Properties of Earthquakes Statistics (*).

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A DARIO GRAFFI per il suo 70° compleanno

Summary. – *The number of earthquakes $\Delta M n(M)$ with magnitude in the range between M and $M + \Delta M$ satisfy the empirical law $\log n(M) = a - bM$ where a and b are parameters which depend on the seismic region. This paper introduces the first mechanical model to explain the empirical law and introduces a physical meaning to the parameters a and b.*

1. – Introduction.

We should first explain why an article on a theoretical seismological problem appears among others of mathematical physics since not all scientist know that theoretical seismology and the theory of elasticity have always been very close. Seismology found in the theory of elasticity the natural environment for the development of its theories and some of its problems found immediate solutions by means of the use of the results of the theory of elasticity; a classical example of this is the determination of the geographical coordinates of the epicenters of earthquakes, based on the different velocity of propagation of shear and compression waves discovered by POISSON in 1830. In other cases theoretical seismology has only found hints in the results of the theory of elasticity to develop new theories based on these results: this was the case of surface waves discovered by RAYLEIGH in 1887 and studied by seismologists who determined their dispersion properties and used them for the study of elastic properties of the crust and the mantle of the earth, for the determination of magnitude and for discriminating between underground nuclear explosions and earthquakes.

In some cases, it was left to seismology to set and solve important problems of the theory of elasticity, this was the case in determining the elastic properties of layered spheres from the travel time of elastic waves between different points of the sphere (HERGLOTZ, 1918) or in the case of determining the eigen frequencies of a layered sphere or of the non uniqueness of the inversion of this problem (e.g. ALTERMAN *et al.*, 1959; BACKUS G. and J. F. GILBERT, 1968; CAPUTO M., 1963).

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In this note, I shall treat tentatively the important problem of the distribution function of the faults which caused earthquakes as a function of the area of the faults. The solution of this problem is of primary importance for the knowledge of the earth's crust and upper mantle and for the study of mechanisms which cause earthquakes. The models considered here are extremely elementary and limited to earthquakes caused by compressive tectonic forces.

The brief discussion of the models presented in this note should be considered only as the beginning of long research that the laboratory research and the observation of natural phenomena, suggest to begin since long. In other words, we should begin a systematic and scientific treatment of the numerous data on earthquakes as it was done for the perfect gas, for the atom and for other physical entities. The problem can be set as follows: let l be the linear dimension of faults of area S, also let

$$W = W(l, x_i) \tag{1}$$

$$\bar{E} = \bar{E}(l, y_i, W) = E(l, x_i, y_i) \tag{2}$$

be the energy stored in the elastic medium before the earthquake and that radiated as elastic energy, where x_i and y_i are parameters which characterize the two energies. Let also $n(l)$ be the number of faults of linear dimension in the range between l and $l + dl$ which caused earthquakes. The total area A of the faults of the earthquakes occurred in a given time interval T_0 sufficiently large and in a given seismic region and the total elastic energy E_0 are

$$A = \int_{l_1}^{l_2} n(l)\, l^2\, dl \tag{3}$$

$$E_0 = \int_{l_1}^{l_2} n(l) E(l, x_i, y_i)\, dl \tag{4}$$

where l_1 and l_2 are the minimum and the maximum linear dimensions of the faults which caused the earthquakes. It is also assumed that the parameters x_i and y_i are constant in the considered time interval and seismic region.

The energy E_0 can be observed on the surface of the earth, whereas the functional A cannot be observed and should be discussed taking into account the laws of physics.

2. – Unidimensional models.

In statistical seismology a very simple empiric formula is used which gave good results approximating the number of earthquakes which occurred in a given region and in a given time interval. The formula expressed as function of the magnitude M is (M. ISHIMOTO, K. IIDA, 1939)

$$\log_{10} n(M) \Delta M = a_i - b_i M + \log_{10} \Delta M \tag{6}$$

where ΔM is a magnitude range and Δn is the number of earthquakes of magnitude in the range between M and $M + \Delta M$ which occurred in a given area and in a given time interval: a_i and b_i are parameters which represent the given seismic region in the given time interval.

It is known that there are various definitions of magnitude, but they are related linearly, therefore it is irrelevant which definition we use in formula (6). The data quoted in this note are referred to the definition based on surface waves. In this case, if the time interval is one year and the unit area is one km² we have for the regions of Europe, North Africa and the Middle East (KARNIK, 1971)

$$0 < a_i < 7 , \quad + 0.5 < b_i < + 1.5$$

per 10^6 km²/year.

Formula (6) has been used by all seismologists, however, to my knowledge, there is no mechanical model to back it.

In this note, we shall briefly discuss a physical model of statistic distribution of tectonic earthquakes and we shall compare the results with (6) obtaining some interesting results.

At this point, we should make clear that it has been understood that tectonic earthquakes, which are the most numerous, believed to be caused relaxation of elastic energy, accumulated in the interior of the earth, by means of slips which occur along fractures inside the earth itself. The slips occur on planes which are called faults and have areas which could be tens of km².

The statistical model considered here is based on the mechanism of tectonic earthquakes and on their energy which is given by the formula (KEILIS BOROK, 1959)

$$W = \frac{\bar{K} p^2 S^{3/2}}{\mu} \tag{7}$$

where $\bar{K}$ is a coefficient of form which is near to unity, μ is the rigidity of rocks, p the relaxation of the elastic stress and S is the area of the slips which caused the earthquake. The associated radiated elastic energy E is

$$E = \bar{\eta} W \tag{8}$$

where $\bar{\eta}$ is a parameter which is estimated to lie between $\frac{1}{3}$ and 1. Let W_0 and E_0 be the total energy and the elastic energy of the earthquakes of a given region in a given time interval, then it must be

$$\int_{l_1}^{l_2} n(l) \frac{\bar{\eta} \bar{K} p^2}{\mu} l^3 \, dl = E_0 \, , \quad \int_{l_1}^{l_2} n(l) \, l^2 \, dl = A \, , \tag{9}$$

Let us set

$$l^3 n(l) = \bar{\varphi}(l) \tag{10}$$

then

$$\int_{l_1}^{l_2} \frac{\bar{\varphi}(l)}{l}\, dl = A\,, \tag{11}$$

$$\int_{l_1}^{l_2} \bar{\varphi}(l)\, \frac{\bar{\eta}\bar{K}p^2}{\mu}\, dl = E_0 \tag{12}$$

the problem is to determine $\bar{\varphi}(l)$ which has an area $E_0\mu/\bar{\eta}\bar{K}p^2$ between l_1 and l_2; or to determine the functional A under the condition E_0.

To solve the problem it is necessary to find the extremals of A. For this purpose, we find that the Euler's equation associated with (11) and (12) is degenerate. However a direct analysis shows that the extremals are

$$\bar{\varphi}(l) = \frac{E_0\mu}{\bar{\eta}p^2\bar{K}}\, \delta(l-l_1) \tag{13}$$

for the maximum, and

$$\bar{\varphi}(l) = \frac{E_0\mu}{\bar{\eta}p^2\bar{K}}\, \delta(l-l_2) \tag{14}$$

for the minimum. From these one obtains

$$n(l) = \frac{E_0\mu}{\bar{\eta}\bar{K}p^2}\, \frac{\delta(l-l_1)}{l_1^3} \tag{15}$$

for the maximum and

$$n(l) = \frac{E_0\mu}{\bar{\eta}\bar{K}p^2}\, \frac{\delta(l-l_2)}{l_2^3} \tag{16}$$

for the minimum.

Both solutions are far from the experimental results and therefore are unacceptable. They prove that the mechanism of energy release does not extremize the functional A which represents the total area of the faults associated with the earthquakes under the hypothesis of constant p.

3. – Experimental checks.

The empiric formula for E and n as functions of M are

$$E = 10^{12+1.44M} = 10^{\alpha+\gamma M} \tag{17}$$

$$n = \bar{a}10^{-bM} \tag{18}$$

from which follows

$$n = \bar{\bar{a}}\, 10^{(b\alpha)/\gamma}\, l^{-(3b)/\gamma} \left(\frac{\bar{\eta} \bar{\bar{K}} p^2}{\mu}\right)^{-b/\gamma} \tag{19}$$

and substituting (9) we obtain

$$\Delta A \simeq l^{2-(3b)/\gamma} \tag{20}$$

We can see then that for $b < 0.96$, ΔA is increasing function of l, for $b = 0.96$, ΔA is constant, and for $b = 0.96$, ΔA is decreasing function of l.

These results which are valid with the hypothesis of constant p are also valid if is p independent of l and has a uniform distribution for all the values of l.

We may now discuss the observed values and the theoretical results. According to formula (15) and (16) one should expect that the elastic energy stored in rocks be released preferably by means of faults with largest area, because in this way the ratio of the work done in the slips which is proportional to the area of the slip S, to the energy released, which is proportional to $S^{3/2}$, is minimal.

The observations instead give (19), which represents a different distribution. Moreover, laboratory experiments on rock samples subjected to compression, have shown that the elastic energy stored is released by fractures whose area follows a law similar to that of earthquakes (MOGI, 1962).

Therefore, there seems to be an apparent contrast between the results of the theory of this paper and the experimental and observed data. But this contrast can be eliminated observing that in the beginning, when tectonic forces begin to act on the intact earth crust, probably the faults have all the largest dimension; the intersections of these faults create a system of minor faults and finally one obtains a fault system in which the number of faults is a decreasing function of the area. At the final stage, earthquakes occur mostly by means of activation of preexisting faults.

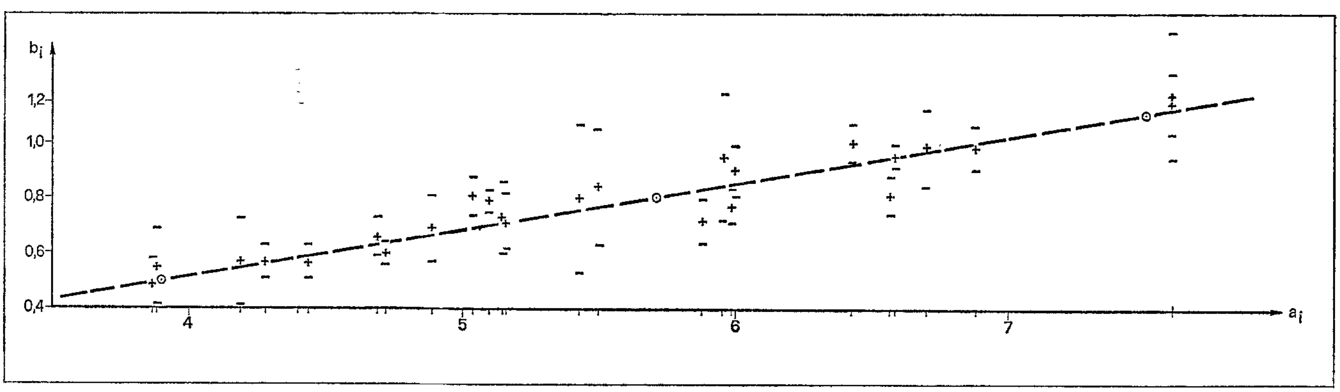

Fig. 1.

The analysis of the value of b for Europe, North Africa and the Middle East, shows that $0.50 < b_i < 1.17$; the average value is 0.94 relatively close to the value of the equipartition of area which is 0.96.

The figure shows the distribution of the a_i, b_i values and one may see that they are linearly related, in the most active areas the value of b_i tends to be larger.

Concerning the laboratory experiments, one should keep in mind that rocks are formed by aggregates of crystals which are joined together along their faces. These are the weak points where fractures occur and which condition the formation of other fractures.

4. – Bidimensional models.

The statistical distribution of earthquakes can be studied also from a geometric and mechanical point of view with an elementary model which gives results very close to reality. In this model earthquakes are represented with a system of faults of area S_i ond direction of unit vectors l_{i1}, l_{i2}, l_{i3} in a solid (the earth crust) subject to uniform force $\bar{f}$ per unit area of direction d_1, d_2, d_3 and intensity linearly increasing with time

$$\bar{f} = \eta\, \bar{d}t = \eta t(d_1 \bar{i}_1 + d_2 \bar{i}_2 + d_3 \bar{i}_3)\,. \tag{21}$$

Given the friction parameter f_a between the faces of the faults, each fault will release energy when the component of the force parallel to the of the fault

$$S_i \eta t\{1 - (d_1 l_{i1} + d_2 l_{i2} + d_3 l_{i3})^2\}^{\frac{1}{2}} \tag{22}$$

will be larger than the friction between the two faces

$$FS_i + S_i f_a \eta t(l_{i1} d_1 + l_{i2} d_2 + l_{i3} d_3) \tag{23}$$

where F is a cohesive constant which depends on the contact between the faces of the faults. Assuming

$$f_a = f_a\{(l_{i1} d_1 + l_{i2} d_2 + l_{i3} d_3)t, t\} = f_a\{t \cos\alpha, t\} \tag{24}$$

this will occur when

$$\eta t\{1 - \cos^2\alpha\}^{\frac{1}{2}} = \eta t \cos\alpha\, f_a\{t \cos\alpha, t\} + F\,. \tag{25}$$

In this mechanism the elastic energy released by a fault of area S in a given time interval T_0, depends on T_0, on F and on the angle between the unit vectors l_{i1}, l_{i2}, l_{i3}, and d_1, d_2, d_3; the energy should be released periodically, and always in the same amount, with a period given by equation (25).

Given the vector d_1, d_2, d_3, the coefficient η of (21), F, and the friction mechanism, the value of T and the energy released by a fault of direction l_{i1}, l_{i2}, l_{i3}, will be inversely proportional to the value of $l_{i1}d_1 + l_{i2}d_2 + l_{i3}d_3$.

To explain this with other words, let us consider two faults with the same area and with directions given by the unit vectors l_{11}, l_{12}, l_{13} and l_{21}, l_{22}, l_{23} such that

$$l_{11}d_1 + l_{12}d_2 + l_{13}d_3 = \omega(l_{21}d_1 + l_{22}d_2 + l_{33}d_3) \tag{26}$$

then the number of earthquakes released by the two faults will be about in the ratio ω. If the elastic energy is supplied to the earth crust continuously the energy will be released periodically and with quanta, but the total elastic energy released by the two faults will be nearly the same. In a system of N faults of different areas and directions one should find that the energy is released with N frequencies which are generally different.

In this model, the swarms of earthquakes are represented by systems of faults whose directions are very close; also the energy released by a system of faults with the same area should depend only on $\eta(d_1\bar{i}_1 + d_2\bar{i}_2 + d_3\bar{i}_3)$.

In the case of a system of faults with the some area, if $\bar{l}_{11}$, $\bar{l}_{12}$, $\bar{l}_{13}$ is the unit vector of the direction of the fault of area S which releases the maximum energy E_{sm}, assuming $\bar{l}_{11}d_1 + \bar{l}_{12}d_2 + \bar{l}_{13}d_3 = 1$ for any other fault it will be

$$E_s = E_{sm}(l_{11}d_1 + l_{12}d_2 + l_{13}d_3)$$

The distribution of earthquakes in a given time interval as function of the energy should be given by the distribution of the cosinus of the angles between the directions of the faults and the direction of the acting tectonic force. If one introduces the return period T of the earthquakes of energy E_s, in a given time interval T_0, the number of earthquakes with energy E_s will be $n(E_s)$

$$n(E_s) = E_{sm}(l_{11}d_1 + l_{12}d_2 + l_{13}d_3)\frac{T_0}{T} \tag{27}$$

where T is given by (25).

As an example let it be

$$\cos\alpha = l_{11}d_1 + l_{12}d_2 + l_{13}d_3\,, \qquad f_\alpha = f(\alpha) \tag{28}$$

then substituting into equation (25) one obtains

$$\eta t \sin\alpha = f(\alpha)\eta t\cos\alpha + F \tag{29}$$

from which the return period and the released energy result

$$t = \frac{F}{\eta(\sin\alpha - f(\alpha)\cos\alpha)}\,, \tag{30}$$

$$E = \frac{\bar{\eta}\bar{K}S^{3/2}\eta^2 t^2}{\mu}\,. \tag{31}$$

Where $\bar{n}$, $\bar{K}$, μ are know with some approximation η and F can be estimated as follows.

A condition could be that the minimum return period T_m the given area correspons to $\alpha = \pi/2$, then

$$T_m = \frac{F}{\eta}\,, \qquad E = \frac{\bar{\eta}\bar{K}l^3F^2}{\mu} \tag{32}$$

in agreement with the experimental results of OHNAKA (1973). Also the number of eartquakes $\bar{N}$ of the given area and time interval T_0 is

$$\bar{N} = \int_{l_1}^{l_2} \frac{T_0\,d\alpha}{t}\,. \tag{33}$$

Equations (32) and (33) could determine F and η and then the return period of any given system of faults.

If we assume that the faults of the system have the same area, the stress drop, wich is p assumed equal to the energy accumulated, is proportional to the time of accumulation and given by (31), then the number of earthquakes of a given energy in a given time interval T_0 is

$$n = \frac{T_0}{t} = \frac{T_0\eta}{F}\,(\sin\alpha - f(\alpha)\cos\alpha)\,, \tag{34}$$

$$E = \frac{\bar{\eta}\bar{K}l^3\eta^2t^2}{\mu} = \frac{\bar{\eta}\bar{K}l^3F^2}{\mu}\,(\sin\alpha - f(\alpha)\cos\alpha)^2\,.$$

If we consider then that l is variable in nature, then the number of earthquakes of a given magnitude M and therefore given energy E in assigned area and time interval T_0, are obtained from (34). For this purpose one should consider the polar coordinate system α, l in a plane and the region R defined by $\alpha_1 < \alpha < \pi/2$, $0 < l < l_2$ where α_1 is the largest zero of t given by (31), in the range 0, $\pi/2$.

To each point of this region, through (34) corresponds a given energy and a given return period; we should therefore consider the line R_E of R for which E is constant and integrate $n(\alpha)$ over that line.

This can be easily obtained by expressing α as a function of l by means of the second of (34), substituting in the first of (34) and then integrating on the path given by the second of (34)

$$n(E) = T_0\eta\left(\frac{\bar{\eta}\bar{K}}{\mu E}\right)^{\frac{1}{2}} D\int l^{\frac{3}{2}-\nu}\left[1+\left(\frac{d\alpha}{dl}\right)^2\right]^{\frac{1}{2}}dl \tag{35}$$

$$= T_0D\left(\frac{\bar{\eta}\bar{K}\eta^2}{E\mu}\right)^{\nu/3}\int t^{\frac{2}{3}\nu-1}\left[1+\left(\frac{dl}{d\alpha}\right)^2\right]^{\frac{1}{2}}d\alpha\,.$$

The factor $Dl^{-\nu}$ has been introduced to take into account the number of faults of linear dimension l which are present in the system.

The interval of integration is 0, l_2 if $E > E_0 = \bar{\eta}\bar{K}l_2^3F^2/\mu$ or 0, $[\bar{\eta}\bar{K}F^2/\mu]^{\frac{1}{3}}$ if $E < E_0$ for the integral in dl.

For the integral in $d\alpha$, instead, the range of integration is (α_1, α_2); where $\alpha_2 = \pi/2$ when $E < E_0$ and α_2 is given by the second of (34) with $l = l_2$ when $E > E_0$.

It is not difficult to verify that for $f(\alpha) = \text{const}$ and E sufficiently large we obtain $n(E)\,E \simeq E^{\frac{1}{3}}$; in agreement with the experimental values.

REFERENCES

BATH M., *Introduction to seismology*, Birkhauser-Verlag, Basel, 1973.

KARNIK, V., *Seismicity of the European area*, 2 parts, Reidel Publ. Co., Dordrecht, Holland, 1971.

KEILIS-BOROK, V. I., *On estimation of the displacement in an earthquake source and source dimensions*, Annali di Geofisica, **12**, 2 (1959).

ISHIMOTO, M. - IIDA, K., Bull. Earth. Res. Inst., **17** (1939), pp. 443-478.

MOGI KIYOO, *Study of elastic shocks caused by the fractures of hetereogeneous materials and its relations to earthquakes phenomena*, Bull. Earth. Res. Inst., **40** (1962), pp. 125-173.

OHNAKA, M., *Experimental studies of stick-slip and their applications to earthquake source mechanism*, J. Phys. Earth., **21** (1973), pp. 285-303.

CAPUTO, M. - KEILIS-BOROK, V. - KRONROD, T. - MOLCHAN, G. - PANZA, G. F. - PIVA, A. - PODGAEZKAYA, V. - POSTPISCHL, D., *Models of earthquake occurrence and isoseismals in Ttaly*, Annali di Geofisica, **26**, n. 2-3 (1973).

CAPUTO, M. - POSTPISCHL, D., *Contour mapping of seismic areas by numerical filtering and geological implications*, Annali di Geofisica, **27**, n. 3-4 (1974).

ALTERMAN, Z. - JAROSH, H. - PEKERIS, C. L., *Oscillation of the earth*, Proc. Roy. Soc., A252 (1959), pp. 80-95.

BACKUS, G. - GILBERT, J. F., Geophys. J.R.A.S., **16** (1968), p. 169.

CAPUTO, M., *Free modes of layered oblate planets*, Journ. Geoph. Res., **68** (1963), pp. 479-503.