On Stability in the Classical Linear Theory of Fluid Mixures.

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Summary. – *In this paper we study the stability* (1) *of linear inviscid fluid mixtures. In particular, we show that the* statical *stability criterion of Gibbs is both necessary and sufficient for the* dynamical *stability of the mixture* (2), *using, as our main hypotheses, only those inequalities and symmetries which are consequences of the second law of thermodynamics* (3).

1. – Notation.

$\mathcal{E}$ will always designate a three-dimensional euclidean point space with $\mathcal{V}$ the corresponding vector space; $\boldsymbol{a}\cdot\boldsymbol{b}$ is the inner product of $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{V}$. We write ∇, div, and Δ, respectively, for the gradient, divergence, and laplacian operators in $\mathcal{E}$.

In matrix operations involving elements $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N) \in \mathbb{R}^N$ the vector $\boldsymbol{\mu}$ will always be identified with a *column vector*, and a similar assertion applies to elements $\mathbf{a} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N) \in \mathcal{V}^N$. In particular, if $\mathbf{a}, \mathbf{b} \in \mathcal{V}^N$, and if $\boldsymbol{M}$ is an $N \times N$ matrix then

$$\mathbf{a} = \boldsymbol{M}\mathbf{b}$$

is equivalent to the following system of *vector* equations:

$$\boldsymbol{a}_i = \sum_{j=1}^{N} \boldsymbol{M}_{ij} \boldsymbol{b}_j \qquad (i = 1, \ldots, N) .$$

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(1) Cf. GALAVAS [1968]. This tract studies existence, uniqueness, and stability for *non-linear* chemically reacting systems using, for the most part, the first method of Liapounov and fixed point arguments.

(2) Apparently the first authors to relate these two notions of stability were COLEMAN and GREENBERG [1967] and COLEMAN [1970] who showed that Gibbs' criterion is *sufficient* for dynamical stability. Their results, which are not for mixtures but rather for general classes of fluids, deal with the full *non-linear* equations of the theory. It is a simple matter to extend the results of Coleman and Greenberg to mixtures. We prefer, however, to work within the *linear* theory, because within this framework we can establish not only the sufficiency of Gibbs' criterion, but also its *necessity*.

(3) Cf. GURTIN and VARGAS [1971].

We write M^T for the transpose of M, and $M > 0$ (resp. $M \geqslant 0$) means that M is positive definite (resp. positive semi-definite). Finally, if $\boldsymbol{\mu}: \mathcal{E} \to \mathbb{R}^N$, then $\nabla\boldsymbol{\mu}: \mathcal{E} \to \mathcal{V}^N$ is defined by

$$\nabla\boldsymbol{\mu} = (\nabla\mu_1, \ldots, \nabla\mu_N) .$$

Throughout this paper B will designate a compact region in $\mathcal{E}$ whose boundary ∂B is sufficiently smooth for the divergence theorem to be applicable (¹). Given fields $\varphi, \psi: B \to \mathbb{R}$; $\boldsymbol{u}, \boldsymbol{v}: B \to \mathcal{V}$; $\boldsymbol{\mu}, \boldsymbol{\sigma}: B \to \mathbb{R}^N$; $\mathbf{u}, \mathbf{v}: B \to \mathcal{V}^N$; we write

$$(\varphi, \psi) = \int_B \varphi\psi , \qquad (\boldsymbol{u}, \boldsymbol{v}) = \int_B \boldsymbol{u}\cdot\boldsymbol{v} , \qquad (\boldsymbol{\mu}, \boldsymbol{\sigma}) = \sum_{i=1}^N (\mu_i, \sigma_i) , \qquad (\mathbf{u}, \mathbf{v}) = \sum_{i=1}^N (\boldsymbol{u}_i, \boldsymbol{v}_i) .$$

Further, for each of the above inner products, $\|\cdot\|$ denotes the corresponding L^2 norm; *i.e.*, *e.g.*, $\|\varphi\|^2 = (\varphi, \varphi)$. Most of the time we will be concerned with functions $\varphi(\boldsymbol{x}, t)$ of $\boldsymbol{x} \in B$ and $t \in [0, \infty)$. In this instance $\|\varphi\|_t$ denotes the norm of $\varphi(\cdot, t)$; that is,

$$\|\varphi\|_t = \left(\int_B \varphi^2(\boldsymbol{x}, t)\, d\boldsymbol{x}\right)^{\frac{1}{2}}.$$

A similar meaning applies to correponding inner products, *e.g.* $(\varphi, \psi)_t$.

Under the above hypotheses on B the Poincaré inequality

$$\|\varphi\| \leqslant \alpha \|\nabla\varphi\| \tag{1.1}$$

holds for every class C^1 field $\varphi: B \to \mathbb{R}$ with compact support in the interior of B. In (1.1) α is a constant independent of φ. More generally, the Poincaré inequality can be extended to the class of all C^1 fields $\varphi: B \to \mathbb{R}$ which vanish on a given *consistent* subset S of ∂B. Roughly speaking, a subset S is consistent if S can be «seen» from any point of B under a cone whose opening admits a lower bound (²).

2. – General theory.

The classical theory of fluid mixtures is based on the following system of balance equations:

(2.1) (³)
$$\begin{gathered}
\dot{\varrho} + \varrho \operatorname{div} \boldsymbol{v} = 0 , \\
\varrho \dot{c}_\alpha = -\operatorname{div} \boldsymbol{h}_\alpha + m_\alpha \qquad (\alpha = 1, \ldots, N) , \\
\varrho \dot{\boldsymbol{v}} + \nabla p = \mathbf{0} , \\
\varrho(\dot{\varepsilon} + \boldsymbol{v}\cdot\dot{\boldsymbol{v}}) = -\operatorname{div}(\boldsymbol{q} + \mu_\alpha \boldsymbol{h}_\alpha + pv) .
\end{gathered}$$

(¹) This will be the case if B is regular in the sense of KELLOGG [1929].
(²) Cf. STAMPACCHIA [1965].
(³) The superposed dot denotes the *material* time derivative.

Here ϱ is the *density*, $v = 1/\varrho$ the *specific volume*, $\boldsymbol{v}$ the *velocity*, p the *pressure*, ε the *internal energy*, $\boldsymbol{q}$ the *heat flux*, c_α the *concentration* (1) of (constituent) α, $\boldsymbol{h}_\alpha$ the *relative mass flux* (2) of α, m_α the *mass supplied* to α (due to chemical reactions), and μ_α the (relative) *chemical potential* (3) of α. These fields are defined for all $\boldsymbol{x}$ in the region of space B occupied by the mixture and for all time t. The mixture is assumed to contain $N+1$ constituents, but only N of the constituent mass balance laws are linearly independent; for this reason the subscript α in $(2.1)_2$ has the range 1 to N. Further, summation from 1 to N over repeated Greek subscripts is implied, so that

$$\mu_\alpha \boldsymbol{h}_\alpha = \sum_{\alpha=1}^{N} \mu_\alpha \boldsymbol{h}_\alpha .$$

To the above list of fields we add the (absolute) *temperature* θ and the *entropy* η. Then, defining the *state* $\boldsymbol{\sigma}$ to be the following vector in $\mathbb{R}^{N+2}$:

$$\boldsymbol{\sigma} = (p, \theta, \boldsymbol{\mu}) , \qquad \boldsymbol{\mu} = (\mu_1, \dots, \mu_N) , \tag{2.2}$$

the constitutive equations of the classical theory take the form

$$\begin{aligned} &\varepsilon = \hat{\varepsilon}(\boldsymbol{\sigma}) , \qquad \eta = \hat{\eta}(\boldsymbol{\sigma}) , \qquad v = \hat{v}(\boldsymbol{\sigma}) , \qquad c_\alpha = \hat{c}_\alpha(\boldsymbol{\sigma}) , \\ &\boldsymbol{q} = \hat{\boldsymbol{q}}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) , \qquad \boldsymbol{h}_\alpha = \hat{\boldsymbol{h}}_\alpha(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) , \qquad m_\alpha = \hat{m}_\alpha(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) . \end{aligned} \tag{2.3}$$

We assume that each of the response functions appearing in (2.3) is of class C^2, and, since the mixture is a fluid, that each of these functions is isotropic.

For our purposes it is convenient to introduce the potential

$$\xi = \varepsilon - \theta\eta + pv - \mu_\alpha c_\alpha , \tag{2.4}$$

so that, by $(2.3)_{1-4}$,

$$\xi = \hat{\xi}(\boldsymbol{\sigma}) . \tag{2.5}$$

GURTIN and VARGAS (4) have shown that for the constitutive equations (2.3) to be compatible with the second law (in the form of the Clausius-Duhem inequality) it

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(1) $c_\alpha = \varrho_\alpha/\varrho$, where ϱ_α is the density of α.

(2) $\boldsymbol{h}_\alpha = \varrho_\alpha(\boldsymbol{v}_\alpha - \boldsymbol{v})$ (no sum on α) with $\boldsymbol{v}_\alpha$ the velocity of α.

(3) Actually, μ_α is the chemical potential of α minus the chemical potential of constituent $N+1$.

(4) [1971], Theorem 4.1. Actually, they take $\boldsymbol{\sigma} = (v, \theta, c_1, \dots, c_N)$ and $\nabla\boldsymbol{\sigma}$ as independent variables. The slightly different version presented above involves only completely trivial modifications.

is necessary and sufficient that

$$\hat{v} = \frac{\partial \hat{\xi}}{\partial p}, \qquad \hat{\eta} = -\frac{\partial \hat{\xi}}{\partial \theta}, \qquad \hat{c}_\alpha = -\frac{\partial \hat{\xi}}{\partial \mu_\alpha}, \tag{2.6}$$

and

$$\frac{1}{\theta}\hat{\boldsymbol{q}}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})\cdot\nabla\theta + \hat{\boldsymbol{h}}_\alpha(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})\cdot\nabla\mu_\alpha + \hat{m}_\alpha(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})\mu_\alpha \leq 0 . \tag{2.7}$$

We henceforth assume that (2.6) and (2.7) are satisfied. Then (2.1)-(2.6) lead to the following alternative form (¹) for the energy equation $(2.1)_4$:

$$\varrho\theta\dot{\eta} = -\operatorname{div}\boldsymbol{q} - \boldsymbol{h}_\alpha\cdot\nabla\mu_\alpha - m_\alpha\mu_\alpha . \tag{2.8}$$

Given a fixed state $\mathring{\boldsymbol{\sigma}} = (\mathring{p}, \mathring{\theta}, \mathring{\boldsymbol{\mu}})$, let

$$\hat{\varphi}(\boldsymbol{\sigma}) = \hat{\varepsilon}(\boldsymbol{\sigma}) - \mathring{\theta}\hat{\eta}(\boldsymbol{\sigma}) + \mathring{p}\hat{v}(\boldsymbol{\sigma}) - \mathring{\mu}_\alpha\hat{c}_\alpha(\boldsymbol{\sigma}) . \tag{2.9}$$

The *classical criterion of Gibbs for the stability of* $\mathring{\boldsymbol{\sigma}}$ is that $\hat{\varphi}(\boldsymbol{\sigma})$ have a strict local minimum at $\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}}$. A simple calculation, based on (2.6), shows that

$$\hat{\varphi}(\boldsymbol{\sigma}) = \hat{\xi}(\boldsymbol{\sigma}) - (\boldsymbol{\sigma} - \mathring{\boldsymbol{\sigma}})^T\hat{\xi}_{\boldsymbol{\sigma}} , \tag{2.10}$$

where $\hat{\xi}_{\boldsymbol{\sigma}}$ is the gradient, in $\mathbb{R}^{N+2}$, of $\hat{\xi}$ with respect to $\boldsymbol{\sigma}$ (and where $\hat{\xi}_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma}$, and $\mathring{\boldsymbol{\sigma}}$ are considered as column vectors in $\mathbb{R}^{N+2}$). Let

$$\boldsymbol{A} = -(\hat{\xi}_{\boldsymbol{\sigma}\boldsymbol{\sigma}})_{\boldsymbol{\sigma}=\mathring{\boldsymbol{\sigma}}} , \tag{2.11}$$

where the $(N+2)\times(N+2)$ matrix $\hat{\xi}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}$ is the second gradient of $\hat{\xi}$. Then (2.10) implies that

$$\hat{\varphi}_{\boldsymbol{\sigma}} = \boldsymbol{0} \quad \text{and} \quad \hat{\varphi}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} = -\hat{\xi}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} \quad \text{at} \quad \boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}} ,$$

and we have

PROPOSITION 2.1. – *A necessary and sufficient condition that the state* $\mathring{\boldsymbol{\sigma}}$ *satisfy the Gibb's criterion for stability is that* $\boldsymbol{A}$ *be positive definite.*

Also note that if

$$\hat{\boldsymbol{f}}(\boldsymbol{\sigma}) = (-\hat{v}(\boldsymbol{\sigma}), \hat{\eta}(\boldsymbol{\sigma}), \hat{c}_1(\boldsymbol{\sigma}), \ldots, \hat{c}_N(\boldsymbol{\sigma})) , \tag{2.12}$$

then (2.6) takes the form

$$\hat{\boldsymbol{f}} = -\hat{\xi}_{\boldsymbol{\sigma}} .$$

(¹) [1971], equation (4.8).

Hence $\boldsymbol{f}_\sigma = -\hat{\xi}_{\sigma\sigma}$ and we conclude from (2.11) that

$$A = (\hat{\boldsymbol{f}}_\sigma)_{\sigma=\mathring{\sigma}} \,. \tag{2.13}$$

Further, it is clear from (2.11) that

$$A = A^T. \tag{2.14}$$

3. – Linear theory.

We now derive the linear theory appropriate to small departures from a strong equilibrium state $\mathring{\boldsymbol{\sigma}}$. Note first that, by (2.13),

$$\hat{\boldsymbol{f}}(\boldsymbol{\sigma}) - \hat{\boldsymbol{f}}(\mathring{\boldsymbol{\sigma}}) = \boldsymbol{A}[\boldsymbol{\sigma} - \mathring{\boldsymbol{\sigma}}] + O(|\boldsymbol{\sigma} - \mathring{\boldsymbol{\sigma}}|^2)$$

as $\boldsymbol{\sigma} \to \mathring{\boldsymbol{\sigma}}$. Thus, if we neglect the term of order $O(|\boldsymbol{\sigma} - \mathring{\boldsymbol{\sigma}}|^2)$, we conclude from (2.12) that the constitutive equations $(2.3)_{2-4}$ have the form

$$\begin{bmatrix} -(v - \mathring{v}) \\ \eta - \mathring{\eta} \\ c_1 - \mathring{c}_1 \\ \vdots \\ c_N - \mathring{c}_N \end{bmatrix} = \boldsymbol{A}[\boldsymbol{\sigma} - \mathring{\boldsymbol{\sigma}}] \,, \tag{3.1}$$

where $\mathring{v} = \hat{v}(\mathring{\boldsymbol{\sigma}})$, etc.

We assume now that $\mathring{\boldsymbol{\sigma}}$ is a *strong equilibrium state* (¹); that is, we suppose that

$$\hat{m}_\alpha(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})\, \mathring{\mu}_\alpha = 0 \tag{3.2}$$

for all possible values of $\boldsymbol{\sigma}$ and $\nabla\boldsymbol{\sigma}$. Then, using an argument based on (2.7), it is not difficult to show that the constitutive equations $(2.3)_{5-7}$ have the following linear approximation (²):

$$\begin{bmatrix} \theta_0^{-1}\boldsymbol{q} \\ \boldsymbol{h}_1 \\ \vdots \\ \boldsymbol{h}_N \end{bmatrix} = -\boldsymbol{L}\begin{bmatrix} \nabla\theta \\ \nabla\boldsymbol{\mu} \end{bmatrix}, \qquad \begin{bmatrix} m_1 \\ \vdots \\ m_N \end{bmatrix} = -\boldsymbol{T}[\boldsymbol{\mu} - \mathring{\boldsymbol{\mu}}] \,, \tag{3.3}$$

and that the $(N+1)\times(N+1)$ matrix $\boldsymbol{L}$ and the $N\times N$ matrix $\boldsymbol{T}$ are positive semi-definite:

$$\boldsymbol{L} \geqslant 0 \,, \qquad \boldsymbol{T} \geqslant 0 \,. \tag{3.4}$$

(¹) Cf. [1971], § 7.

(²) Cf. [1971], Theorem 7.1.

The linearized versions of $(2.1)_{1\text{-}3}$ and (2.8) are

$$
\begin{aligned}
\varrho \dot{v} &= \operatorname{div} \boldsymbol{v}\,, \\
\varrho \dot{\eta} &= -\operatorname{div} \boldsymbol{q}\,, \\
\varrho \dot{c}_\alpha &= -\operatorname{div} \boldsymbol{h}_\alpha + m_\alpha\,, \\
\varrho \dot{\boldsymbol{v}} &= -\nabla p\,,
\end{aligned}
\tag{3.5}
$$

where we have written ϱ for the constant density $\mathring{\varrho}$ in $\mathring{\boldsymbol{\sigma}}$. To the same degree of approximation the material time derivative and the spatial time derivative coincide; thus, *e.g.*, in $(3.5)_1$ $\dot{v}(\boldsymbol{x}, t)$ is the derivative of $v(\boldsymbol{x}, t)$ with respect to t holding $\boldsymbol{x}$ fixed.

Equations (3.3) and (3.5) constitute the complete system of field equations for the linear theory; they are easily combined to form the following matrix equation:

$$
\left[\begin{array}{c|c} \varrho \boldsymbol{A} & 0 \\ \hline 0 & \varrho \end{array}\right]
\left[\begin{array}{c} \dot{p} \\ \dot{\theta} \\ \dot{\boldsymbol{\mu}} \\ \hline \dot{\boldsymbol{v}} \end{array}\right]
+
\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \boldsymbol{T} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]
\left[\begin{array}{c} p \\ \theta \\ \hline \boldsymbol{\mu} \\ \hline \boldsymbol{v} \end{array}\right]
=
\tag{3.6}
$$

$$
= \Delta \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \boldsymbol{L} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]
\left[\begin{array}{c} p \\ \hline \theta \\ \boldsymbol{\mu} \\ \hline \boldsymbol{v} \end{array}\right]
-
\left[\begin{array}{c} \operatorname{div} \boldsymbol{v} \\ 0 \\ 0 \\ \nabla p \end{array}\right],
$$

where, for convenience, we have written

$$
p \text{ for } p - \mathring{p}\,, \qquad \theta \text{ for } \theta - \mathring{\theta}\,, \qquad \boldsymbol{\mu} \text{ for } \boldsymbol{\mu} - \mathring{\boldsymbol{\mu}}\,. \tag{3.7}
$$

Equation (3.6) can be written more succintly in the form

$$
\mathsf{A}\dot{\mathsf{u}} + \mathsf{T}\mathsf{u} = \Delta \mathsf{L}\mathsf{u} - \mathsf{D}\mathsf{u}\,, \tag{3.8}
$$

where the $(N+3)\times(N+3)$ matrices A, T, and L, the column vector u, and the matrix differential operator D have obvious definitions. Note that the field u has values in the vector space

$$
\mathcal{U} = \mathbb{R}^{N+2} \times \mathcal{V}\,.
$$

To this system of equations we add the (homogeneous) boundary conditions (cf. (3.7)):

$$
\begin{aligned}
p &= 0 \quad \text{on } S_1 \times [0, \infty)\,, & \boldsymbol{v}\cdot\boldsymbol{n} &= 0 \quad \text{on } (\partial B \setminus S_1) \times [0, \infty)\,, \\
\theta &= 0 \quad \text{on } S_2 \times [0, \infty)\,, & \boldsymbol{q}\cdot\boldsymbol{n} &= 0 \quad \text{on } (\partial B \setminus S_2) \times [0, \infty)\,, \\
\mu_\alpha &= 0 \quad \text{on } S_3 \times [0, \infty)\,, & \boldsymbol{h}_\alpha\cdot\boldsymbol{n} &= 0 \quad \text{on } (\partial B \setminus S_3) \times [0, \infty)\,,
\end{aligned}
\tag{3.9}
$$

with S_2 and S_3 consistent subsets of ∂B (1). Note that, by $(3.3)_1$, (3.9) implies that

$$(3.10)\qquad p\boldsymbol{v}\cdot\boldsymbol{n}-\begin{bmatrix}\theta\\ \mu\end{bmatrix}^T \boldsymbol{L}\begin{bmatrix}\dfrac{\partial\theta}{\partial n}\\ \dfrac{\partial\mu}{\partial n}\end{bmatrix}=0 \qquad \text{on } \partial B\times[0,\infty)\,.$$

By a *process* (2) we mean a class C^2 function $\boldsymbol{u}=(p,\theta,\mu,\boldsymbol{v})$ from B into $\mathcal{U}$ that satisfies (3.8) and (3.9) (with $\boldsymbol{q}$ and $\boldsymbol{h}_\alpha$ given by $(3.3)_1$). Basic to the proof of stability given in the next section is

THEOREM 3.1 (CONSERVATION LAW). – *Every process* $\boldsymbol{u}$ *satisfies*

$$(3.11)\qquad \tfrac{1}{2}(\boldsymbol{u},\boldsymbol{A}\boldsymbol{u})_t=\tfrac{1}{2}(\boldsymbol{u},\boldsymbol{A}\boldsymbol{u})_0-\int_0^t\{(\boldsymbol{u},\boldsymbol{T}\boldsymbol{u})_\tau+(\nabla\boldsymbol{u},\boldsymbol{L}\nabla\boldsymbol{u})_\tau\}\,d\tau\,.$$

PROOF. – Since

$$(\boldsymbol{u},\boldsymbol{D}\boldsymbol{u})=(p,\operatorname{div}\boldsymbol{v})+(\nabla p,\boldsymbol{v})\,,$$

the divergence theorem and (3.10) yield

$$(3.12)\qquad (\boldsymbol{u},\boldsymbol{L}\Delta\boldsymbol{u})-(\boldsymbol{u},\boldsymbol{D}\boldsymbol{u})=-(\nabla\boldsymbol{u},\boldsymbol{L}\nabla\boldsymbol{u})\,.$$

Also, by (2.14) and the definition of $\boldsymbol{A}$, $\boldsymbol{A}=\boldsymbol{A}^T$, and it follows that

$$(3.13)\qquad \frac{d}{dt}(\boldsymbol{u},\boldsymbol{A}\boldsymbol{u})_t=2(\boldsymbol{u},\boldsymbol{A}\dot{\boldsymbol{u}})_t\,.$$

If we take the L^2 inner product of (3.8) with $\boldsymbol{u}$ and use (3.12) and (3.13), we arrive at

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{u},\boldsymbol{A}\boldsymbol{u})_t=-(\boldsymbol{u},\boldsymbol{T}\boldsymbol{u})_t-(\nabla\boldsymbol{u},\boldsymbol{L}\nabla\boldsymbol{u})\,,$$

which clearly implies (3.11). □

4. – Stability.

In this section we shall establish necessary and sufficient conditions for the stability of solutions to the system (3.8), (3.9), using, for the most part, only the

(1) The consistency requirement is needed only for Theorem 4.2. For the remaining results it suffices to use the weaker boundary condition (3.10).

(2) The requirement that $\boldsymbol{u}$ be class C^2 is far stronger than needed; indeed, it suffices to have $\boldsymbol{u}$ a weak solution (in the usual sense) of the system (3.8), (3.9).

conditions (2.14) and (3.4) and the usual requirement that ϱ be positive:

$$(4.1) \qquad A = A^T, \qquad L \geqslant 0, \qquad T \geqslant 0, \qquad \varrho > 0 .$$

As explained in Section 3, $(4.1)_{1\text{-}3}$ are consequences of the second law of thermodynamics. It is important to note that we make no assumptions whatsoever concerning the *symmetry* of the matrices L and T.

Theorem 4.1 (Liapounov Stability). – *Let* $\mathring{\sigma}$ *be Gibbs stable. Then given any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that any process* $\mathbf{u}$ *with* $\|\mathbf{u}\|_0 < \delta$ *satisfies* $\|\mathbf{u}\|_t < \varepsilon$ *for all* $t \geqslant 0$.

Proof. – Since $\mathring{\sigma}$ is Gibbs stable, Proposition 2.1 implies that $A > 0$; hence $(4.1)_4$ and the definition of $\mathbf{A}$ imply that $\mathbf{A} > 0$. Thus there exist constants $\alpha, \beta > 0$ such that

$$(4.2) \qquad \alpha \|\mathbf{w}\|^2 \leqslant (\mathbf{w}, \mathbf{A}\mathbf{w}) \leqslant \beta \|\mathbf{w}\|^2$$

for any L^2 function $\mathbf{w}: B \to \mathcal{U}$. On the other hand, (3.11) and $(4.1)_{2\text{-}3}$ imply that, for any process $\mathbf{u}$,

$$(\mathbf{u}, \mathbf{A}\mathbf{u})_t \leqslant (\mathbf{u}, \mathbf{A}\mathbf{u})_0 ,$$

and hence, by (4.2),

$$\|\mathbf{u}\|_t \leqslant \gamma \|\mathbf{u}\|_0 ,$$

where $\gamma = (\beta/\alpha)^{\frac{1}{2}} > 0$. This inequality clearly yields the desired result. □

Theorem 4.1 has the obvious

Corollary (Uniqueness). – *Let* $\mathring{\sigma}$ *be Gibbs stable. Then any process which vanishes at time* $t = 0$ *must be identically zero for all time.*

By a *strongly-compatible initial function* we mean a C^∞ function $\mathbf{u}_0: B \to \mathcal{U}$ with compact support in the interior of B. Clearly, $\mathbf{u}_0$ satisfies the boundary conditions (3.9). In fact, if $\mathbf{u}_0 = (p, \theta, \boldsymbol{\mu}, \boldsymbol{v})$, then p, θ, $\boldsymbol{\mu}$, $\boldsymbol{v}$, $\nabla\theta$, and $\nabla\boldsymbol{\mu}$ all vanish on $(\partial B) \times [0, \infty)$, so that $\mathbf{u}_0$ satisfies the boundary conditions (3.9) for any possible choice of the sets S_1, S_2, and S_3.

Our next result shows that Gibbs' criterion is also *necessary* for stability.

Theorem 4.2 (Instability). – *Assume that* $\mathring{\sigma}$ *is not Gibbs stable. Assume further that* A *and* L *are invertible and that*

$$(4.3) \qquad \beta = -\left(\frac{\partial \hat{v}}{\partial p}\right)_{\sigma = \mathring{\sigma}} > 0 .$$

Then there exists a strongly-compatible initial function $\boldsymbol{u}_0$ *and constants* $C, \lambda > 0$ *such that any process* $\boldsymbol{u}$ *with* $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in B$ *has*

$$\|\boldsymbol{u}\|_t \geqslant Ce^{\lambda t}. \tag{4.4}$$

The proof of this theorem is based on two lemmas.

LEMMA 4.1. – *Let* (4.3) *hold. Then there exists an* $\alpha > 0$ *such that*

$$\boldsymbol{\tau}^T \boldsymbol{A} \boldsymbol{\tau} \geqslant -\alpha |\boldsymbol{\nu}|^2 \tag{4.5}$$

for every $\boldsymbol{\tau} = (\pi, \boldsymbol{\nu}) \in \mathbb{R}^{N+2}$.

PROOF. – By (2.2), (2.12), (2.13), and (4.3), β is the entry in the first row and first column of $\boldsymbol{A}$. Thus $\boldsymbol{\tau}^T \boldsymbol{A} \boldsymbol{\tau}$ admits the representation

$$\boldsymbol{\tau}^T \boldsymbol{A} \boldsymbol{\tau} = \beta\pi^2 + \pi\gamma_0(\boldsymbol{\nu}) + \gamma_1(\boldsymbol{\nu}, \boldsymbol{\nu}) , \qquad \boldsymbol{\tau} = (\pi, \boldsymbol{\nu}) , \tag{4.6}$$

where $\gamma_0(\boldsymbol{\nu})$ is linear in $\boldsymbol{\nu}$ and γ_1 is quadratic in $\boldsymbol{\nu}$. Let $\pi_0 = \pi/\beta + \gamma_0(\boldsymbol{\nu})/2\beta^2$. Then (4.6) implies that

$$\boldsymbol{\tau}^T \boldsymbol{A} \boldsymbol{\tau} = \beta^3\pi_0^2 + \varphi(\boldsymbol{\nu}, \boldsymbol{\nu}) \geqslant \varphi(\boldsymbol{\nu}, \boldsymbol{\nu}) , \qquad \varphi(\boldsymbol{\nu}, \boldsymbol{\nu}) = \gamma_1(\boldsymbol{\nu}, \boldsymbol{\nu}) - \frac{1}{4\beta}\gamma_0(\boldsymbol{\nu})^2 , \tag{4.7}$$

where we have used (4.3). Since $\varphi(\boldsymbol{\nu}, \boldsymbol{\nu})$ is quadratic in $\boldsymbol{\nu}$, there exists an $\alpha > 0$, independent of $\boldsymbol{\nu}$, such that

$$\varphi(\boldsymbol{\nu}, \boldsymbol{\nu}) \geqslant -\alpha |\boldsymbol{\nu}|^2 \tag{4.8}$$

for all $\boldsymbol{\nu} \in \mathbb{R}^{N+1}$, and (4.7), (4.8) imply (4.5). □

LEMMA 4.2. – *Suppose that* $\mathring{\boldsymbol{\sigma}}$ *is not Gibbs stable and* $\boldsymbol{A}$ *is invertible. Then there exists a strongly-compatible initial function* $\boldsymbol{u}_0$ *such that*

$$(\boldsymbol{u}_0, \boldsymbol{A}\boldsymbol{u}_0) < 0 . \tag{4.9}$$

PROOF. – Since $\mathring{\boldsymbol{\sigma}}$ is not Gibbs stable, we conclude from Proposition 2.1 that $\boldsymbol{A}$ is not positive-definite. Thus, since $\boldsymbol{A}$ is symmetric and invertible, $\boldsymbol{A}$ must have a strictly negative eigenvalue. Let $\boldsymbol{\tau} \in \mathbb{R}^{N+2}$ denote a corresponding eigenvector, so that

$$\boldsymbol{\tau}^T \boldsymbol{A} \boldsymbol{\tau} < 0 .$$

Let $\varphi: B \to \mathbb{R}$ $(\varphi \neq 0)$ be a C^∞ function with compact support in the interior of B, and let $\boldsymbol{u}_0: B \to \mathcal{U}$ be defined by

$$\boldsymbol{u}_0(\boldsymbol{x}) = \varphi(\boldsymbol{x})(\tau_1, \tau_2, \ldots, \tau_{N+2}, \boldsymbol{0}) ,$$

(so that the initial velocity field corresponding to $\boldsymbol{u}_0$ is zero). Then $\boldsymbol{u}_0$ is strongly compatible, and, in view of the definition of $\boldsymbol{A}$,

$$(\boldsymbol{u}_0, \boldsymbol{A}\boldsymbol{u}_0) = \varrho\|\varphi\|^2 \boldsymbol{\tau}^T \boldsymbol{A}\boldsymbol{\tau} < 0 ,$$

so that $\boldsymbol{u}_0$ has all of the desired properties. □

PROOF OF THEOREM 4.2. – By (3.11), $(4.1)_3$, and the definition of $\boldsymbol{L}$,

$$(4.10) \qquad (\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u})_t \leqslant (\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u})_0 - \int_0^t (\nabla\boldsymbol{\nu}, \boldsymbol{L}\nabla\boldsymbol{\nu})_\tau \, d\tau$$

in any process $\boldsymbol{u}$, provided $\boldsymbol{\nu} = (\theta, \boldsymbol{\mu})$. Since $\boldsymbol{L}$ is invertible, $(4.1)_2$ implies that $\boldsymbol{L} > 0$. Thus there exists a $\delta > 0$ such that for any vector $\boldsymbol{\varkappa} \in \mathbb{R}^{N+1}$

$$\boldsymbol{\varkappa}^T \boldsymbol{L}\boldsymbol{\varkappa} \geqslant \delta|\boldsymbol{\varkappa}|^2;$$

hence

$$(\nabla\boldsymbol{\nu}, \boldsymbol{L}\nabla\boldsymbol{\nu}) \geqslant \delta\|\nabla\boldsymbol{\nu}\|^2$$

in any process $\boldsymbol{u}$. This result, (3.9), and Poincaré's inequality (1.1) imply

$$(\nabla\boldsymbol{\nu}, \boldsymbol{L}\nabla\boldsymbol{\nu}) \geqslant \omega\|\boldsymbol{\nu}\|^2,$$

where $\omega > 0$ is independent of $\boldsymbol{u}$; hence, by (4.10),

$$(4.11) \qquad (\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u})_t \leqslant (\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u})_0 - \omega\int_0^t \|\boldsymbol{\nu}\|_\tau^2 \, d\tau .$$

Next, letting $\boldsymbol{\tau} = (p, \boldsymbol{\nu})$, $(4.1)_4$, (4.5), and the definition of $\boldsymbol{A}$ imply

$$\varrho^{-1} \boldsymbol{u}^T \boldsymbol{A}\boldsymbol{u} = \boldsymbol{\tau}^T \boldsymbol{A}\boldsymbol{\tau} + |\boldsymbol{v}|^2 \geqslant -\alpha|\boldsymbol{\nu}|^2,$$

and we conclude from (4.11) that

$$(4.12) \qquad \gamma\|\boldsymbol{\nu}\|_t^2 \geqslant -(\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u})_0 + \omega\int_0^t \|\boldsymbol{\nu}\|_\tau^2 \, d\tau , \qquad \gamma = \alpha\varrho > 0$$

in every process $\boldsymbol{u}$. Let $\boldsymbol{u}_0$ be the initial field established in Lemma 4.2, and let

$$(4.13) \qquad 2\lambda = \omega/\gamma > 0 , \qquad C^2 = -\gamma^{-1}(\boldsymbol{u}_0, \boldsymbol{A}\boldsymbol{u}_0) > 0 ,$$

where we have used (4.9). Then if $\boldsymbol{u}$ is any process with $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$,

$$(\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u})_0 = (\boldsymbol{u}_0, \boldsymbol{A}\boldsymbol{u}_0)\,,$$

and (4.12), (4.13) yield

$$\|\boldsymbol{v}\|_t^2 \geqslant C^2 + 2\lambda\int_0^t \|\boldsymbol{v}\|_\tau^2\, d\tau\,, \tag{4.14}$$

which is a standard Gronwall inequality. Letting $\varphi(t)$ denote the right-hand side of (4.14), we have $\|\boldsymbol{v}\|_t^2 \geqslant \dot{\varphi}(t)$ and $\dot{\varphi} = 2\lambda\|\boldsymbol{v}\|^2 \geqslant 2\lambda\varphi$. This differential inequality can be integrated to give $\varphi(t) \geqslant \varphi(0)\exp[2\lambda t]$ which yields

$$\|\boldsymbol{v}\|_t^2 \geqslant C^2 \exp[2\lambda t]\,. \tag{4.15}$$

If we take the square root of both sides of (4.15) and use the obvious inequality $\|\boldsymbol{u}\|^2 \geqslant \|\boldsymbol{v}\|^2$ we arrive at (4.4). □

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