On analytic primals.

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Summary. - *Cartain standard forms of equations of primals with singularities of a given type are investigated. Also the relation between the geometric and analytic case is given and some analytic invariants are found. The application of these methods to the resolution of singularities is indicated.*

1. In a paper on resolution of singularities [3] B. SEGRE used certain forms of local equations of d-folds lying on non-singular $(d+1)$-folds. He derived, by means of these methods, some results about the geometric properties of the variety. Some properties of analytic primals were later on discussed by him in [4].

In the extensive paper [3] only a small part of it was devoted to a detailed discussion of the relation between the geometric and analytic methods. In this paper we show, by means of an example, that the method of choosing local equations may change in some cases the geometric properties of the variety at the point in question. Furthermore we derive a set of conditions under which we can apply the methods of analytic transfosmations, thereby simplifying, and not altering, the geometric situation. We also obtain some new results in this connection.

2. In this paragraph we give an example of a possible behaviour, under analytic transformations, of a surface $\boldsymbol{F}$ in S_3 with a double curve $\boldsymbol{C}$. For simplicity we use non-homogeneous coordinates $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

Let $\boldsymbol{F}$ be the surface whose equation is

$$(x^2+2xy+y^3)^2 f_1(x, y, z) + 2(x^2+2xy+y^3) z f_2(x, y, z) +$$

$$z^2 f_3(x, y, z) = 0,$$

where $f_3(0, 0, 0) \neq 0$ and

$$[f_2(0, 0, 0)]^2 - f_3(0, 0, 0) f_1(0, 0, 0) \neq 0,$$

but f_1, f_2, f_3 are otherwise general.

This surface $\boldsymbol{F}$ has a double point at the origin and on $\boldsymbol{F}$ there is a double curve $\boldsymbol{C}$ whose equations are

$$x^2 + 2xy + y^3 = 0, \quad z = 0.$$

The curve $\boldsymbol{C}$ has a double point at $\boldsymbol{O}$ and the use of dilatations [3] is restricted to bases which are non-singular. Thus to resolve the singularity of $\boldsymbol{F}$ at $\boldsymbol{O}$ (or, strictly speaking, the singularity of $\boldsymbol{F}$ through $\boldsymbol{O}$) we first have to apply a dilatation with $\boldsymbol{O}$ as base. We obtain thereby the proper transform of S_3 which is a non-singular threefold M_1 and the proper transform F_1 of $\boldsymbol{F}$ on it. On F_1 there is a double curve C_1 which is the proper transform of $\boldsymbol{C}$ and a double burve E_1 corresponding to the point $\boldsymbol{O}$. To resolve the singularities of F_1 we have to apply, at least, two more dilatations, one with C_1 as base and one with E_1 (or, strictly speaking, with its proper transform) as base, in total, at least three dilatations.

Now in the problem of resolution of singularities one often deals with a sequence of consecutive points and whith the behaviour of a given variety at these points. To facilitate the analysis one often allows the use of local coordinates which are analytic transforms of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, regular at the point in question. Returning now to the surface $\boldsymbol{F}$, suppose we are interested in the behaviour of $\boldsymbol{F}$ at $\boldsymbol{O}$ and we allow the following analytic transformation:

$$x_1 = x + 2y - \frac{1}{2}y^2 - \frac{1}{8}y^3 - \frac{1}{16}y^4 - \frac{5}{128}y^5 - \ldots = x + y + \sqrt{1-y},$$

$$y_1 = x + \frac{1}{2}y^2 + \frac{1}{8}y^3 + \frac{1}{16}y^4 + \frac{5}{128}y^5 + \ldots = x + y - y\sqrt{1-y},$$

$$z_1 = z.$$

This transformation is certainly regular at $\boldsymbol{O}$. Then $x_1 y_1 = x^2 + 2xy + y^3$ and the «local equation» of $\boldsymbol{F}$ may be written in the form

$$x_1^2 y_1^2 F_1(x_1, y_1, z_1) + 2x_1 y_1 z_1 F_2(x_1, y_1, z_1) + z_1^2 F_3(x_1, y_1, z_1) = 0,$$

where $F_i(x_1, y_1, z_1)$ is a power series, convergent for small

$$|x_1|, |y_1|, |z_1| \quad \text{for} \quad i = 1, 2, 3, \text{ and } F_3(0, 0, 0) \neq 0$$

$$[F_2(0, 0, 0)]^2 - F_1(0, 0, 0) . F_3(0, 0, 0) \neq 0.$$

Here we have two double «lines» through $\boldsymbol{O}$, given by $x_1 = z_1 = 0$ and $y_1 = z_1 = 0$ and to resolve the singularity we first apply the local dilatation

$$x_1 = x_2, \; y_1 = y_2, \; z_1 = z_2 y_2,$$

so that the local equation of the proper transform F_1 of $\boldsymbol{F}$ becomes

$$x_1^2 F_1(x_2, y_2, y_2 z_2) + 2x_2 z_2 F_2(x_2, y_2, y_2 z_2) + z_2^2 F_3(x_2, y_2, y_2 z_2) = 0$$

and here the tangent cone at the origin 0_1 is

$$x_2^2 F_1(0, 0, 0) + 2x_2 z_2 F_2(0, 0, 0) + z_2^2 F_3(0, 0, 0) = 0$$

which is a pair of *distinct planes* and on subsequent application of local dilatation with the line $x_2 = 0 = z_2$ as base, there will correspond *two simple points* to the point 0_1. Thus from the above example it can be seen that if we allow a choice of local coordinates the number of dilatations required is *two*, whilst in the geometric case the number is *three* at least. Also the dimension of the base can be altered. Hence the two situations are not the same and the complications may be greater and may occur in various neighbourhoods of the point with which we are concerned. Of course, the application of analytic transformations simplifies the algebra considerably in most cases and we establish here which geometric properties can be preserved, and also obtained, when analytic transformations and local dilatations are allowed. Thus in this parer we consider the relation between the analytic and geometric case, and the conditions under which we can derive geometric results by applying analytic transformations and local dilatations.

Throughout we suppose that the ground field $\boldsymbol{K}$ is the field of complex numbers.

3. We now give precise definitions to the terms geometric and analytic properties and to geometric and analytic methods.

Let $\boldsymbol{V}$ be a $(d-1)$-fold lying on a non-singular $\boldsymbol{d}$-fold $\boldsymbol{U}$ situated in A_N. Any global property of $\boldsymbol{V}$ or its birational transform will be called a geometric property. Furthermore only birational transformations of $\boldsymbol{V}$ into V_1 which are regular at a generic point of $\boldsymbol{V}$ and V_1 will be allowed in the geometric case and such will be called geometric methods. Let $\boldsymbol{O}$ be a point of $\boldsymbol{V}$ and $x_1, \ldots, x_d$ be local uniformising parameters of U at $\boldsymbol{O}$. Then $\boldsymbol{V}$ is represented in the quotient ring of $\boldsymbol{O}$, denoted by $Q(O/U)$, by a single equation which may be written in terms of the local uniformising parameters $x_1, \ldots, x_d$ as a power series

$$f(x) = f_s(x) + f_{s+1}(x) + \ldots \quad = 0,$$

where $f_i(x)$ is a form of degree $\boldsymbol{i}$ in $x_1, \ldots, x_d$, $s \geq 1$ and $f_s(x) \neq 0$. This is called the *local equation of* V *at* O *and* $\boldsymbol{O}$ is an $\boldsymbol{s}$-ple point of $\boldsymbol{V}$. We write $O[f(x)] = s$, where in general $O[F(x)]$ denotes the order of $F(x)$. If

$$(1) \qquad x_i = \sum_{\lambda=1}^{d} a_{i\lambda} x'_\lambda + \varphi_2^{(i)}(x') + \varphi_3^{(i)}(x') + \dots, \ i = 1, 2, \dots, d,$$

where $\varphi_n^{(i)}(x')$ is a form of degree n in $x_1', \dots, x'_d$ and $\Delta = | a_{ij} | \neq 0$, then $x_1', \dots, x'_d$ may be taken as new local coordinates and the local equation of V then becomes

$$f^*(x') = f_s^*(x') + f_{s+1}^*(x') + \dots \quad = 0.$$

We say that $f(x)$ and $f^*(x')$ are analyticallly equivalent at O, or just equivalent at O. If $f^*(x') = O$ rappresents an analytic primal W, then we say that W is an *analytic transform* of V *regular at* O. We use the same symbol O as a point of V and W. Similar procedure will be adopted throughout, *i. e.* O_i will denote a point of V_i and W_i if W_i is an analytic transform of V_i regular at O_i. We do not introduce new symbols as no possible confusion can arise.

The following properties may be easily verified;

(i) the transformation (1) is reversible and x'_i may be expressed as a power series in $x_1, \dots, x_d$ for $i = 1, 2. \dots, d$,

(ii) the analytic equivalence at O of the power series is a true equivalence relation,

(iii) $f_s(x)$ and $f_s^*(x')$ are *projectively* equivalent.

Also, if in $Q(O_1/U_1)$, a $(d-1)$-fold V_1 is given by the vanishing of a *polynomial* $f(x_1, \dots, x_d)$ in the local uniformising parameters $x_1, \dots, x_d$ of U_1 at O_1 and (x), (x') are related by (1), then we say that W_1 given by $f^*(x') = 0$ is an analytic transform of V_1 regular al O_1. In virtue of (ii) and (iii), the multiplicities of V_1 and W_1 at O_1 are the same. Also the tangent cones are projectively equivalent. Any property of V_1 at O_1 invariant under the transformation (1) will be referred to as an analytic property of V_1 a ξO_1. Methods which allow transformations of the type (1) will be called analytic methods. Also any local dilatation [3], [5] of W_1 wil be allowed when analytic methods are applied.

Before we investigate further properties, we first consider the geometric case in some detail.

4. Let V be a primal in A_d having an s-ple point at O. Suppose C is a subvariety of V of dimension δ such that (i) each point of C is s-ple on V, (ii) C is a simple point of C, (iii) there is no s-ple subvariety D of V,

other than C, which is of dimension δ_1 $(\delta_1 > 0)$ and which passes through O. Then we say that V has $M(s, \delta)$ at O. In this paper we shall investigate certain properties of points of this type only. Throughout we denote by Δ the integer $d - \delta$. In $Q(O/A_d)$ the subvariety C is represented by an ideal P_δ and since C has a simple point at O, therefore the extended ideal $Q(O/A_d)$. P_δ of non-units in (Q/OA_d) has a basis consisting of Δ elements ([2], 132), and, in particular, we may choose the basis to be polynomials in $x_1, \ldots, x_d$. This can be easily done by multiplying each element of the basis by a suitable unit of $Q(O/A_d)$. Indeed, if we arrange the coordinate system so that the tangent [δ] to C at O has the equations $x_i = 0$, $i = 1, 2, \ldots, \Delta$, then we may choose as the basis of P_δ the polynomials $\psi_1, \ldots, \psi_\Delta$, where $\psi_i = x_i + \varphi_i$ and $O(\varphi_i) \geq 2$ for $i = 1, 2, \ldots, \Delta$. We observe that $\psi_1, \ldots, \psi_\Delta$ may be taken as a system of parameters in $Q(C_\delta/A_d)$. Hence ([6], 292), if $f_i(\psi_1, \ldots, \psi_\Delta)$ is a form of degree i in $\psi_1, \ldots, \psi_\Delta$ with coefficients $a_j(x_1, \ldots, x_d)$ in the polynomial ring $K[x_1, \ldots, x_d]$, written shortly $K[x]$, then $f_i(\psi) = 0\,[\mathrm{mod}(P_\delta^{i+1})]$ if and only if all the coefficients $a_j(x)$ in the form $f_i(\psi)$ are members of P_δ.

We now write the equation of V in the form

$$f(x) = f_s(\psi_1, \ldots, \psi_\Delta) + f_{s-1}(\psi_1, \ldots, \psi_\Delta) + \ldots + f_0 = 0,$$

where $f_i(\psi)$ is a form of degre i in $\psi_1, \ldots, \psi_\Delta$ with coefficients in $K[x]$ and we suppose that $f_i(\psi) \neq 0\,[\mathrm{mod}(P_\delta^{i+1})]$ for $i = 0, 1, \ldots, s-1$.

Then $f(x) = 0\,[\mathrm{mod}(P_\delta^s)]$ if and only if $f_i(\psi) = 0$ for $i = 0, 1, \ldots, s-1$. But C_δ is s-ple on V. Thus the equation of V may be written in the form

$$f(x) = f_s(\psi_1, \ldots, \psi_\Delta) = 0. \tag{2}$$

We call it the *standard equations* of V with $M(s, \delta)$ at O.

We point out that the converse is not true. Howewer, if the equation of V can be put in the form (2) and $0\,[f_s(\psi)] = s$, and Δ, as defined previously, is the least integer with this property, then V has an s-ple point at O and, furthermore, any point of the variety C given by $\psi_1 = 0 = \psi_2 = \ldots = \psi_\Delta$ is s-ple on V. But in this case V has only $M(s, \delta)$ at O if

(i) C has a simple point at O,

(ii) there is no other subvariety D of V through O which is of dimension δ_1 $(\delta_1 > 0)$ and which is s-ple on V.

5. Let $f(x) = 0$ be an equation of an analytic primal W. Suppose there is an analytic subvariety D of dimension δ through O which has simple point at O and whose points are s-ple on W. By this we mean that the

equation of W can be written, by a suitable choice of the local coordinates, in the form

(3) $$f(x) = F(\psi_1, \ldots, \psi_\Delta) = 0,$$

where

$$\psi_i = x_i + \varphi_2^{(i)} + \ldots, \; i = 1, 2, \ldots, \Delta, \quad 0[f(x)] = s,$$

and, as before, $\Delta = d - \delta$ and Δ is the least integer with this property. Suppose that there is no other subvariety of dimension $\delta_1 \geq \delta$ through O which is s-ple on W. Then we say that W has $M(s, \delta)$ at O. Thus, if V is any algebraic primal having $M(s, \delta)$ at O and W is an analytic transform of V regular at O and having $M(s, \delta_1)$ at O, then certainly $\delta_1 \geq \delta$.

We also extend the definitions of multiple subvarieties to analytic primals. We first introduce some general notation which will be used throughout. Let $\omega^{(r)}$ denote some differential operator of the form $\frac{\partial^r}{\partial x_1^{\lambda_1} \ldots \partial x_d^{\lambda_d}}$, with $\lambda_1 + \lambda_2 + \ldots + \lambda_d = r$. Then for a fixed r there are $\binom{d+r-1}{r}$ distinct forms that $\omega^{(r)}$ can take. Let $\omega^{(r)} f(x)$, $(r = 0, 1, \ldots, \sigma)$ be all possible derivatives of $f(x)$, $\omega^{(0)} f(x) = f(x)$ $\left[\text{atogether} \binom{d+\sigma}{\sigma} \text{terms}\right]$.

The ideal generated by all these polynomials will be denoted by $\Omega(\sigma)$. If $f(x)$ is a power series, we give the same definition to $\omega^{(r)}$ and we define in a similar way an ideal $\Pi(\sigma)$ generated by $\omega^{(r)} f(x)$ $[r = 0, 1, \ldots, \sigma]$ in $K\{x\}$, where, in general, $K\{x\}$ denotes the ring of power series $K\{x_1, \ldots, x_d\}$. Suppose that $f(x) = 0$ represents an analytic primal of dimension $d - 1$ and suppose that $\Pi(s-1)$ determines in $K\{x\}$ an analytic variety (not necessarily pure) of dimension δ and $\Pi(s)$ is the unit ideal. Then we say that W has an s-ple subvariety through O of dimension δ.

Suppose that $f(x) = F(x')$, where (x), (x') are related by (1), $f(x)$ is a polynominal and $F(x')$ is possibly a power series. Then

$$\frac{\delta f(x)}{\delta x_i} = \sum_{\lambda=1}^{d} \frac{\partial F}{\partial x'_\lambda} \frac{\partial x_\lambda^1}{\partial x^i}$$

and hence, if

$$\omega^{(r)} f(x) = f^{(r)}(x) = g(x'),$$

say, it follows that $g(x') = 0\,[\text{mod } \Pi(s)]$ for $r \leq s$. We shall use this result in § 10.

6. We now suppose that V has $M(s, \delta)$ at O, but the vertex of the tangent cone of V at O has dimension τ and $\tau > \delta$. This is the case, for example, with a surface F having a binode B_s at O which is of the type $M(2, 0)$, but the vertex of the nodal tangent cone is of dimension one if $s > 2$.

We discuss the general problem first. Suppose that $f(x)$ is a polynomial of the form

$$f(x) = f_s(x_1, \ldots, x_\sigma) + f_{s+1}(x) + \ldots + f_n(x), \tag{4}$$

where $1 \leq \sigma < d$, $s > 1$. Then it may be possible to write the polynominal $f(x)$ in the form

$$f_s(X_1^{(2)}, \ldots, X_\sigma^{(2)}) + f_{s+2}(x) + \ldots + f_m(x), \tag{5}$$

where $X_i^{(2)} = x_i + \varphi_2^{(i)}$ for $i = 1, 2, \ldots, \sigma$, and $\varphi_2^{(i)}$ is a form of degree 2 in x_1, $x_2, \ldots, x_d$ and in (5) possibly $f_i(x)$ is different from that in (4) for

$$i = s + 2, \ldots, 2s,$$

and also m may be different from n if $n \leq 2s$.

Let $X_i^{(\rho)}$ be a general notation for any polynomial of the form

$$X_i^{(\rho)} = x_i + \varphi_2^{(i)} + \ldots + \varphi_\rho^{(i)}, \; i = 1, 2, \ldots, d,$$

where $\varphi_r^{(i)}$ is a form of degree r in $x_1, x_2, \ldots, x_d$. Suppose there exist polynomials $X_1^{(\rho)}, \ldots, X_\sigma^{(\rho)}$ such that $f(x)$ may be written in the form

$$f(x) = f_s(X_1^{(\rho)}, \ldots, X_\sigma^{(\rho)}) + f_{s+\rho} + \ldots + F_{N_\rho}, \tag{6}$$

where here f_i is possibly different from that in (4) for $i = s + \rho, \ldots, \rho s$ and N may be differnt from n if $n \leq \rho s$. We call it the *process of partial factorisation and if* $\rho > 2$, we say that for the polynomial (5) the process can be continued. This method was first introduced by MISS H. P. HUDSON [1] in the study of surfaces with binodes.

If in the expression (6) the process cannot be continued and $f_i(x) \neq 0$ for some i, $s + \rho \leq i \leq N$, then we say that the *partial factorisation terminates.* If $f_i(x) = 0$ for $i = s + \rho, s + \rho + 1, \ldots, N_\rho$, then we say that the *partial factorisation is completed.*

Similarly, if

$$f(x) = f_s(x_1, \ldots, x_\sigma) + f_{s+1} + \ldots$$

is a power series, we may «partially factorise» this expression and write

$$f(x) = f_s(X_1^{(\rho)}, \ldots, X_\sigma^{(\rho)}) + f_{s+\rho} + \ldots,$$

where $X_i^{(\rho)}$ $(i = 1, 2, \ldots, \sigma)$ are defined as above.

If there exist power series $X_i = x_i + \varphi_2^{(i)} + \varphi_3^{(i)} + \ldots$ for $i = 1, 2, \ldots, \sigma$ such that $f(x) = f_s(X_1, \ldots, X_\sigma)$, then we say that the process of partial factorisation is *completed* for $f(x)$, and if it is not completed and cannot be continued, we say that it *terminates*.

We first prove

THEOREM I. - *If V, given by $f(x) = 0$, is a primal in A_d and it has $M(s, \delta)$ at O, and if the vertex of the tangent cone of V at O has dimension τ, where $\tau > \delta$, then the partial factorisation of $f(x)$ must terminate.*

PROOF. - Since the vertex of the tangent cone is of dimension τ, by a suitable choice of the coordinate system, we may write the equation of $\boldsymbol{V}$ in the form $f(x) = 0$, where $f(x)$ is given by (4) and $\sigma = d - \tau < d$. Also we may suppose that the tangent $[\delta]$ of $\boldsymbol{C}$ (cf. § 4) has the equations

$$x_1 = 0 = x_2 = \ldots = x_\Delta .$$

Clearly the process of partial factorisation is independent of the coordinate system. Let $\Omega(s-1) = p$ be the ideal defined in § 5. Then the ideal $\boldsymbol{p}$ determines a variety in A_d and the only component of it through $\boldsymbol{O}$ is the irreducible variety $\boldsymbol{C}$. Now $x_\Delta = 0$ is an equations of a prime which passes through the tangent $[\delta]$ of $\boldsymbol{C}$ at $\boldsymbol{O}$, hence there exists a polynomial of the form $x_\Delta + \varphi(x) = p(x)$, with $O[\varphi(x)] \geq 2$, which vanishes on $\boldsymbol{C}$. The extended ideal $\boldsymbol{p}_0 = \boldsymbol{p} \cdot Q(O/A_d)$ is a δ-dimensional ideal determining C' in $Q(O/A_d)$. Hence $[p(x)]^m = O\,[\mathrm{mod}\,(\boldsymbol{p}_0)]$ for some positive interger $\boldsymbol{m}$ ([2], 16). Hence

$$[p(x)]^m = \varphi_1(x)\, F_1(x) + \ldots + \varphi_\alpha(x) F_\alpha(x), \tag{7}$$

where $\varphi_i(x) \in Q(O/A_d)$ and $F_i(x)$ denotes some polynomial of the form

$$\omega^{(r)} f(x), \quad O \leq r \leq s-1, \quad i = 1, 2, \ldots, \alpha .$$

Since $\boldsymbol{V}$ has $M(s, \delta)$ at $\boldsymbol{O}$, we assert that the partial factorisation cannot be completed. For suppose we can complete it. Then $f(x) = f_s(X_\sigma^{(\rho)}, \ldots, X_\sigma^{(\rho)})$ for some ρ and thus the $(d-1)$-fold $\boldsymbol{V}$ has an $\boldsymbol{s}$-ple subvariety of dimension $d - \sigma = \tau > \delta$, given by $X_i^{(\rho)} = 0$, $i = 1, 2, \ldots, \sigma$, which passes through $\boldsymbol{O}$ and this contradicts the assumption that $\boldsymbol{V}$ has $M(s. \delta)$ at $\boldsymbol{O}$. This proves the assertion. Suppose now that the partial factorisation does not terminate and cannot be completed. Then we can write $f(x)$ in the form (6) for every $\rho \geq 1$ and N_ρ is an integer which depends on ρ an $f_i(x) \neq 0$ for some i,

$$s + \rho \leq i \leq N_\rho .$$

In particular, in (6) we take $\rho = m$. Then consider the algebraic curve Γ whose equalions are

$$X_i^{(m)} = 0 \quad i = 1, 2, ..., \sigma$$

$$x_j = 0, \quad j = \sigma + 1, \ldots, \Delta - 1, \Delta + 1, \ldots, d.$$

This curve has a simple point at O and its tangent at O is the axis Ox_Δ. Hence its Puiseux's expansion at O is of the form

$$x_i = a_{i2}t^2 + a_{i3}t^3 + \ldots, \; i = 1, 2, \ldots, \Delta - 1, \Delta + 1, \ldots, d, \; x_\Delta = t,$$

where a_{ij} is an element of K. Substituting this expansion in (7) we obtain

$$t^m + \ldots = \bar{\varphi}_1(t)\, \bar{F}_1(t) + \ldots + \bar{\varphi}_\alpha(t)\, \bar{F}_\alpha(t),$$

where

$$\bar{\varphi}_i(t) = \varphi_i(a_{12}t^2 + \ldots, \ldots, a_{d2}t^2 + \ldots)$$

and $\bar{F}_i(t)$ is similary defined for $i = 1, 2, \ldots, \alpha$. We observe that since $\varphi_i(x) \in Q(O/A_d)$, hence $O[\bar{\varphi}(t)] \geq 0$. Let $\omega^{(r)}f(x) = f^{(r)}(x)$. By the definition of Γ we have $\{\bar{X}_i^{(m)}(t)\} = 0$. Hence $f^{(r)}(t) = q(t)$, where $O[q(t)] \geq s + m - r$, and therefore

$$O[\bar{\varphi}_1(t)\bar{F}_1(t) + \ldots + \bar{\varphi}_\alpha(t)\bar{F}_\alpha(t)] \geq s + m - (s - 1) = m + 1 > m$$

which leads to a contradiction. Thus the partial factorisation must terminate in this coordinate system and hence also in any other coordinate system. This establishes Theorem I.

7. We may obtain Theorem I in a slighlty different form. Since V $M(s, \delta)$ at O, the equation of V is of the form (2). We may write it

$$f_s(\psi_1, \ldots, \psi_\sigma) + f_{s+1}(\psi_1, \ldots, \psi_\Delta) + \ldots + f_N(\psi_1, \ldots, \psi_\Delta) = 0, \tag{8}$$

where σ, Δ have the same meaning as in the previous paragraph, and $f_i(\psi)$ is a form in $\psi_1, \ldots, \psi_\Delta$ if degree i with coefficient in $K[x]$.

Let $\chi_i^{(\rho)}$ be a general notation for any polynomial of the form

$$\chi_i^{(\rho)} = \psi_i + \theta_2^{(i)}(\psi) + \ldots + \theta_\rho^{(i)}(\psi), \quad i = 1, 2, \ldots, \sigma,$$

where $\theta_j^{(i)}(\psi)$ is a homogeneous polynomial in $\psi_1, \ldots, \psi_\Delta$ of degree j with coefficients in $K[x]$. Then we may write

$$f(x) = f_s(\chi_1^{(\rho)}, \ldots, \chi_\sigma^{(\rho)}) + f_{s+\rho}(\psi) + \ldots + f_{N\rho}(\psi), \tag{9}$$

for some ρ, and here $f_i(\psi)$ is not, in general, the same as $f_i(\psi)$ in (8) for $i = s + \rho, \ldots, s\rho$. We call it the *process of partial factorisation with respect to* $\psi_1, \ldots, \psi_\Delta$. As before, since V has $M(s, \delta)$ at O and $\Delta > \sigma$, we may show that this process must terminate but we do not include the proof as it is similar to the one of Theorem I. Thus Theorem I can now be stated in the form of

THEOREM II. - *With the hypothesis of Theorem* I, *the process of partial factorisation with respect to* $\psi_1, \ldots, \psi_\Delta$ *must terminate.*

8. In this paragraph we prove various results in connection with analytic transformations. We observe that every analytic transformation of the type (1) may be written as a product of a linear transformation

$$x_i'' = \left(\sum_{\lambda=1}^{d} A_{\lambda i} x_\lambda \right) / \Delta, \tag{10}$$

where A_{ij} is the cofactor of a_{ij} in the determinant $| A_{ij} | = \Delta$, followed by the analytic transformation

$$x_i'' = x_i' + g_2^{(i)} + \ldots, \tag{11}$$

where

$$g_j^{(i)} = \left(\sum_{\lambda=1}^{d} A_{\lambda i} \varphi_j^{(\lambda)} \right) / \Delta.$$

In what follows we shall assume that the analytic transformations are of the type (11). Let $f(x)$ be given by (4). Let $\mathfrak{t}$ be the ideal generated by

$$\frac{\partial f_s}{\partial x_1}, \ldots, \frac{\partial f_s}{\partial x_\sigma}.$$

We first prove

LEMMA 1. - *Suppose that* $f(x)$ *can be written in the form* (6) *for some* $\rho \geq 1$. *Then the partial factorisation can be continued if and only if* $f_{s+\rho} \equiv 0 \pmod{\mathfrak{t}}$.

PROOF. - If $f_{s+\rho} = O(\mathrm{mod}\ t)$, then $f_{s+\rho} = a_1 w_1 + \dots + a_\sigma w_\sigma$, where $w_i = \dfrac{\partial f_s}{\partial x_i}$ and a_i is a form of degree $\rho + 1$ for $i = 1, 2, \dots, \sigma$. Then

$$f_s(X_1^{(\rho)} + a_1, \dots, X_\sigma^{(\rho)} + a_\sigma) = f_s(X^{(\rho)}) + \sum_{i=1}^{\sigma} a_i w_i + \varphi(x),$$

where $O[\varphi(x)] \geq s + \rho + 1$.

Hence $f(x)$ can be written in the form (6) with ρ replaced by $\rho + 1$.

Conversely, if $f(x)$ is represented by (6) with ρ replaced by $\rho + 1$, then

$$X_i^{(\rho+1)} = X_i^{(\rho)} + \varphi_{\rho+1}^{(i)} \ (i = 1, 2, \dots, \sigma)$$

and

$$f(x) = f_s(X^{(\rho)}) + \sum_{i=1}^{\sigma} \varphi_{\rho+1}^{(i)} w_i + f^*_{s+\rho+1} + \dots + f^*_{N_\rho}$$

$$= f_s(X^{(\rho)}) + f_{s+\rho} = f^*_{s+\rho+1} + \dots + f^*_{N_\rho}.$$

Thus $f_{s+\rho} = \sum_{i=1}^{\sigma} \varphi_{\rho+1}^{(i)} w_i = 0$ (mod. $\boldsymbol{t}$) which proves the Lemma.

We now show that if the partial factorisation is completed in two ways, *i. e.* if

$$f(x) = f_s(X_1^{(\rho)}, \dots, X_\sigma^{(\rho)}) + \dots + f_{N_1} \tag{12}$$

$$= f_s((X_1^*)^{(r)}, \dots, (X_\sigma^*)^{(r)}) + f^*_{s+r} + \dots + f^*_{N_2}, \tag{13}$$

then $r = \rho$. To prove this it will be sufficient to show that if $f(x)$ is represented by both (12) and (13), and $r = \rho$, and if also the partial factorisation can be continued in one case, then it can be continued in the other case too.

We prove it in

LEMMA 2. *Let $f(x)$ be represented by the two equations* (12) *and* (13) *with $r = \rho$. Then $f_{s+\rho} = 0$(mod $\boldsymbol{t}$) if and only if $f^*_{s+\rho} = 0$ (mod $\boldsymbol{t}$).*

PROOF. - Suppose $f_{s+\rho} = 0$ (mod $\boldsymbol{t}$). Then we may continue the process of partial factorisation in (12). In this case put $X_i^{(\rho+1)} = \bar{x}_i$, for $i = 1, 2, \dots \sigma$ and $x_i = \bar{x}_i$ for $i = \sigma + 1, \dots, d$. This is an analytic transformation. regular $\boldsymbol{O}$. Then

$$f(x) = f_s(\bar{x}_1, \dots, \bar{x}_\sigma) + f_{s+\rho+1}(\bar{x}) + \dots \tag{14}$$

Also

$$(X^*)^{(r)} = \bar{x}_i + \varphi_2^{(i)}(\bar{x}) + \dots, \ i = 1, 2, \dots, \sigma$$

and hence

$$f(x)=f_s(\bar{x}_1+\varphi_2^{(1)}(\bar{x})+\dots,\ \dots,\ \bar{x}_\sigma+\varphi_2^{(\sigma)}(\bar{x})+\dots)+f^*_{s+\rho}(\bar{x})+\dots$$

We now express

$$f_s(\bar{x}_1+\varphi_2^{(1)}(\bar{x})+\dots,\ \dots,\ \bar{x}_\sigma+\varphi_2^{(\sigma)}(\bar{x})+\dots)$$

in the form

$$f_s(\bar{x}_1,\ \dots,\ \bar{x}_\sigma)+f_{s+1}(\bar{x})+\dots$$

and comparing it with (14) we find that

$$\varphi_2^{(i)}(\bar{x})=\varphi_3^{(i)}(\bar{x})=\dots=\varphi_\alpha^{(i)}(\bar{x})=0 \text{ for } i=1,\ 2,\ \dots,\ \sigma,$$

where $\alpha\geq\rho$. Thus

$$f(x)=f_s(\bar{x}_1+\varphi_{\rho+1}^{(1)}(\bar{x})+\dots,\ \dots,\ \bar{x}_\sigma+\varphi_{\rho+1}^{(\sigma)}(\bar{x})+\dots)+f^*_{s+\rho}(\bar{x})+\dots$$

and equating the forms of degree $s+\rho$ in this expression and in (14) we find that

$$\varphi_{\rho+1}^{(1)}(\bar{x})\frac{\partial f_s(\bar{x})}{\partial\bar{x}_1}+\dots+\varphi_{\rho+1}^{(\sigma)}(\bar{x})\frac{\partial f_s(\bar{x})}{\partial\bar{x}_\sigma}+f^*_{s+\rho}(\bar{x})=0$$

and returning to the original coordinates we obtain the congruence

$$f^*_{s+\rho}(x)=0 \pmod{\boldsymbol{t}}.$$

Similarly we can show that if $f^*_{s+\rho}(x)=0 \pmod{\boldsymbol{t}}$ then also $f_{s+\rho}=0 \pmod{\boldsymbol{t}}$. This proves the Lemma.

As a consequence of the previous results we thus state.

THEOREM III. - *If $f(x)$ is represented by (12) and the partial factorisation cannot be continued, then ρ is unique, independent of the way the partial factorisation has been carried out.*

9. We extend here the previous, without proofs, to varieties with $M(s,\ \delta)$ at $\boldsymbol{O}$. We use the notation of § 7 and we thus state.

THEOREM IV. - *Let V have $M(s,\ \delta)$ at O and let the equation of V be $f(x)=0$, where $f(x)$ is given by (9). If the process of partial factorisation with respect to $\psi_1,\ \dots,\ \psi_\Delta$ terminates, then ρ is unique, independent of the way the partial factorisation has been carried out.*

10. In this paragraph we prove certain results about isolated s-ple points. We first state and prove

THEOREM V. - *If V has an isolated s-ple point at O, then so has any analytic transform regular at O.*

PROOF. - Suppose that this theorem is not true. Then we can find an analytic transform W of V which is regular at O, such that the point O is not of the type $M(s, O)$. Then on W there exists an analytic curve Γ which is s-ple on W. Suppose that W is given by $f^{*}(x')=0$, where (x), (x^1) are related by the equation (1). Let $P_1(t)$, ..., $P_d(t)$ be the Puiseux's expansion of Γ, where

$$P_i(t) \in K\{t\},\ i=1, 2, ..., d.$$

Let Γ_1, given by $Q_1(t)$, ..., $Q_d(t)$ be its transform in A_d. Then Γ_1 is an analytic curve (possibly even an algebraic curve). Let γ be the tangent to Γ_1 at O. Suppose γ does not lie in the prime Π given by

$$\pi(x) \equiv \pi_1 x_1 + ... + \pi_d x_d = 0.$$

Now O is an isolated s-ple point of V. Let p be the maximal prime ideal of $Q(O/A_d)$. Then the extended ideai $p_0 = \Omega(s-1)$. $Q(O/A_d)$ is p-primary (cf. § 6) and thus for some positive integer m we have $[\pi(x)]^m = 0 \,(\mathrm{mod}\ p_0)$. Now (x), (x') are related by (1), hence on substitution we obtain $\pi(x) = \pi_1(x^1)$, where $\pi_1(x') \in K\{x'\}$. If follows from § 5 that

$$[\pi_1(x')]^m = 0 [\mathrm{mod}\ \Pi(s-1)].$$

Now

$$[\bar{\pi}_1(t)]^m = ct^M + ...,\ c \in K,\ c \neq 0 \text{ and } M = nm,$$

where n is a positive integer. But if $F(x)$ is any member of $\Pi(s-1)$, then $F(t) = 0$. This leads to a contradiction and establishes the truth of Theorem V.

We point out that the analogous result is not true in the theory of valuations. Let $R = K[x, y]$ be the ring of polynomials in x, y over K. Let Σ be the quotient field of R and R' be the completion of R. Let p be the maximal prime ideal (x, y) in R. Suppose R_1 is the quotient ring R_p. Then R_1 is isomorphic with a subring of R'. Suppose B is a zero dimensional valuation of rank one and the value group Γ of B is isomorphic to the additive group of integers. Let $v(\xi)$ denote the value of ξ in B and suppose $v(x) > 0$, $v(y) > 0$. Let Σ' be the quotient field of R'. Then $\Sigma \subseteq \Sigma'$ and B extends in Σ' to a valuation B' of dimension zero and rank ***two*** *i.e.* if we identify Σ with a subfield of Σ', then there exists an element ξ of Σ' such that $v(\xi) > v(\eta)$

for any $\eta \in \Sigma$. Thus there exists a one-dimensional valuation $\boldsymbol{B}'$ of Σ' such that the valuation ring $\boldsymbol{R}'$ of $\boldsymbol{B}'$ contains $\boldsymbol{R}$.

That the similar result is not true in the case of singularities, *i.e.* that in our particular case we cannot have an «analytic curve» which is $\boldsymbol{s}$-ple on $\boldsymbol{W}$, has been shown above.

We finish the paragraph by stating without proof (¹)

Theorem VI. - *If V has $M(s, \delta)$ at O, then so has any analytic transform W of V regular at O.*

11. In this paragraph we study the effects of dilatations on both, geometric and analytic primals. Suppose $\boldsymbol{V}$ is a primal in A_d and $\boldsymbol{V}$ has $M(s, \delta)$ at $\boldsymbol{O}$ and $\boldsymbol{W}$ is any analytic transform of $\boldsymbol{V}$ regular at $\boldsymbol{O}$. Suppose that $\boldsymbol{C}$ is of dimension δ, each point of it is $\boldsymbol{s}$-ple on $\boldsymbol{V}$, it passes through $\boldsymbol{O}$ and is non-singular. We apply a dilatation to $\boldsymbol{V}$ with $\boldsymbol{C}$ as the base. Let Π be the tangent $[\delta]$ to $\boldsymbol{C}$ at $\boldsymbol{O}$. Let V_1 and U_1 be the transforms of $\boldsymbol{V}$ and A_d respectively. Then to every $[\delta+1]$ through Π there corresponds a point of U_1. Let the equation of $\boldsymbol{V}$ be respresented by $f(x)=0$, where $f(x)$ is given by (6). We recall that Π has the equations $x_1=0=x_2=\ldots=x_\Delta$. Let O_1 of V_1 correspond to the $[\delta+1]$ given by the equations $x_2=0=x_3=\ldots=x_\Delta$. Let $\psi_1, \ldots, \psi_\Delta$ be the polynomials defined in § 4 and let m_i be the degree of ψ_i $(i=1, 2, \ldots, \Delta)$. Let ν_1 be the order of $\boldsymbol{C}$ and put $\nu=\max\limits_{1\leq i\leq\Delta}(\nu_1, m_i)+1$. Suppose $\varphi_1, \ldots, \varphi_\rho$ is the complete set of forms vanishing on $\boldsymbol{C}$ which are of degree ν in the homogeneous coordinates $x_0, x_1, \ldots, x_d$. Suppose $\boldsymbol{V}$ lies in S_d and A_d is obtained from it by taking $x_0=0$ as the prime at infinity. If we transform $\boldsymbol{V}$ and S_d by means of all forms of degree $\boldsymbol{m}$, then we say that the dilatation is of degree $\boldsymbol{m}$ (cf. also [2], 246).

Suppose, in general, we apply a dilatation of degree μ to S_d and $\boldsymbol{V}$ and we obtain thereby $U^{(\mu)}$ and $V^{(\mu)}$ $(\mu\geq\nu)$. Then $U^{(\nu+1)}$ is in regular correspondence ([2], 248) with the join of $U^{(\nu)}$ and S_d and $V^{(\nu+1)}$ is also in regular correspondence with the join of $V^{(\nu)}$ and $\boldsymbol{V}$. We shall thus discuss, for simplicity, the join of $U^{(\nu)}$ and S_d which we denote by U_1 and we denote the transform of $\boldsymbol{V}$ on U_1 by V_1. Suppose that in non-homogeneous coordinates we write the polynomials $\varphi_1, \ldots, \varphi_\rho$ in the form $\psi_1, \ldots, \psi_\rho$, where $\psi_1, \ldots, \psi_\Delta$ are defined in § 5. Then $\psi_1, \ldots, \psi_\Delta$ form a basis of the extended ideal $Q(O/A_d)\cdot\Omega(s-1)$ and hence

$$\psi_i=\sum_{j=1}^{\Delta}\lambda_j^{(i)}\psi_j,\ \text{where } \lambda_j^{(i)}\in Q(O/A_d),\ j=1, 2, \ldots, \Delta$$

(¹) The proof is similar to the proof of Theorem V.

and $i = \Delta + 1, \ldots, \rho$. Suppose U_1 is situated in A_N, where $N = d\rho + \rho - 1$. There by a suitable arrangement of the coordinate system of A_N a generic point of U_1 may be taken in the form

$$x_1, \ldots, x_d, \frac{\psi_2}{\psi_1}, \ldots, \frac{\psi_\rho}{\psi_1} \frac{x_i \psi_j}{\psi_1} \; (1 \leq i \lesssim d, \; 2 \leq j \leq \rho),$$

or shortly $\eta_1, \ldots, \eta_N$. The origin O_1 is a simple point of U_1. We assert that

$$\psi_1, \frac{\psi_2}{\psi_1}, \ldots, \frac{\psi_\Delta}{\psi_1}, x_{\Delta+1}, \ldots, x_d$$

may be taken as uniformising parameters of U_1 at O_1. Let K^* be the ring of power series in

$$\psi_1, \frac{\psi_2}{\psi_1}, \ldots, \frac{\psi_\Delta}{\psi_1}, x_{\Delta+1}, \ldots, x_d.$$

We first show that $x_i \in K^*$ for $i = 1, 2, \ldots, \Delta$. Let $\varphi(x)$ be any form of degree λ in $x_1, \ldots, x_d$. Then

$$\varphi(x) = \varphi(x_1, \ldots, x_d) = \varphi\left(\psi_1, \frac{\psi_2}{\psi_1} \cdot \psi_1, \ldots, \frac{\psi_\Delta}{\psi_1} \psi_1, x_{\Delta+1}, \ldots, x_d\right) + \varphi^{(1)}(x),$$

where $0[\varphi^{(1)}(x)] \geq \lambda + 1$. We may now proced in the same way with $\varphi^{(1)}(x)$ and we find that any form $\varphi(x)$ is a member of K^*. In particular, $x_i \in K^*$ for $i = 1, 2, \ldots, d$. Hence $Q(O/A_d) \subseteq K^*$. We know that

$$\psi_i = \lambda_1^{(i)} \psi_1 + \ldots + \lambda_\Delta^{(i)} \psi_\Delta,$$

and thus

$$\frac{\psi_i}{\psi_1} = \lambda_1^{(i)} + \lambda_2^{(i)} \frac{\psi_2}{\psi_1} + \ldots + \lambda_\Delta^{(i)} \frac{\psi_\Delta}{\psi_1},$$

where $\lambda_j^{(i)} \in Q(O/A_d) \subseteq K^*$. Hence $\frac{\psi_i}{\psi_1} \in K^*$ for all i. Let ζ be any element of $Q(P_1/U)$. Then ζ may be represented in the form $\frac{F(\eta)}{G(\eta)}$ where $G(0, 0, \ldots, 0) \neq 0$, and thus we may express ζ as a power series $F_1(\eta_1, \ldots, \eta_N)$. From the previous results it follows that $\xi \in K^*$ and this proves the assertion.

Let $\boldsymbol{W}$ be any analytic transform of $\boldsymbol{V}$ regular at $\boldsymbol{O}$. Then, by Theorem VI we know that $\boldsymbol{W}$ has $M(s, \delta)$ at $\boldsymbol{O}$. Suppose that we apply a dilatation and for one choice of the coordinate system the dilatation takes the form

$$X_1 = x_1, X_i = \frac{x_i}{x_1} \; (i = 2, 3, \ldots, \Delta), \; X_j = x_j, \; j = \Delta + 1, \ldots, d.$$

Let (x') be any other system of coordinates where (x), (x') are related by (11) with (x''), (x') replaced by (x'), (x), respectively. Let

$$X_1' = x_1', \ X_i' = \frac{x_i}{x'_i} \ (i = 2, 3, \ldots, \Delta), \ X_j' = x_j',$$

$j = \Delta + 1, \ldots, d$, be the equations of a dilatation in the new coordinates. Then

$$X_1' = x_1' = x_1 + \psi_2^{(1)}(x) + \ldots = X_1 + \bar{\psi}_2^{(1)}(X) + \ldots$$

$$X_i' = \frac{x_i'}{x_1'} = \frac{x_i + \psi_2^{(i)}(x) + \ldots}{x_1 + \psi_2^{(1)}(x) + \ldots}$$

$$= \frac{X_i + [\psi_2^{(i)}(x)]/x_1 + \ldots}{1 \ + [\psi_2^{(1)}(x)]/x_1 + \ldots}$$

$$= X_i + c_i X_1 + \Phi_2^{(1)}(X) + \ldots, \ i = 2, 3, \ldots, \Delta$$

$$X'_{(i)} = x_{(i)} = x_i' + \psi_2^{(i)}(x) + \ldots$$

$$= X_i + \bar{\psi}_2^{(i)}(X) + \ldots, \quad i = \Delta + 1, \ldots, d,$$

where $\Phi_j^{(i)}(X)$, $\bar{\psi}_j^{(i)}(X)$ are forms in $X_1, \ldots, X_d$ of degree j and $c_i \in K$. Hence (X), (X') are related by an equation of the type (1) and here $| a_{ij} | = 1$. Thus local dilatation is independent of the set of local coordinates and, in particular, we may take the new coordinates in the form

$$X_i' = \psi_i, \ i = 1, 2, \ldots, \Delta, \text{ and } x_i' = x_i \text{ for } i \geq \Delta + 1.$$

Hence we have.

Theorem VII. - *Let V have $M(s, \delta)$ at O and W be any analytic transform of V regular at O. Suppose that C through O which is s-ple on V is non-singular. If we apply dilatations to V and W and O_1 corresponds on both, V_1 and W_1 to the same $[\delta + 1]$ through the tangent $[\delta]$ to C at O, then W_1 is an analytic transform of V_1 regular at O_1.*

Suppose now that V has an isolated s-ple point at O and V_1 is obtained by a dilatation with O as base and O_1 corresponds to O. Suppose we form a sequence $\{V_i, O_i\}$ with O_i as the base of a dilatation of V_i, and O_{i+1} corresponds to O_i. Hence as a consequence of Theorem V and Theorem VII we have:

THEOREM VIII. - *Let* $\{V_i, O_i\}$ *be a sequence. Let* $\{W_i, O_i\}$ *be any analytic sequence such that* W_i *is an analytic transform of* V_i *regular at* O_i *for* $i = 0, 1, \ldots,$ *and* $V_0 = V$, $W_0 = W$. *Then* O_i *is an isolated s-ple point of* $\boldsymbol{W}_i$ *if and only if it is an isolated s-ple point of* W_i.

12. We end the paper by giving some results about isolated s-ple points. Let $\boldsymbol{V}$ be a primal with an isolated s-ple point at $\boldsymbol{O}$. Let $\boldsymbol{W}$ be any analytic transform of $\boldsymbol{V}$ regular at $\boldsymbol{O}$. We now state and prove.

THEOREM IX. - *If* V *has an isolated s-ple point at* O *and* $O_1, O_2, \ldots,$ *is a sequence of free points, then* O_r *is* s_1*-ple point on* V_r *for some* r, *where* $1 \leq s_1 < s$.

PROOF. - Suppose this is not the case. Then there exists an analytic curve γ on $\boldsymbol{W}$ through $\boldsymbol{O}$ which is s-ple on $\boldsymbol{W}$ and thus $\boldsymbol{W}$ does not have an isolated s-ple point at $\boldsymbol{O}$; this contradicts Theorem V and establishes the truth of the theorem.

We shall apply subsequently the above results to sequences of isolated s-ple points on primals where not all the points are free points.

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