$-\frac{1}{d(\sigma;\lambda)}\frac{\partial}{\partial\lambda}d(\sigma;\lambda) = \tau\{\sigma G'(\lambda) + \sigma^2 G'(\lambda)\,V(\sigma,\lambda)\} \tag{5.77}$$

for $|\sigma| + |\lambda| < \varepsilon$. Hence (5.58) follows for $|\lambda| < \varepsilon/2$ from (5.68), from (5.77), and from the equations

$$d'(\lambda) = \frac{\partial}{\partial\sigma}d(\sigma;\lambda)\Big|_{\sigma=\lambda} + \frac{\partial}{\partial\lambda}d(\sigma,\lambda)\Big|_{\sigma=\lambda} \tag{5.78}$$

and (see (5.62))

$$[\lambda G(\lambda)]'[I - \lambda G(\lambda)]^{-1} = V(\lambda;\lambda) + \lambda G'(\lambda) + \lambda^2 G'(\lambda)\,V(\lambda;\lambda)\,. \tag{5.79}$$

The reader may verify the continuity of $(\partial/\partial\sigma)\,d(\sigma,\lambda)$ and $(\partial/\partial\lambda)\,d(\sigma,\lambda)$ in both variables together which is necessary for the validity of (5.78).

Up to now, (5.58) is established only for $\lambda$ such that $|\lambda|$ is small. We wish to establish (5.58) for all complex $\lambda$ such that $\lambda$ is in the complement of the set (5.80)

$$\Big[\bigcup_i \{\lambda_i\}\Big] \cup \{\hat\lambda : f(\hat\lambda) = 0\} \cup \{\tilde\lambda : d(\tilde\lambda) = 0\} \tag{5.80}$$

where the left most union in (5.80) is over the whole sequence $\{\lambda_i\}$. We may appeal to analytic continuation in order to establish (5.58) for all $\lambda$ not in the set (5.80), provided we show that the set (5.80) is countable, and can have only $\lambda = \infty$ as an accumulation point. To show the latter, it suffices to show that the zeroes of $d(\lambda)$

are countable, and may accumulate only at $\lambda=\infty$. Note that the function

$$[\Delta(\lambda;k)f(\lambda)]^l d(\lambda) \tag{5.81}$$

has only removable singularities (and so its appropriate analytic extension is an entire function). This follows from the definition of $d(\lambda)$, and from part (vii) of theorem IV.3. The set of zeroes of $d(\lambda)$ is a subset of the set of zeroes of the extension of (5.81), which is an entire function. Hence the set of zeroes of the extension of (5.81), and thus of $d(\lambda)$, are countable, and may accumulate only at $\lambda=\infty$. Thus the set (5.80) has the required properties. (*) ■

COROLLARY V.4-1. – Suppose $a$ is a complex number such that

$$a\notin\bigcup_i\{\lambda_i\}. \tag{5.82}$$

If $f(a)=0$, then each of the functions $d(\lambda)$, $G(\lambda)$, $[I-\lambda G(\lambda)]^{-1}$, $[I-\lambda K(\lambda)]^{-1}$ has (at worst) a pole at $\lambda=a$.

PROOF. – The assertion about $d(\lambda)$ follows from the fact that (5.81) has only removable singularities; the assertion about $G(\lambda)$ follows from (5.37) and (5.38); the assertion about $[I-\lambda G(\lambda)]^{-1}$ follows from (5.57) and from the fact that

$$[f(\lambda)]^{l-1}[\Delta(\lambda;k)]^{l-1}D_j(\lambda) \tag{5.83}$$

and

$$\Delta(\lambda;k)z_j(\lambda)=\Delta(\lambda;k)[I-\lambda H(\lambda)]^{-1}x_j(\lambda) \tag{5.84}$$

have only removable singularities. The assertion about $[I-\lambda K(\lambda)]^{-1}$ follows from (5.29) and from the assertion about $[I-\lambda G(\lambda)]^{-1}$. ■

COROLLARY V.4-2. – The equation

$$\begin{aligned}\tau\Big\{[\lambda K(\lambda)]'[I-\lambda K(\lambda)]^{-1}-\sum_{j=0}^{k-1}\lambda^j \boldsymbol{m}_j[[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}]\Big\}&\\ =-\frac{\Delta'(\lambda;k)}{\Delta(\lambda;k)}-\frac{d'(\lambda)}{d(\lambda)}-\sum_{j=0}^{k-1}\lambda^j\tau_j\{[\boldsymbol{\lambda} G(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} G(\boldsymbol{\lambda})]^{-1}\}&\end{aligned} \tag{5.85}$$

holds for all integers $k\geqslant k_0-1$ and for all $\lambda$ in the set (5.24) such that $d(\lambda)\neq 0$.

PROOF. – Note that equation (2.31) yields

$$\tau\{[I-\lambda H(\lambda)][\lambda G(\lambda)]'[I-\lambda G(\lambda)]^{-1}[I-\lambda H(\lambda)]^{-1}\}=\tau\{[\lambda G(\lambda)]'[I-\lambda G(\lambda)]^{-1}\} \tag{5.86}$$

(*) Note that (5.81) is non-vanishing at $\lambda=0$.

since the maps $\lambda \to G(\lambda)$ and $\lambda \to [\lambda G(\lambda)]'$ are analytic with respect to $|\ |_1$ for each $\lambda$ in (5.24), and hence $G(\lambda) \in C(1)$ and $[\lambda G(\lambda)]' \in C(1)$ for $\lambda$ in (5.24). The other operators in (5.86) are evidently in $C(\infty)$ for $\lambda$ in (5.24) such that $d(\lambda) \neq 0$. Note that the restriction on $\lambda$ that $d(\lambda) \neq 0$, along with the condition that $\lambda$ be in (5.24) guarantees existence of the inverses in (5.86) (see assertion (i) of lemma V.4). Also note that the equation (3.10) holds if $A(\lambda)$ in (3.10) is analytic with respect to $|\ |_1$; multiplying (5.86) through by $\boldsymbol{m}_j$ we obtain the relation

$$\tau_j\{[I-\lambda H(\lambda)][\lambda G(\lambda)]'[I-\lambda G(\lambda)]^{-1}[I-\lambda H(\lambda)]^{-1}\} = \tau_j\{[\lambda G(\lambda)]'[I-\lambda G(\lambda)]^{-1}\} \tag{5.87}$$

which is valid for each non-negative integer $j$, since (5.86) is valid for each $\lambda$ in (5.24) such that $d(\lambda) \neq 0$. The existence of the polynomial on the right side of (5.85) follows.

Existence of the trace on the left side of (5.85) follows from the results of lemma V.2; the restrictions imposed on $\lambda$ for the validity of (5.85) guarantee existence of $[I-\lambda K(\lambda)]^{-1}$ (see assertion (i), lemma V.4).

From (5.26), (5.86), (5.87), and from the linearity of $\boldsymbol{m}_j$, we have

$$\begin{aligned}\tau\Big\{[\lambda K(\lambda)]'[I-\lambda K(\lambda)]^{-1} - \sum_{j=0}^{k-1}\lambda^j \boldsymbol{m}_j[[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}]\Big\} \\ = \tau\Big\{[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1} - \sum_{j=0}^{k-1}\lambda^j \boldsymbol{m}_j[[\boldsymbol{\lambda} H(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} H(\boldsymbol{\lambda})]^{-1}]\Big\} \\ + \tau\{[\lambda G(\lambda)]'[I-\lambda G(\lambda)]^{-1}\} - \sum_{j=0}^{k-1}\lambda^j \tau_j\{[\boldsymbol{\lambda} G(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} G(\boldsymbol{\lambda})]^{-1}\}\end{aligned} \tag{5.88}$$

provided $\lambda$ and $k$ satisfy the restrictions imposed in the statement of the corollary. Hence (5.85) follows from (4.97), (5.58), and (5.88). ∎

We can now state the analogue of lemma V.4 for the function $H^{\{m\}}(\lambda)$.

LEMMA V.5. – Let $\lambda_m$ be an element of the sequence $\{\lambda_i\}$, where $m$ henceforth denotes a fixed positive integer.

We shall *always* assume that $\lambda$ is an element of the set $(5.43)_m$. Let $d^{\{m\}}(\lambda)$ be the $l^{\{m\}} \times l^{\{m\}}$ determinant with (see (5.51)-(5.53))

$$\delta_{ij} - \frac{\lambda}{f(\lambda)}\langle z_j^{\{m\}}(\lambda), y_i^{\{m\}}(\bar{\lambda})\rangle \tag{5.89}$$

in its $i$-th row, $j$-th column, where (see (5.53))

$$z_j^{\{m\}}(\lambda) = [I-\lambda H^{\{m\}}(\lambda)]^{-1} x_j^{\{m\}}(\lambda)\,. \tag{5.90}$$

Then:

(i) Let $a$ be a point of the set $(5.43)_m$. The following assertions are equivalent:

1) $d^{\{m\}}(a)=0$;

2) $\lambda=a$ is an eigenvalue of (5.2);

3) $\lambda=a$ is an eigenvalue of the equation (5.91);

$$\lambda G^{\{m\}}(\lambda)z=z \qquad (z\in\mathscr{H}). \tag{5.91}$$

Let $b$ be a point of the set $(5.43)_m$. The following assertions are equivalent:

4) $d^{\{m\}}(b)\neq 0$;

5) the inverse $[I-bK(b)]^{-1}$ exists and is defined on all of $\mathscr{H}$;

6) the inverse $[I-bG^{\{m\}}(b)]^{-1}$ exists and is defined on all of $\mathscr{H}$.

(ii) Let $D_j^{\{m\}}(\lambda)$ be the transformation (from $\mathscr{H}$ onto the set of complex numbers) such that $D_j^{\{m\}}(\lambda)u$ is the determinant obtained by replacing the $j$-th column of $d^{\{m\}}(\lambda)$ by (see (5.53))

$$(\langle u, y_1^{\{m\}}(\bar\lambda)\rangle, \ldots, \langle u, y_{l\{m\}}^{\{m\}}(\bar\lambda)\rangle)^T \qquad (u\in\mathscr{H}).$$

The equation

$$[I-\lambda G^{\{m\}}(\lambda)]^{-1}u=u+\frac{\lambda}{f(\lambda)}\sum_{j=1}^{l\{m\}} z_j^{\{m\}}(\lambda)\,\frac{D_j^{\{m\}}(\lambda)u}{d^{\{m\}}(\lambda)} \tag{5.92}$$

holds at all points $\lambda$ in the set $(5.43)_m$ such that $d^{\{m\}}(\lambda)\neq 0$.

(iii) The equation

$$\frac{-[d^{\{m\}}(\lambda)]'}{d^{\{m\}}(\lambda)}=\tau\{[\lambda G^{\{m\}}(\lambda)]'[I-\lambda G^{\{m\}}(\lambda)]^{-1}\} \tag{5.93}$$

holds at all points $\lambda$ in the set $(5.43)_m$ such that $d^{\{m\}}(\lambda)\neq 0$.

COROLLARY V.5-1. – Let $\lambda$ and $a$ be complex numbers satisfying $|\lambda-\lambda_m|<\varepsilon_m$ and $|a-\lambda_m|<\varepsilon_m$, where $\varepsilon_m$ is the number defined in lemma V.3. If $f(a)=0$, then each of the functions

$$d^{\{m\}}(\lambda), \quad G^{\{m\}}(\lambda), \quad [I-\lambda G^{\{m\}}(\lambda)]^{-1}, \quad \text{and} \quad [I-\lambda K(\lambda)]^{-1}$$

has at worst a pole at $\lambda=a$.

From corollaries V.4-1 and V.5-1, we are able to state the following result.

COROLLARY V.5-2. – If $a$ is any complex number such that $f(a)=0$, then $[I-\lambda K(\lambda)]^{-1}$ has (at worst) a pole at $\lambda=a$. Furthermore, the function $(5.27)_k$ is analytic with respect to $|\ |_1$ on the open set of all points $\lambda$ such that $\lambda$ is not an eigenvalue of (5.2), and such that $f(\lambda)\neq 0$, provided $k\geqslant k_0-1$ in $(5.27)_k$.

COROLLARY V.5-3. – The formula

$$(5.94)\qquad \tau\Big\{[\lambda K(\lambda)]'[I-\lambda K(\lambda)]^{-1}-\sum_{j=0}^{k-1}\lambda^j \boldsymbol{m}_j[[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}]\Big\}$$
$$=-\frac{[\Delta^{\{m\}}(\lambda;k)]'}{\Delta^{\{m\}}(\lambda;k)}-\frac{[d^{\{m\}}(\lambda)]'}{d^{\{m\}}(\lambda)}-\sum_{j=0}^{k-1}\lambda^j\tau_j\{[\boldsymbol{\lambda} G^{\{m\}}(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} G^{\{m\}}(\boldsymbol{\lambda})]^{-1}\}$$

holds for integers $k\geqslant k_0-1$ and for each $\lambda$ in $(5.43)_m$ such that $d^{\{m\}}(\lambda)\neq 0$.

PROOF. – Use (5.46), (5.93), and (4.97) with $\Delta^{\{m\}}(\lambda;k)$ and $H^{\{m\}}(\lambda)$ replacing $\Delta(\lambda;k)$ and $H(\lambda)$ respectively. ∎

A complex valued function $q(\lambda)$ is said to be a meromorphic function of maximal domain if there exists a (non-empty) countable set $\Phi$ of isolated points such that $q(\lambda)$ is defined and analytic for all complex $\lambda\notin\Phi$, and if $q(\lambda)$ has a pole of positive multiplicity at each point $\lambda\in\Phi$ (i.e. $q(\lambda)$ may not have a removable or essential singularity at points $\lambda\in\Phi$).

THEOREM V.6. – Let $H(\lambda)$ be given by (4.1), and let the operators $H_i$ satisfy the assumption (4.96) for each integer $i$ in $0\leqslant i\leqslant s$, where the numbers $\alpha_i$ are real and positive. Let $f(\lambda)$ be an entire, complex-valued function such that $f(0)\neq 0$. Let $x_j(\lambda)\in\mathscr{H}$ and $y_j(\lambda)\in\mathscr{H}$ for each complex $\lambda$ and each integer $j$ in $1\leqslant j\leqslant l$, where $l$ is some fixed positive integer. We suppose that the mappings $\lambda\to x_j(\lambda)$ and $\lambda\to y_j(\lambda)$ are entire functions with respect to the norm $|\ |$ of the Hilbert space $\mathscr{H}$ for each integer $j$ in $1\leqslant j\leqslant l$. Let the function $K(\lambda)$ be given by (5.1).

(i) The (distinct) eigenvalues of (5.2) from a denumerable (or possibly empty) set of isolated points.

(ii) For each fixed integer $k\geqslant k_0-1$, there exists a unique entire or meromorphic function $\mathscr{D}(\lambda;k)$ of maximal domain such that

$$(5.95)\qquad \{\lambda|f(\lambda)\neq 0\}\subseteq \text{domain }(\mathscr{D}(\lambda;k))$$

and such that the equation

$$(5.96)\qquad -\frac{\mathscr{D}'(\lambda;k)}{\mathscr{D}(\lambda;k)}=\tau\Big\{[\lambda K(\lambda)]'[I-\lambda K(\lambda)]^{-1}-\sum_{j=0}^{k-1}\lambda^j\boldsymbol{m}_j[[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I-\boldsymbol{\lambda} K(\boldsymbol{\lambda})^{-1}]\Big\}$$

holds for each integer $k\geqslant k_0-1$ and each complex $\lambda$ which is not an eigenvalue of (5.2) and which is not a zero of the function $f$.

(iii) If $a$ is a complex number such that $f(a)\neq 0$, and if $\lambda=a$ is an eigenvalue of (5.2), then $\mathscr{D}(a;k)=0$ for each integer $k$ satisfying $k\geqslant k_0-1$.

(iv) If $b$ is a complex number such that $\mathscr{D}(b; k) = 0$ for some integer $k$ satisfying $k \geq k_0 - 1$, then exactly one of the following occur.

(1) $f(b) \neq 0$ and $\lambda = b$ is an eigenvalue of (5.2);

(2) $f(b) = 0$.

(v) The (possibly finite or empty) ordered sequences of zeroes and poles of $\mathscr{D}(\lambda; k)$, written according to multiplicity, are independent of $k$, provided $k \geq k_0 - 1$. Hence the domain of $\mathscr{D}(\lambda; k)$ is independent of $k$ for $k \geq k_0 - 1$. The ordered sequences of zeroes and poles of $\mathscr{D}(\lambda; k)$, taken according to algebraic multiplicity, will be denoted by $\{\nu_i\}$ and $\{\varrho_i\}$ respectively. The sequences $\{\nu_i\}$ and $\{\varrho_i\}$ have no finite limit points.

(vi) The equation

$$(5.97) \qquad -\frac{\mathscr{D}'(\lambda; k)}{\mathscr{D}(\lambda; k)} = \sum_{j=k}^{\infty} \lambda^j \tau_j\{[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}\}$$

holds for each integer $k \geq k_0 - 1$ and each $\lambda$ such that $|\lambda| < \infty$ if $\mathscr{D}(\lambda; k)$ has no zeroes or poles, or such that

$$(5.98) \qquad |\lambda| < \min\left\{\left[\bigcup_i \{|\nu_i|\}\right] \cup \left[\bigcup_j \{|\varrho_i|\}\right]\right\}$$

if $\mathscr{D}(\lambda; k)$ has zeroes or poles. The minimum on the right side of (5.98) is a positive number.

(vii) Let the polynomial $p(\lambda; k)$ be defined by

$$(5.99) \qquad p(\lambda; k) = \sum_{j=0}^{k-1} \frac{\lambda^{j+1}}{j+1} \tau_j\{[\boldsymbol{\lambda} G(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} G(\boldsymbol{\lambda})]^{-1}\}.$$

Furthermore, let $\tilde{g}(\lambda; k)$ be the $l \times l$ determinant with

$$(5.100) \qquad \delta_{ij} f(\lambda) \Delta(\lambda; k) - \langle \Delta(\lambda; k)[I - \lambda H(\lambda)]^{-1} x_j(\lambda), y_i(\bar{\lambda}) \rangle$$

in its $i$-th row, $j$-th column for each complex $\lambda \notin \bigcup_i \{\lambda_i\}$, where the latter union is taken over the whole sequence $\{\lambda_i\}$. The function $\tilde{g}(\lambda; k)$ has removable singularities at the points $\lambda = \lambda_i$, and hence has an extension $g(\lambda; k)$ which is an entire function. The equation

$$(5.101) \qquad \mathscr{D}(\lambda; k) = \frac{g(\lambda; k)}{[f(\lambda)]^l [\Delta(\lambda; k)]^{l-1}} \exp p(\lambda; k)$$

holds for each $\lambda$ in the set (5.24).

COMMENTS. – The reader will note that the Fredholm function $\mathscr{D}(\lambda; k)$ of the operator $K(\lambda)$ is asserted to be defined and analytic (at least) for points $\lambda$ such that $f(\lambda) \neq 0$. In addition, the function $\mathscr{D}(\lambda; k)$ either will have poles or will actually be analytic at points such that $f(\lambda) = 0$. In parts (i) through (vi), we did not wish to make reference to the auxiliary functions $G(\lambda)$, $d(\lambda)$, etc.; hence the assertions (i) through (vi) will not necessarily be proved in the order in which they are stated.

We will begin by constructing $\mathscr{D}(\lambda; k)$ in terms of the functions $\Delta(\lambda; k)$ and $d(\lambda)$ (see (5.102)), which is a generalization of a classical « multiplication » theorem of Fredholm. In using the definition (5.102), we shall restrict $\lambda$ to the set (5.24); the restriction of $\mathscr{D}(\lambda; k)$ to the set (5.24) is called $\tilde{\mathscr{D}}(\lambda; k)$. The function $\tilde{\mathscr{D}}(\lambda; k)$ has all of the properties we want it to have at points of the set (5.24), i.e. it is analytic on (5.24) and the eigenvalues of (5.2) which occur in the set (5.24) are also the zeroes of $\tilde{\mathscr{D}}(\lambda; k)$. But we claim that the functions $\tilde{\mathscr{D}}(\lambda; k)$ should not really be badly behaved about the points $\lambda = \lambda_m$ unless, of course, it should happen that $f(\lambda_m) = 0$. As we pointed out earlier, the problem with $G(\lambda)$ and $d(\lambda)$ (and hence with $\tilde{\mathscr{D}}(\lambda; k)$) is that they were constructed by using the operator $[I - \lambda H(\lambda)]^{-1}$; hence these functions are not defined at the points $\lambda = \lambda_m$. So we look at the function $\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$ constructed from $\Delta^{\{m\}}(\lambda; k)$ and $d^{\{m\}}(\lambda)$ (see (5.103)). Heuristically, $\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$ has all the properties we want it to have about the point $\lambda = \lambda_m$ (provided $f(\lambda_m) \neq 0$), i.e. it is analytic about $\lambda = \lambda_m$ and it vanishes at the eigenvalues of (5.2) which are near the point $\lambda_m$. But we will show that $\tilde{\mathscr{D}}(\lambda; k)$ and $\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$ are equal on their domains of definition (which, as we indicated comprise all but a countable number of points, possibly accumulating at $\lambda = \infty$). Hence $\tilde{\mathscr{D}}(\lambda; k)$ has a removable singularity at $\lambda = \lambda_m$ if $f(\lambda_m) \neq 0$, and we may extend the definition of $\tilde{\mathscr{D}}(\lambda; k)$ to the largest set on which it will be analytic. The resulting function is, of course, $\mathscr{D}(\lambda, k)$, which now has the properties we want, i.e. it is analytic on its domain of definition, and it vanishes if $\lambda$ is an eigenvalue of (5.2). One shortcoming about $\mathscr{D}(\lambda; k)$ is that it may also have zeroes at some points $\lambda$ such that $f(\lambda) = 0$; aside from this, and one other critical property, $\mathscr{D}(\lambda; k)$ shares many of the properties that $\Delta(\lambda; k)$ has. The other one property we refer to is that up to now, we cannot guarantee (at least yet) that the sums of reciprocal powers of the eigenvalues of (5.2) (or of the zeroes of $\mathscr{D}(\lambda; k)$) are convergent, or, if convergent, can be evaluated. For this, certain additional assumptions are needed on the growth of the functions $|x_j(\lambda)|$, $|y_j(\lambda)|$ and $|f(\lambda)|$; this will be pursued later on. The relation (5.101) will be important for the establishment of the analogue of (4.102); it expresses $\mathscr{D}(\lambda; k)$ as a quotient of two entire functions.

PROOF. – Let $p(\lambda; k)$ be the polynomial defined in (5.99), and let

$$\tilde{\mathscr{D}}(\lambda; k) = \Delta(\lambda; k)\, d(\lambda) \exp p(\lambda; k) \tag{5.102}$$

for each integer $k \geq k_0 - 1$ and for each $\lambda$ in the open set (5.24). (The set (5.24) coincides with the domain of definition of the function $d(\lambda)$).

Let $\lambda_m$ be an element of the sequence $\{\lambda_i\}$, where $m$ is a fixed positive integer, and let $p^{\{m\}}(\lambda; k)$ be the polynomial obtained by replacing $G(\lambda)$ by $G^{\{m\}}(\lambda)$ in (5.99). We define

$$\tilde{\mathscr{D}}^{\{m\}}(\lambda; k) = \Delta^{\{m\}}(\lambda; k)\, d^{\{m\}}(\lambda) \exp p^{\{m\}}(\lambda; k) \tag{5.103}$$

for each integer $k \geqslant k_0 - 1$ and each complex $\lambda$ in the set $(5.43)_m$; here $\Delta^{\{m\}}(\lambda; k)$ is the function defined in lemma V.3. (The set $(5.43)_m$ is precisely the domain of definition of the function $d^{\{m\}}(\lambda)$.) If $f(\lambda_m) \neq 0$, then $\lambda_m$ is an element of the open set $(5.43)_m$ by the results of lemma V.3; hence if $f(\lambda_m) \neq 0$, then the function $\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$ is defined and analytic for each integer $k \geqslant k_0 - 1$ and for each $\lambda$ in a small open disc (which does not depend on $k$) about the point $\lambda_m$.

For convenience, let $\tilde{\mathscr{S}}$ and $\tilde{\mathscr{S}}^{\{m\}}$ denote the sets (5.24) and $(5.43)_m$ respectively.

Let $\tilde{\mathscr{T}}$ $(\tilde{\mathscr{T}}^{\{m\}})$ denote the set of all points $\lambda$ in $\tilde{\mathscr{S}}(\tilde{\mathscr{S}}^{\{m\}})$ such that $d(\lambda) \neq 0$ $(d^{\{m\}}(\lambda) \neq 0)$. By the definition of $\tilde{\mathscr{S}}(\tilde{\mathscr{S}}^{\{m\}})$, we have that $\Delta(\lambda; k) \neq 0$ $(\Delta^{\{m\}}(\lambda; k) \neq 0)$ for each integer $k \geqslant k_0 - 1$ and each $\lambda$ in $\tilde{\mathscr{S}}(\tilde{\mathscr{S}}^{\{m\}})$. Hence $\tilde{\mathscr{D}}(\lambda; k) \neq 0$ $(\tilde{\mathscr{D}}^{\{m\}}(\lambda; k) \neq 0)$ for each $\lambda$ in $\tilde{\mathscr{T}}(\tilde{\mathscr{T}}^{\{m\}})$.

Note that the set $\tilde{\mathscr{T}}$ is precisely the complement of the set described in (5.80); hence $\tilde{\mathscr{T}}$ consists of all points in the plane except possibly for a countable number of points which are isolated, i.e. which may accumulate only at $\lambda = \infty$. Hence $\tilde{\mathscr{T}}$ is a connected open subset of the complex plane. The same conclusions evidently apply to the set $\tilde{\mathscr{T}}^{\{m\}}$, and hence to $\tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}}$.

By (5.85), (5.94), (5.102), and (5.103), we have

$$\frac{\tilde{\mathscr{D}}'(\lambda; k)}{\tilde{\mathscr{D}}(\lambda; k)} = \frac{(\tilde{\mathscr{D}}^{\{m\}}(\lambda; k))'}{\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)} \tag{5.104}$$

for each $\lambda \in \tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}}$ and each $k \geqslant k_0 - 1$. But $0 \in \tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}}$, and

$$\tilde{\mathscr{D}}(0; k) = \tilde{\mathscr{D}}^{\{m\}}(0; k) = 1 \tag{5.105}$$

so that for each integer $k \geqslant k_0 - 1$, we have

$$\tilde{\mathscr{D}}(\lambda; k) = \tilde{\mathscr{D}}^{\{m\}}(\lambda; k) \tag{5.106}$$

provided $\lambda \in \tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}}$, since the latter set is a connected open set. By continuity, the equation (5.106) is valid for each $\lambda \in \tilde{\mathscr{S}} \cap \tilde{\mathscr{S}}^{\{m\}}$; in order to show this, the reader will recall that the complement of $\tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}}$ consists of a denumerable number of isolated points, so that if $\lambda \in (\tilde{\mathscr{S}} \cap \tilde{\mathscr{S}}^{\{m\}}) - (\tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}})$, then there is a punctured disc centered about $\lambda$ consisting entirely of points in $\tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}}$. Equation (5.106) is then established for $\lambda \in (\tilde{\mathscr{S}} \cap \tilde{\mathscr{S}}^{\{m\}}) - (\tilde{\mathscr{T}} \cap \tilde{\mathscr{T}}^{\{m\}})$ by continuity arguments.

Note that the points in the complement of $\tilde{\mathscr{S}} \cap \tilde{\mathscr{S}}^{\{m\}}$ are isolated; hence there exists a small punctured open disc $\Theta_m$ centered about $\lambda_m$ such that $\Theta_m \subset \tilde{\mathscr{S}} \cap \tilde{\mathscr{S}}^{\{m\}}$.

If $f(\lambda_m) \neq 0$, then $\lambda_m \in \tilde{\mathscr{S}}^{\{m\}}$, as we have already seen. Thus if $f(\lambda_m) \neq 0$, the function $\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$ is analytic for each $\lambda \in \Theta_m \cup \{\lambda_m\}$ and each integer $k \geqslant k_0 - 1$; also $\tilde{\mathscr{D}}(\lambda; k)$ is analytic for each $\lambda \in \Theta_m$ and each $k \geqslant k_0 - 1$. Furthermore, (5.106) holds for each $\lambda \in \Theta_m$ and each integer $k \geqslant k_0 - 1$; hence if $f(\lambda_m) \neq 0$, the function $\tilde{\mathscr{D}}(\lambda; k)$ has a removable singularity at the point $\lambda = \lambda_m$. If $f(\lambda_m) = 0$, then by corollary V.5-1 and by (5.103), we conclude that $\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$ has (at worst) a pole at the point $\lambda_m$; since (5.106) is valid for each $\lambda \in \Theta_m$, we conclude that the function $\tilde{\mathscr{D}}(\lambda; k)$ has (at worst) a pole at the point $\lambda_m$. By the preceding, and by the results of corollary V.4-1, we may conclude that the complement of $\tilde{\mathscr{S}}$ consists of a denumerable number of isolated points at which $\tilde{\mathscr{D}}(\lambda; k)$ has, at worst, poles; hence there exists an entire or meromorphic function $\mathscr{D}(\lambda; k)$ of maximal domain $\mathscr{S}_k$ such that $\mathscr{D}(\lambda; k)$ is an analytic extension of $\tilde{\mathscr{D}}(\lambda; k)$. In fact, we will now show that $\mathscr{S}_k$ is independent of $k$; at this point, it is clear that (5.95) holds.

Since $\mathscr{D}(\lambda; k) = \tilde{\mathscr{D}}(\lambda; k)$ for each $\lambda \in \tilde{\mathscr{S}}$, and hence for each $\lambda \in \tilde{\mathscr{T}}$, we can show that the equation (5.96) follows from (5.85) and (5.102), provided $\lambda \in \tilde{\mathscr{T}}$ and $k \geqslant k_0 - 1$. (The validity of (5.96) in part (ii) is asserted for all $\lambda$ belonging to a set somewhat different than $\tilde{\mathscr{T}}$; at this point, we cannot consider the proof of (5.96) as being complete).

Let $k$ and $k'$ be integers such that $k' > k \geqslant k_0 - 1$. We claim that

$$\frac{\mathscr{D}'(\lambda; k)}{\mathscr{D}(\lambda; k)} - \frac{\mathscr{D}'(\lambda; k')}{\mathscr{D}(\lambda; k')} = \sum_{j=k}^{k'-1} \lambda^j \tau_j\{[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}\} \tag{5.107}$$

for each $\lambda \in \tilde{\mathscr{T}}$. First we show that right side of (5.107) is well defined. Since $(5.27)_k$ is analytic with respect to $|\ |_1$ for (say) $\lambda$ near the point zero, provided $k \geqslant k_0 - 1$, the difference $(5.108)_j$

$$\lambda^j \boldsymbol{m}_j\{[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}\} \tag{$5.108_j$}$$

of $(5.27)_{j+1}$ and $(5.27)_j$ is analytic with respect to $|\ |_1$ for $\lambda$ near zero and for integers $j \geqslant k_0 - 1$; a fortiori, the operator $(5.108)_j$, and hence the operator

$$\boldsymbol{m}_j\{[\lambda K(\lambda)]'[I - \lambda K(\lambda)]^{-1}\}$$

is an element of the space $C\{1; \mathscr{H}\}$ for integers $j \geqslant k_0 - 1$. Hence the right side of (5.107) is well defined. Equation (5.107) is then established for $\lambda \in \tilde{\mathscr{T}}$ by subtracting the relation $(5.96)_{k'}$ from the relation $(5.96)_k$. Since $\tilde{\mathscr{T}}$ is a connected open set, we may conclude from (5.107) that

$$\mathscr{D}(\lambda; k) = \mathscr{D}(\lambda; k') \exp \sum_{j=k}^{k'-1} \frac{\lambda^{j+1}}{j+1} \tau_j\{[\boldsymbol{\lambda} K(\boldsymbol{\lambda})]'[I - \boldsymbol{\lambda} K(\boldsymbol{\lambda})]^{-1}\} \tag{5.109}$$

provided $\lambda \in \tilde{\mathscr{T}}$ and $k' > k \geqslant k_0 - 1$. But the complement of $\tilde{\mathscr{T}}$ is a denumerable set of isolated points; since $\mathscr{D}(\lambda; k)$ and $\mathscr{D}(\lambda; k')$ are either entire functions or meromorphic functions of maximal domain, we may conclude (by arguments of the type previously employed) from (5.109) that $\mathscr{S}_k = \mathscr{S}_{k'}$. In the sequel, we shall denote $\mathscr{S}_k$ or $\mathscr{S}_{k'}$ by the symbol $\mathscr{S}$. Note that the zeroes and poles of an entire or meromorphic function of maximal domain must be countable and isolated. The results of assertion (v) in the statement of this theorem are therefore proved.

The proofs of parts (iii) and (iv) are related; since the proof of (iv) is a bit more difficult, we shall prove (iv) and leave the proof of (iii) to the reader.

Suppose $b \in \mathscr{S}$ and $\mathscr{D}(b; k) = 0$ for some integer $k \geqslant k_0 - 1$. If $f(b) = 0$, we are finished, so let us suppose henceforth that $f(b) \neq 0$. We consider two subcases: either $b \neq \lambda_m$ for any positive integer $m$ or $b = \lambda_m$ for some positive integer $m$.

Suppose $b = \lambda_m$ for some positive integer $m$. Since we have assumed that $f(b) \neq 0$, by the results of lemma V.3, we may conclude that $b \in \tilde{\mathscr{S}}^{\{m\}}$ and that $\Delta^{\{m\}}(b, k) \neq 0$. But $\mathscr{D}(\lambda; k)$ is an analytic extension of $\tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$, and $\mathscr{D}(\lambda; k) = \tilde{\mathscr{D}}^{\{m\}}(\lambda; k)$ for $\lambda \in \tilde{\mathscr{S}}^{\{m\}}$. Hence $\tilde{\mathscr{D}}^{\{m\}}(b, k) = \mathscr{D}(b; k) = 0$; by (5.103), we have that $d^{\{m\}}(b) = 0$ since $\tilde{\mathscr{D}}^{\{m\}}(b, k) = 0$ and since $\Delta^{\{m\}}(b, k) \neq 0$. Hence by assertion (i) of lemma V.5, we may conclude that $\lambda = b$ is an eigenvalue of (5.2).

If $b \neq \lambda_m$ for any positive integer $m$, we may prove (iv) in a similar fashion, by using the results of lemma V.4. The result (i) is an immediate consequence of parts (iii), (iv), and (v).

Up to now, the relation $(5.96)_k$ has been established for each $\lambda \in \tilde{\mathscr{T}}$. Let $\mathscr{T}$ denote the set of all points $\lambda \in \mathscr{S}$ such that $\mathscr{D}(\lambda; k) f(\lambda) \neq 0$. We wish to establish $(5.96)_k$ for each $\lambda \in \mathscr{T}$. (Using the results of parts (iii) and (iv), the reader may show that $\mathscr{T}$ is precisely the set of all $\lambda$ such that $\lambda$ is not an eigenvalue of (5.2) and $\lambda$ is not a zero of the function $f$). We note by the definitions of $\mathscr{T}$ and $\tilde{\mathscr{T}}$ that $\tilde{\mathscr{T}} \subseteq \mathscr{T}$; since the complement of $\tilde{\mathscr{T}}$ is a set of isolated points, each point $\lambda$ in $\mathscr{T} - \tilde{\mathscr{T}}$ is at the center of a punctured open disc comprised entirely of points in $\tilde{\mathscr{T}}$. The validity of $(5.96)_k$ for each $\lambda$ in $\mathscr{T} - \tilde{\mathscr{T}}$ may be established by continuity arguments.

Assertion (vi) in the statement of this theorem follows directly from the results of assertion (ii).

The fact that the function $\tilde{g}(\lambda; k)$ has an extension $g(\lambda; k)$ which is an entire function follows from the fact that $\Delta(\lambda; k)[I - \lambda H(\lambda)]^{-1}$ has removable singularities at the points $\lambda = \lambda_i$.

We note that $g(\lambda; k)$ is precisely the expression (5.81), provided $\lambda \in \tilde{\mathscr{S}}$ (i.e. $\lambda$ is an element of the set (5.24)). Hence (5.101) follows directly from (5.102) for $\lambda \in \tilde{\mathscr{S}}$. ■

The situation (2) described in part (iv) of theorem V.6 occasionally does occur, as illustrated in the following example. The computation of $\mathscr{D}(\lambda; 0)$ for the operator $K(\lambda)$ described in the example may be done in many different ways.

EXAMPLE. – Let $\{\varphi_1, \varphi_2, \varphi_3\}$ be any orthonormal set in the space $\mathscr{H}$. Let

$$K(\lambda) u = \varphi_1 \langle u, \varphi_1 \rangle + \varphi_2 \langle u, \varphi_2 \rangle + \frac{1}{(\lambda - 1)} \varphi_3 \langle u, \varphi_3 \rangle .$$

We set $H(\lambda)=0$, $P(\lambda)=K(\lambda)$, $l=3$, and

$$f(\lambda) = (\lambda-1)$$

$$x_i(\lambda) = (\lambda-1)\varphi_i \qquad (i=1,2)$$

$$x_3(\lambda) = \varphi_3$$

$$y_i(\lambda) = \varphi_i \qquad (i=1,2,3)\,.$$

Since $H(\lambda)=0$, we have $\Delta(\lambda;0)\equiv 1$ and $p(\lambda,0)\equiv 0$. Hence $\tilde{\mathscr{D}}(\lambda;0)$ and $d(\lambda)$ are defined for all $\lambda\neq 1$, and $\tilde{\mathscr{D}}(\lambda;0)=d(\lambda)$ for $\lambda\neq 1$. Setting $z_j(\lambda)=x_j(\lambda)$ in (5.54), we have that $d(\lambda)=1-\lambda$. Hence $\mathscr{D}(\lambda;0)$ is defined for all complex $\lambda$, and $\mathscr{D}(\lambda;0)=(1-\lambda)$. We see that $\mathscr{D}(\lambda;0)$ has a zero at $\lambda=1$, but $\lambda=1$ is not an eigenvalue of (5.2), because $\lambda=1$ is a pole of $K(\lambda)$. ∎

The first result of corollary V.5-2 and the results of the next corollary are essentially contained in TAMARKIN [14; p. 148] for the special case $\mathscr{H}=L_2[0,1]$ and $0<\alpha_i\leqslant 2$.

COROLLARY V.6-1. – Let $a$ be a complex number such that $f(a)\neq 0$. If $\lambda=a$ is an eigenvalue of (5.2), then $\lambda=a$ is a pole of $[I-\lambda K(\lambda)]^{-1}$, (where the Laurent series for the latter is convergent with respect to $|\;|_\infty$ in a punctured disc centered about $\lambda=a$).

PROOF. – Suppose $a\notin\bigcup_i\{\lambda_i\}$. Since $f(a)\neq 0$, we conclude that $a$ is an element of the set (5.24). By assertion (i) of lemma V.4, we have that $d(a)=0$. By (5.57), we conclude that $[I-\lambda G(\lambda)]^{-1}$ has, at worst, a pole at $\lambda=a$. Hence by (5.29), we conclude that $[I-\lambda K(\lambda)]^{-1}$ has, at worst, a pole at $\lambda=a$. If $[I-\lambda K(\lambda)]^{-1}$ has a removable singularity at $\lambda=a$, then it is easy to show that $\lim\limits_{\lambda\to a}[I-\lambda K(\lambda)]^{-1}$ provides an inverse to $[I-aK(a)]$, contradicting the fact that $\lambda=a$ is an eigenvalue of (5.2).

If $a=\lambda_m$ for some $m$, then similar arguments apply. Here one would use part (i) of lemma V.5, equation (5.92), and the relation obtained by replacing $H(\lambda)$ and $G(\lambda)$ in (5.29) by $H^{\{m\}}(\lambda)$ and $G^{\{m\}}(\lambda)$ respectively. The latter relation, i.e. the analogue of (5.29), is valid because of (5.44)-(5.45). ∎

We have arrived at the point where we will be able to establish an analogue of (4.102) for the function $K(\lambda)$. The equation (5.101) will be a key result here, since we have the meromorphic function $\mathscr{D}(\lambda;k)$ expressed as a quotient of entire functions.

We will briefly recall a few facts about entire functions of finite order. Let $W(\lambda)\not\equiv 0$ be an entire complex valued function. Let $\delta\geqslant 0$. We recall that $W(\lambda)$ is said to be a function of order $\delta$ if, for each $\varepsilon'>0$, there exists $A(\varepsilon')>0$ such that the inequality

$$|W(\lambda)|\leqslant A(\varepsilon')\exp\left[|\lambda|^{\delta+\varepsilon'}\right] \tag{5.110}$$

holds for all complex $\lambda$, and if $\delta$ is the smallest nonnegative real number with the property (5.110). If $W_i(\lambda)$ has order $\delta_i$ for $i=1,2$, then the functions $W_1(\lambda)+W_2(\lambda)$ and $W_1(\lambda)\,W_2(\lambda)$ have orders not exceeding $\max\{\delta_1, \delta_2\}$. If $W(\lambda)$ has order $\delta \geqslant 0$, and if $\{\omega_i\}$ is the sequence of non-zero roots of the equation $W(\lambda)=0$, taken according to algebraic multiplicity, then

$$W(\lambda)=B\lambda^n e^{q(\lambda)}\prod_i\left[1-\frac{\lambda}{\omega_i}\right]\exp\sum_{j=1}^{h}\frac{1}{j}\left[\frac{\lambda}{\omega_i}\right]^j \tag{5.111}$$

where $B$ is some non-zero constant, where $n$ is some non-negative integer, where $(h+1)$ is the smallest integer larger than $\delta$, and where $q(\lambda)$ is a certain polynomial whose degree does not exceed $h$. The product appearing in (5.111) is taken over the whole sequence $\{\omega_i\}$, and is uniformly convergent on compact subsets of the complex plane. The product in (5.111) is not quite the same as the canonical product discussed in TITCHMARSH [15; p. 250]; however, we will not need this concept here. The inequality

$$\sum_i|\omega_i|^{-(\delta+\varepsilon)}<+\infty \tag{5.112}$$

holds for each $\varepsilon>0$; furthermore, a computation similar to the one needed to obtain (4.147) will yield (*)

$$-\frac{W'(\lambda)}{W(\lambda)}=-q'(\lambda)+\sum_{i=h}^{\infty}\lambda^i\sum_j\omega_j^{-(i+1)}\,. \tag{5.113}$$

Since $q'(\lambda)$ is a polynomial of degree $(h-1)$ or less, we note that the relation

$$-\boldsymbol{m}_i\left(\frac{W'}{W}\right)=\sum_j\omega_j^{-(i+1)} \tag{5.114}$$

holds for each integer $i>\delta-1$. We can now state the analogue of (4.102).

THEOREM V.7. – Suppose $H(\lambda)$, $K(\lambda)$, $x_j(\lambda)$, $y_j(\lambda)$, and $f(\lambda)$ satisfy the assumptions of theorem V.6.

Let $\varkappa\geqslant 0$ and let $A(\varepsilon')>0$ for each $\varepsilon'>0$. Suppose that the inequality

$$|f(\lambda)|+\sum_{j=0}^{l}|x_j(\lambda)|+|y_j(\lambda)|\leqslant A(\varepsilon')\exp\left[|\lambda|^{\varkappa+\varepsilon'}\right] \tag{5.115}$$

holds for each $\varepsilon'>0$ and for each complex $\lambda$. Let

$$k_2=\max\{k_0,\varkappa\} \tag{5.116}$$

and let $\{\nu_i\}$ and $\{\varrho_i\}$ be the ordered sequences of zeroes and poles of the function $\mathscr{D}(\lambda;k)$ (for $k\geqslant k_0-1$), taken according to multiplicity. Then:

(*) We assume $W(0)\neq 0$ in (5.113) and (5.114).

(i) We have

$$\sum_i |\nu_i|^{-(k_2+\varepsilon)} < +\infty \tag{5.117}$$

and

$$\sum_i |\varrho_i|^{-(k_2+\varepsilon)} < +\infty \tag{5.118}$$

for each $\varepsilon > 0$, where the sums (5.117) and (5.118) are taken over the whole sequences $\{\nu_i\}$ and $\{\varrho_i\}$ respectively.

(ii) The formula

$$\sum_i \nu_i^{-(k+1)} - \sum_i \varrho_i^{-(k+1)} = \tau_k\{[\lambda K(\lambda)]^l[I-\lambda K(\lambda)]^{-1}\} \tag{5.119}$$

holds for each integer $k$ such that $k > k_2 - 1$.

NOTE. – The series in (5.117) and (5.118) are not necessarily asserted to converge if $\varepsilon = 0$. Consequently, if $k_2$ is an integer, the formula (5.119) is not necessarily valid for $k = k_2 - 1$. This contrasts somewhat with the results obtained in part (vi) of theorem IV.3. In this respect, it may be that this theorem can be slightly refined if $\varkappa < k_0$ (see theorem V.8).

PROOF. – As we have previously done, we let $k_1$ be the smallest integer such that $k_1 \geqslant k_0 - 1$. We let $k = k_1$ in (5.99), (5.100), and (5.101). Recall that any determinant can be expressed in terms of sums and differences of products of its component entries. From the results of assertion (viii) of theorem IV.3, and from (5.115), we are able to obtain estimates on the expression (5.100) and hence on the function $g(\lambda; k)$. Hence there exists a function $\hat{B}(\varepsilon') > 0$ defined for $\varepsilon' > 0$ such that the inequality

$$|g(\lambda; k_1)| \leqslant \hat{B}(\varepsilon') \exp[|\lambda|^{k_2+\varepsilon'}] \tag{5.120}$$

holds for each complex $\lambda$ and each $\varepsilon' > 0$. Since $p(\lambda; k_1)$ in (5.99) is a complex valued polynomial of degree $k_1$ or less, we have

$$|\exp p(\lambda; k_1)| \leqslant \tilde{B}(\varepsilon') \exp[|\lambda|^{k_1+\varepsilon'}] \tag{5.121}$$

where $\tilde{B}(\varepsilon') > 0$ for each $\varepsilon' > 0$. By definition of $k_1$, we can show that $k_1 < k_0$; hence from (5.116), (5.120), and (5.121), we obtain

$$|g(\lambda; k_1) \exp p(\lambda; k_1)| \leqslant B(\varepsilon') \exp[|\lambda|^{k_2+\varepsilon'}] \tag{5.122}$$

where $B(\varepsilon') > 0$ for each $\varepsilon' > 0$. Finally, (4.104) and (5.115) yield the inequality

$$|[f(\lambda)]^l[\Delta(\lambda; k_1)]^{l-1}| \leqslant D(\varepsilon') \exp[|\lambda|^{k_2+\varepsilon'}] \tag{5.123}$$

where $D(\varepsilon') > 0$ for each $\varepsilon' > 0$.

Let $\{\tilde{\nu}_i\}$ be the ordered sequence of zeroes of the function $r(\lambda)$, taken according to multiplicity, where

$$r(\lambda) = g(\lambda; k_1) \exp p(\lambda; k_1). \tag{5.124}$$

Let $\{\tilde{\varrho}_i\}$ be the ordered sequence of zeroes of the function $t(\lambda)$, taken according to multiplicity, where

$$t(\lambda) = [f(\lambda)]^l [\Delta(\lambda; k_1)]^{l-1}. \tag{5.125}$$

From (5.122), (5.123), and (5.112), we have

$$\sum_i |\tilde{\nu}_i|^{-(k_2+\varepsilon)} < +\infty \tag{5.126}$$

and

$$\sum_i |\tilde{\varrho}_i|^{-(k_2+\varepsilon)} < +\infty \tag{5.127}$$

for each $\varepsilon > 0$. Furthermore, by (5.122), (5.123), and by (5.114), the equations

$$-\boldsymbol{m}_k\left(\frac{r'}{r}\right) = \sum_i \tilde{\nu}_i^{-(k+1)} \tag{5.128}$$

and

$$-\boldsymbol{m}_k\left(\frac{t'}{t}\right) = \sum_i \tilde{\varrho}_i^{-(k+1)} \tag{5.129}$$

holds for integers $k > k_2 - 1$. Now by (5.101), (5.124), and (5.125), we have

$$\mathscr{D}(\lambda; k_1) = r(\lambda)/t(\lambda) \tag{5.130}$$

so that the ordered sequences $\{\nu_i\}$ and $\{\varrho_i\}$ are subsequences of the ordered sequences $\{\tilde{\nu}_i\}$ and $\{\tilde{\varrho}_i\}$ respectively. Hence (5.117) and (5.118) follow from (5.126) and (5.127) respectively. The equation

$$\sum_i \nu_i^{-(k+1)} - \sum_i \varrho_i^{-(k+1)} = \sum_i \tilde{\nu}_i^{-(k+1)} - \sum_i \tilde{\varrho}_i^{-(k+1)} \tag{5.131}$$

follows from (5.130) if $k > k_2 - 1$ by arguments of an algebraic sort. Also we have from (5.130).

$$-\boldsymbol{m}_k\left[\frac{\mathscr{D}'(\lambda; k_1)}{\mathscr{D}(\lambda; k_1)}\right] = -\boldsymbol{m}_k\left[\frac{r'(\lambda)}{r(\lambda)}\right] + \boldsymbol{m}_k\left[\frac{t'(\lambda)}{t(\lambda)}\right]. \tag{5.132}$$

Equation (5.119) follows from (5.97), (5.128), (5.129), (5.131), and (5.132). ■

The next theorem represents a slight refinement of theorem V.7. If the functions $x_i(\lambda)$, $y_i(\lambda)$, and $f(\lambda)$, are all polynomials in $\lambda$, then (5.115) holds with $\varkappa = 0$. Here $k_2 = k_0$, but the series in (5.117)-(5.118) are actually convergent if $\varepsilon = 0$. If $k_0$ is an integer, then it turns out that we have validity of (5.119) for $k = k_0 - 1$, and not just for $k > k_0 - 1$.

THEOREM V.8. Let $K(\lambda)$ be given by (5.1), and let all the assumptions of theorem V.6 be satisfied. In addition, let us suppose that $x_i(\lambda)$ and $y_i(\lambda)$ are $\mathscr{H}$-valued polynomials in $\lambda$ for each integer $i$ satisfying $1 \leqslant i \leqslant l$, and let us assume that $f(\lambda)$ is a complex-valued polynomial in $\lambda$ with $f(0) \neq 0$. Then (5.117) and (5.118) with $k_2 = k_0$ hold for $\varepsilon = 0$ and (5.119) holds for each integer $k \geqslant k_0 - 1$.

PROOF. – We assume that $f(0) = 1$ and that $x_j(0) = \vec{0}$ for $j = 1, \ldots, l$, where $\vec{0}$ is the zero-vector in $\mathscr{H}$. To see that the latter is really no restriction, let us first define the polynomial $g(\lambda)$ by

$$\begin{cases} g(\lambda) = \lambda^{-1}[1 - f(\lambda)], & \lambda \neq 0 \\ g(0) = -f'(0). \end{cases} \tag{5.133}$$

Let the operator $\hat{H}(\lambda)$ be defined by

$$\hat{H}(\lambda)v = H(\lambda)v + \sum_{j=1}^{l} x_j(\lambda)\langle v, y_j(\bar{\lambda})\rangle, \qquad (v \in \mathscr{H}) \tag{5.134}$$

and let

$$\hat{x}_j(\lambda) = \lambda g(\lambda)\, x_j(\lambda). \tag{5.135}$$

Using the relation

$$\frac{1}{f(\lambda)} = \frac{\lambda g(\lambda)}{f(\lambda)} + 1 \tag{5.136}$$

we see that

$$K(\lambda)v = \hat{H}(\lambda)v + \frac{1}{f(\lambda)} \sum_{j=1}^{l} \hat{x}(\lambda)\langle v, y_j(\bar{\lambda})\rangle \tag{5.137}$$

for each $v \in \mathscr{H}$. We note that $\hat{H}(\lambda)$ is an operator-valued *polynomial*, and that $\hat{x}_j(\lambda)$ is an $\mathscr{H}$-valued *polynomial*. Furthermore, $\hat{x}_j(0) = \vec{0}$ for $j = 1, \ldots, l$. If $k_0$ is the number defined in (4.16), we shall temporarily write $k_0(H)$ instead of $k_0$. By the remarks made in Corollary IV.3, we may write $k_0(H) = k_0(\hat{H})$. We shall henceforth drop the carats in (5.137), and we shall henceforth assume that

$$x_i(\lambda) = \lambda w_i(\lambda) \qquad (i = 1, \ldots, l), \tag{5.138}$$

where $w_i(\lambda)$ is an $\mathscr{H}$ valued *polynomial* in $\lambda$.

We will rewrite (5.2) as a matrix system. Let $h$ be a fixed element in $\mathscr{H}$ with $|h| = 1$. We shall see that (5.2) can be written in the « equivalent » form,

$$u_0 = \lambda H(\lambda) u_0 + \lambda \sum_{i=1}^{l} w_i(\lambda) \langle u_i, h \rangle \tag{5.139}$$

$$u_i = \lambda g(\lambda) \langle u_i, h \rangle h + \lambda \langle u_0, y_i(\bar{\lambda}) \rangle h \qquad (i = 1, \ldots, l) \tag{5.140}$$

where $u_i \in \mathscr{H}$. More precisely, let $a$ be a complex number such that $f(a) \neq 0$. Then if $\lambda = a$ and $(u_0, u_1, \ldots, u_l)$ is a non-trivial solution of (5.139)-(5.140), we claim that $\lambda = a$ and $u = u_0$ must be a non-trivial solution of (5.2). For we must have

$$u_i = \frac{a}{f(a)} \langle u_0, y_i(\bar{a}) \rangle h \qquad (i = 1, \ldots, l) \tag{5.141}$$

from (5.133), and from (5.140). Hence the requirement that $u_i \neq \vec{0}$ for some $i$ implies that $u_0 \neq \vec{0}$. Using (5.139) and (5.141), we see that $\lambda = a$ and $u = u_0$ is a solution of (5.2). Conversely, if $\lambda = a$ and $u$ is a solution of (5.2), then if $u_0 = u$ and if $u_i$ is given by (5.141), we can show that $(u_0, u_1, \ldots, u_l)$ is a non-trivial solution of (5.139)-(5.140) with $\lambda = a$.

If $f(a) = 0$, and if $\lambda = a$ in (5.139)-(5.140), then (5.139)-(5.140) may possibly have a non-trivial solution; however, this is of no consequence here.

We define the operators $A_i(\lambda)$, $B_i(\lambda)$, and $D_i(\lambda)$ for $i = 1, \ldots, l$ by the equations (the $B_i$ are subscripted for convenience):

$$A_i(\lambda) v = w_i(\lambda) \langle v, h \rangle \tag{5.142}$$

$$B_i(\lambda) v = g(\lambda) \langle v, h \rangle h \tag{5.143}$$

$$D_i(\lambda) v = \langle v, y_i(\bar{\lambda}) \rangle h, \tag{5.144}$$

where $v \in \mathscr{H}$. Let $\boldsymbol{H}(\lambda)$ be the $(l+1) \times (l+1)$ square matrix of operators given by

$$\boldsymbol{H}(\lambda) = \operatorname{diag}(H(\lambda), B_1(\lambda), \ldots, B_l(\lambda)) \tag{5.145}$$

and let $\boldsymbol{P}(\lambda)$ denote the $(l+1) \times (l+1)$ matrix of operators given by

$$\boldsymbol{P}(\lambda) = \begin{pmatrix} 0 & A_1(\lambda) & \cdots & A_l(\lambda) \\ D_1(\lambda) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ D_l(\lambda) & 0 & \cdots & 0 \end{pmatrix}. \tag{5.146}$$

Let

$$\boldsymbol{K}(\lambda) = \boldsymbol{H}(\lambda) + \boldsymbol{P}(\lambda) \tag{5.147}$$

and let

$$(u_0, u_1, \ldots, u_l)^T = \boldsymbol{u} \qquad (u_i \in \mathscr{H}) \tag{5.148}$$

where $\boldsymbol{u}$ is an element of the product space $\mathscr{H}^{[l+1]}$. Clearly (5.139)-(5.140) is equivalent to

$$\lambda \boldsymbol{K}(\lambda)\boldsymbol{u} = \boldsymbol{u}\,. \tag{5.149}$$

The simplification here is that $\boldsymbol{K}(\lambda)$, $\boldsymbol{P}(\lambda)$ and $\boldsymbol{H}(\lambda)$ are polynomial operators; the theory that we developed for $H(\lambda)$ is applicable to the operators $\boldsymbol{K}(\lambda)$ and $\boldsymbol{H}(\lambda)$, since

$$k_0(\boldsymbol{K}) = k_0(H) \tag{5.150}$$

$$= k_0(\boldsymbol{H}) \tag{5.151}$$

by the results of Corollary IV.3 and lemma III.3. We shall continue to denote the quantities in (5.150)-(5.151) by $k_0$ for simplicity.

If we did not assume that $x_j(0) = \vec{0}$, we would not have been able to transform (5.2) into a system of the form (5.149); the appearance of $\lambda$ as a coefficient in (5.149) is most convenient here, since it enables us to use the results of theorem IV.3 easily.

We recall that $\boldsymbol{K}(\lambda)$ is an operator-valued *polynomial* in $\lambda$, and that (5.150) holds. Let $\boldsymbol{I}$ denote the identity operator on $\mathscr{H}^{[l+1]}$. For each integer $k \geq k_0 - 1$, let $\mathscr{E}(\lambda; k)$ denote the entire function satisfying $\mathscr{E}(0; k) = 1$ and the equation obtained by replacing $\mathscr{D}(\lambda; k)$, $K(\lambda)$, and $I$ in $(5.96)_k$ by $\mathscr{E}(\lambda; k)$, $\boldsymbol{K}(\lambda)$ and $\boldsymbol{I}$ respectively. The function $\mathscr{E}(\lambda; k)$ exists by the results of theorem IV.3; its zeroes are independent of $k$ and coincide with the eigenvalues of (5.149). We seek to develop a relation between $\mathscr{D}(\lambda; k)$ and $\mathscr{E}(\lambda; k)$; using the analogue of (5.97), we could theoretically, at least, compute $\mathscr{E}(\lambda; k)$; however, the computation of $[\boldsymbol{I} - \lambda\boldsymbol{K}(\lambda)]^{-1}$ is simply too involved because of the presence of off-diagonal terms in the matrix $\boldsymbol{K}(\lambda)$. Instead, we will employ an appropriate analogue of (5.102) with the tilde removed.

Let $\boldsymbol{x}_i(\lambda)$ be the $(l+1)\times 1$ column vector given by

$$\boldsymbol{x}_i(\lambda) = (w_i(\lambda), \vec{0}, \ldots, \vec{0})^T \qquad (i = 1, \ldots, l) \tag{5.152}$$

$$\boldsymbol{x}_i(\lambda) = (\vec{0}, \ldots, \vec{0}, h, \vec{0}, \ldots, \vec{0})^T \qquad (i = l+1, \ldots, 2l) \tag{5.153}$$

where $h$ is the element in the $(i+1-l)$-th row of $\boldsymbol{x}_i(\lambda)$ in (5.153). Let $\boldsymbol{y}_i(\lambda)$ be the $(l+1)\times 1$ column vectors given by

$$\boldsymbol{y}_i(\lambda) = \boldsymbol{x}_{i+l}(\lambda) \qquad (i = 1, \ldots, l) \tag{5.154}$$

and

$$\boldsymbol{y}_i(\lambda) = (y_{i-l}(\lambda), \vec{0}, \ldots, \vec{0})^T \qquad (i = l+1, \ldots, 2l)\,. \tag{5.155}$$

Note that the zero vector (in $\mathscr{H}$) is the element in the first row of $\boldsymbol{x}_i(\lambda)$ and $\boldsymbol{y}_j(\lambda)$ for $l+1 \leqslant i \leqslant 2l$ and for $1 \leqslant j \leqslant l$.

We may write (see (5.146))

$$\boldsymbol{P}(\lambda)\boldsymbol{v} = \sum_{i=1}^{\tilde{l}} \boldsymbol{x}_i(\lambda)\langle \boldsymbol{v}, \boldsymbol{y}_i(\bar{\lambda})\rangle \tag{5.156}$$

where $\boldsymbol{v} \in \mathscr{H}^{[l+1]}$ and where

$$\tilde{l} = 2l\,. \tag{5.157}$$

A simple calculation shows that the $(l+1)\times(l+1)$ matrix $[\boldsymbol{I} - \lambda \boldsymbol{H}(\lambda)]^{-1}$ is given by

$$[\boldsymbol{I} - \lambda \boldsymbol{H}(\lambda)]^{-1} = \operatorname{diag}\left([I - \lambda H(\lambda)]^{-1}, I + \frac{\lambda}{f(\lambda)} B_1(\lambda), \dots, I + \frac{\lambda}{f(\lambda)} B_l(\lambda)\right). \tag{5.158}$$

We write

$$\boldsymbol{G}(\lambda) = [\boldsymbol{I} - \lambda \boldsymbol{H}(\lambda)]^{-1}\boldsymbol{P}(\lambda) \tag{5.159}$$

and we define

$$\boldsymbol{z}_i(\lambda) = [\boldsymbol{I} - \lambda \boldsymbol{H}(\lambda)]^{-1}\boldsymbol{x}_i(\lambda) \qquad (i = 1, \dots, \tilde{l})\,. \tag{5.160}$$

The Fredholm function $\boldsymbol{d}(\lambda)$ of $\boldsymbol{G}(\lambda)$ (see the statement of lemma V.4) is the determinant of the $\tilde{l}\times\tilde{l}$ matrix containing

$$\delta_{ij} - \lambda\langle \boldsymbol{z}_j(\lambda), \boldsymbol{y}_i(\bar{\lambda})\rangle \tag{5.161}$$

in its $i$-th row, $j$-th column. From (5.152), (5.153), (5.154), (5.155), (5.138), and (5.158), we have for $i, j = 1, \dots, l$:

$$\langle \boldsymbol{z}_j(\lambda), \boldsymbol{y}_i(\bar{\lambda})\rangle = 0 \tag{5.162}$$

$$\langle \boldsymbol{z}_{j+l}(\lambda), \boldsymbol{y}_{i+l}(\bar{\lambda})\rangle = 0 \tag{5.163}$$

$$\langle \boldsymbol{z}_j(\lambda), \boldsymbol{y}_{i+l}(\bar{\lambda})\rangle = \lambda^{-1}\langle z_j(\lambda), y_i(\bar{\lambda})\rangle \tag{5.164}$$

$$\langle \boldsymbol{z}_{j+l}(\lambda), \boldsymbol{y}_i(\bar{\lambda})\rangle = \delta_{ij}/f(\lambda)\,. \tag{5.165}$$

We write $a_{ij}$ instead of $\langle z_j(\lambda), y_i(\bar{\lambda})\rangle$ for brevity. The function $\boldsymbol{d}(\lambda)$ is thus equal to

$$\boldsymbol{d}(\lambda) = \det\begin{pmatrix} \mathscr{I}_l & -\dfrac{\lambda}{f(\lambda)}\mathscr{I}_l \\ -A & \mathscr{I}_l \end{pmatrix} \tag{5.166}$$

where $\mathscr{I}_l$ is the $l\times l$ identity matrix over the set of complex numbers, and $A$ is the $l\times l$ matrix of elements $a_{ij}$. By elementary operations on determinants, it can be seen that

$$d(\lambda) = \det \begin{pmatrix} \frac{f(\lambda)}{\lambda}\mathscr{I}_l & -\mathscr{I}_l \\ -\frac{\lambda}{f(\lambda)}A & \frac{\lambda}{f(\lambda)}\mathscr{I}_l \end{pmatrix} \tag{5.167}$$

$$= \det \begin{pmatrix} \frac{f(\lambda)}{\lambda}\mathscr{I}_l & -\mathscr{I}_l \\ \mathscr{I}_l - \frac{\lambda}{f(\lambda)}A & \Theta_l \end{pmatrix} \tag{5.168}$$

$$= (-1)^l \det \begin{pmatrix} \Theta_l & \mathscr{I}_l \\ \mathscr{I}_l - \frac{\lambda}{f(\lambda)}A & \Theta_l \end{pmatrix} \tag{5.169}$$

where $\Theta_l$ is the zero matrix of size $l\times l$. By writing out the last determinant in (5.169), and repeatedly expanding along elements of the last column, we obtain

$$d(\lambda) = \det\left[\mathscr{I}_l - \frac{\lambda}{f(\lambda)}A\right] \tag{5.170}$$

$$= d(\lambda)\,. \tag{5.171}$$

The next step is to compute a relation between $\varDelta(\lambda;k)$ and $\boldsymbol{\Delta}(\lambda;k)$ for integers $k\geqslant k_0-1$; according to the conventions observed, the latter function satisfies the analogue of (4.97) with $\varDelta(\lambda;k)$, $H(\lambda)$ and $I$ replaced by $\boldsymbol{\Delta}(\lambda;k)$, $\boldsymbol{H}(\lambda)$, and $\boldsymbol{I}$ respectively, and is assumed to have value of one at $\lambda=0$. The existence of $\boldsymbol{\Delta}(\lambda;k)$, of course, follows from the fact that $\boldsymbol{H}(\lambda)$ is an operator-valued polynomial, from (5.151), and from theorem IV.3. A simple computation shows that

$$[\lambda\boldsymbol{H}(\lambda)]'[\boldsymbol{I}-\lambda\boldsymbol{H}(\lambda)]^{-1} = \operatorname{diag}\left([\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1},\quad -\frac{f'(\lambda)}{f(\lambda)}E_1, \ldots, -\frac{f'(\lambda)}{f(\lambda)}E_l\right) \tag{5.172}$$

where

$$E_i v = \langle v, h\rangle h\,, \quad v\in\mathscr{H},\ i=1,\ldots,l\,. \tag{5.173}$$

The relation (5.136) is useful here. Extending $\{h\}$ to an orthonormal basis for $\mathscr{H}$, and using the definition of trace, we obtain

$$\tau_k\left[\frac{-f'}{f}E_i\right] = \boldsymbol{m}_k\left[\frac{-f'}{f}\right] \qquad k=0,1,\ldots;\ i=1,\ldots,l\,. \tag{5.174}$$

Hence by (5.172), (5.174), and (3.37), we obtain the relation

$$\tau_k\{[\lambda\boldsymbol{H}(\lambda)]'[\boldsymbol{I}-\lambda\boldsymbol{H}(\lambda)]^{-1}\} = \tau_k\{[\lambda H(\lambda)]'[I-\lambda H(\lambda)]^{-1}\} - l\boldsymbol{m}_k\left\{\frac{f'(\lambda)}{f(\lambda)}\right\} \tag{5.175}$$

valid for integers $k \geqslant k_0 - 1$. Using (5.175), (4.98), and the appropriate analogue of (4.98), we have

$$\frac{\boldsymbol{\Delta}'(\lambda; k)}{\boldsymbol{\Delta}(\lambda; k)} = \frac{\Delta'(\lambda; k)}{\Delta(\lambda; k)} + \frac{lf'(\lambda)}{f(\lambda)} - l\sum_{j=0}^{k-1} \lambda^j \boldsymbol{m}_j \left\{\frac{f'}{f}\right\}. \tag{5.176}$$

Hence

$$\boldsymbol{\Delta}(\lambda; k) = \Delta(\lambda; k)[f(\lambda)]^l \exp\left\{-l\sum_{j=0}^{k-1} \frac{\lambda^{j+1}}{j+1} \boldsymbol{m}_j \left[\frac{f'}{f}\right]\right\} \tag{5.177}$$

which is valid for each complex $\lambda$ and for each integer $k \geqslant k_0 - 1$. Let $\boldsymbol{p}(\lambda; k)$ be the polynomial obtained by replacing $G$ and $I$ in (5.99) by $\boldsymbol{G}$ and $\boldsymbol{I}$ respectively. From (5.170)-(5.171), from (5.102) (without the tilde), from the analogue of (5.102) obtained by replacing $\tilde{\mathscr{D}}$, $p$, $\Delta$, and $d$ by $\boldsymbol{\mathscr{E}}$, $\boldsymbol{p}$, $\boldsymbol{\Delta}$, and $\boldsymbol{d}$, and from (5.177), we can state the relation

$$\boldsymbol{\mathscr{E}}(\lambda; k) = \mathscr{D}(\lambda; k)[f(\lambda)]^l \exp q(\lambda; k) \tag{5.178}$$

where

$$q(\lambda; k) = \exp \boldsymbol{p}(\lambda; k) - p(\lambda; k) - l\sum_{j=0}^{k-1} \frac{\lambda^{j+1}}{j+1} \boldsymbol{m}_j \left\{\frac{f'}{f}\right\}\Bigg] . \tag{5.179}$$

The relation (5.178) is valid for each integer $k \geqslant k_0 - 1$ and each $\lambda$ which is not a pole of $\mathscr{D}(\lambda; k)$. It is *important* to note here that $q(\lambda; k)$ is a polynomial of degree $k$ or less.

Let $\{\boldsymbol{\nu}_i\}$ and $\{r_i\}$ be the ordered sequences of zeroes of $\boldsymbol{\mathscr{E}}(\lambda; k)$ and $f(\lambda)$ respectively, where each zero is taken according to algebraic multiplicity. Then since $\boldsymbol{K}(\lambda)$ is an operator-valued polynomial, and since (5.150) holds, we may apply the results of theorem IV.3 to conclude that

$$\sum_i |\boldsymbol{\nu}_i|^{-k_0} < +\infty \tag{5.180}$$

and that

$$\sum_i \boldsymbol{\nu}_i^{-(k+1)} = \tau_k\{[\lambda \boldsymbol{K}(\lambda)]'[\boldsymbol{I} - \lambda \boldsymbol{K}(\lambda)]^{-1}\} \tag{5.181}$$

provided $k$ is an integer such that $k \geqslant k_0 - 1$ in (5.181). Let $\{\nu_i\}$ and $\{\varrho_i\}$ be the ordered sequences defined in assertion (v) of theorem V.6. By (5.178), we see that $\{\nu_i\}$ is a subsequence of the sequence $\{\boldsymbol{\nu}_i\}$. Hence, by (5.180), the relation (5.117) with $k_2 = k_0$ holds for $\varepsilon = 0$. The sequences $\{\varrho_i\}$ and $\{r_i\}$ are finite sequences, since $f(\lambda)$ is a polynomial. Hence by (5.178), we have by arguments of an algebraic sort:

$$\sum_i \boldsymbol{\nu}_i^{-(k+1)} = \sum_i \nu_i^{-(k+1)} - \sum_i \varrho_i^{-(k+1)} + l\sum_i r_i^{-(k+1)} \tag{5.182}$$

valid for integers $k \geqslant k_0 - 1$. If $k_1$ is the smallest integer such that $k_1 \geqslant k_0 - 1$ then we have from (5.178)

$$\boldsymbol{m}_k\left[\frac{\mathscr{E}'(\lambda;k_1)}{\mathscr{E}(\lambda;k_1)}\right] = \boldsymbol{m}_k\left[\frac{\mathscr{D}'(\lambda;k_1)}{\mathscr{D}(\lambda;k_1)}\right] + l\boldsymbol{m}_k\left[\frac{f'}{f}\right] + \boldsymbol{m}_k[q'(\lambda;k_1)] \tag{5.183}$$

which is valid for integers $k \geqslant k_0 - 1$, that is for integers $k \geqslant k_1$. But the degree of the polynomial $q'(\lambda; k_1)$ is smaller than $k_1$, so

$$\boldsymbol{m}_k[q'(\lambda;k_1)] = 0 \tag{5.184}$$

if $k \geqslant k_0 - 1$. From the analogue of (5.97) with $k = k_1$ and with $\boldsymbol{\mathscr{D}}$, $\boldsymbol{K}$, and $\boldsymbol{I}$ replacing $\mathscr{D}$, $K$, and $I$ respectively, and from (5.181), we conclude that if $k \geqslant k_0 - 1$, then

$$\sum_i \boldsymbol{\nu}_i^{-(k+1)} = -\boldsymbol{m}_k\left[\frac{\boldsymbol{\mathscr{E}}'(\lambda;k_1)}{\boldsymbol{\mathscr{E}}(\lambda;k_1)}\right]. \tag{5.185}$$

Since $f(\lambda)$ is a polynomial, we have

$$\sum_i r_i^{-(k+1)} = -\boldsymbol{m}_k\left[\frac{f'}{f}\right] \tag{5.186}$$

for each non-negative integer $k$. Hence by (5.182)-(5.186), and by (5.97) with $k = k_1$, we have

$$\sum_i \nu_i^{-(k+1)} - \sum_i \varrho_i^{-(k+1)} = -\boldsymbol{m}_k\left[\frac{\mathscr{D}'(\lambda;k_1)}{\mathscr{D}(\lambda;k_1)}\right] \tag{5.187}$$

$$= \tau_k\{[\lambda K(\lambda)]'[I - K(\lambda)]^{-1}\} \tag{5.188}$$

valid for integers $k \geqslant k_0 - 1$. ■

**VI.** – We wish to evaluate the quantities appearing on the right sides of equations (4.102) and (5.119). We shall do this for the operator $K(\lambda)$ by means of a recursion formula. Let

$$L(\lambda) = [\lambda K(\lambda)]'[I - \lambda K(\lambda)]^{-1}. \tag{6.1}$$

We have from (6.1)

$$L(\lambda) = \lambda L(\lambda)K(\lambda) + [\lambda K(\lambda)]'. \tag{6.2}$$

Let us assume that

$$K(\lambda) = \sum_{j=0}^{\infty} \lambda^j K_j \tag{6.3}$$

and that

$$L(\lambda) = \sum_{j=0}^{\infty} \lambda^j L_j \tag{6.4}$$

where convergence of (6.3) and (6.4) is with respect to (say) $|\ |_\infty$. From (6.2)-(6.4), and from (3.19), we obtain

$$L_k = (k+1)K_k + \sum_{i=0}^{k-1} L_i K_{k-1-i}\,. \tag{6.5}$$

Hence we have

$$L_0 = K_0 \tag{6.6}$$

$$L_1 = 2K_1 + K_0^2 \tag{6.7}$$

$$L_2 = 3K_2 + (K_0K_1 + 2K_1K_0) + K_0^3 \tag{6.8}$$

$$L_3 = 4K_3 + (K_0K_2 + 3K_2K_0) + 2K_1^2 + (K_0^2K_1 + K_0K_1K_0 + 2K_1K_0^2) + K_0^4\,. \tag{6.9}$$

We write

$$\sum_i \nu_i^{-1} - \sum_i \varrho_i^{-1} = \tau\{K_0\} \tag{6.10}$$

$$\sum_i \nu_i^{-2} - \sum_i \varrho_i^{-2} = \tau\{2K_1 + K_0^2\} \tag{6.11}$$

$$\sum_i \nu_i^{-3} - \sum_i \varrho_i^{-3} = \tau\{3K_2 + 3K_1K_0 + K_0^3\} \tag{6.12}$$

$$\sum_i \nu_i^{-4} - \sum_i \varrho_i^{-4} = \tau\{4K_3 + 4K_2K_0 + 2K_1^2 + 4K_1K_0^2 + K_0^4\}\,. \tag{6.13}$$

Equation (6.10) is valid if $k_0 < 1$ and $\varkappa < 1$ (see (5.115)). Equation (6.11) is valid if $k_0 < 2$ and if $\varkappa < 2$, etc.

If $K(\lambda)$ satisfies the conditions of theorem V.8 (here the functions $x_j(\lambda)$, $y_j(\lambda)$, and $f(\lambda)$ are polynomials), then equations (6.10) is valid if $k_0 \leqslant 1$, equation (6.11) is valid if $k_0 \leqslant 2$, etc.

In order to obtain (say) equation (6.12) from (6.8), we would have to show that if $k_0 \leqslant 3$, then

$$\tau\{K_0K_1\} = \tau\{K_1K_0\}\,. \tag{6.14}$$

We will prove (say) equation (6.14) in more general form. Each term in the expression for $L_k$ is of the form

$$K_{i_1}^{j_1} \dots K_{i_q}^{j_q} \tag{6.15}$$

where $j_1, \dots, j_q$ denote powers. Here $q$ is some integer such that $q \leqslant k+1$; also, we have $0 \leqslant i_r \leqslant k$ and $1 \leqslant j_r \leqslant k+1$ for each integer $r$ in $1 \leqslant r \leqslant q$. Furthermore, we have that

$$\sum_{r=1}^{q}(i_r+1)j_r=k+1 \tag{6.16}$$

as the reader will note by examining (6.6) through (6.9). These facts may be proved by induction if one uses (6.5). Equation (6.16) states that the sum on the left side of (6.16) is invariant for each term (6.15) which appears in the expression for $L_k$.

If $i_r>s$ for some $r$ (see (4.1)), then (see (4.1) and (5.1)) $K_{i_r}$ is precisely the coefficient of $\lambda^{i_r}$ in the Maclaurin expansion of $P(\lambda)$ in (5.3); since $P(\lambda)$ is analytic (with respect to $|\ |_1$ about $\lambda=0$, we have that $K_{i_r}\in C\{1;\mathscr{H}\}$ if $i_r>s$, so that (2.40) yields the fact that the trace of (6.15) is equal to the trace of any permutation of (6.15).

If $i_r\leqslant s$ for each $r$ in $1\leqslant r\leqslant q$, then (6.15) is in $C(p)$, where

$$\frac{1}{p}=\sum_{r=1}^{q}\frac{j_r}{\alpha_{i_r}}. \tag{6.17}$$

The condition $k\geqslant k_0-1$ implies that (see 4.16)

$$(i_r+1)\alpha_{i_r}\leqslant k+1\,. \tag{6.18}$$

From (6.16)-(6.18), we obtain

$$\frac{1}{p}\geqslant\sum_{r=0}^{q}\frac{j_r(i_r+1)}{k+1}=1 \tag{6.19}$$

provided $k\geqslant k_0-1$. Hence if $k\geqslant k_0-1$, the trace of (6.15) is equal to the trace of any of its permutations by (6.19) and (2.40). From these results, incidentally, we can see that $[\lambda K(\lambda)]'$ and $[I-\lambda K(\lambda)]^{-1}$ may be commuted in (5.119), provided the condition $k\geqslant k_0-1$ is satisfied.

If $K(\lambda)=H(\lambda)$, then we observe that equation (4.2) has an eigenvalue if and only if $\tau(L_k)\neq 0$ for some integer $k\geqslant k_0-1$. This follows directly from (4.98) and (4.102). If $f(\lambda)$ has only positive zeroes, and if $\tau(L_k)>0$ for some integer $k>k_2-1$ (see (5.116)) (or some integer $k\geqslant k_0-1$ if the assumptions of theorem V.8 are satisfied), then we can guarantee that (5.2) has at least one eigenvalue.

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