Algebraic Cones With Linear Vertex (1).

by JACK OHM (Madison 6, U. S. A.)

Summary - *Some basic properties of algebraic cones are derived using methods of an elementary geometric nature.*

Introduction.

We prove here some fundamental theorems involving the projection of a variety from a linear variety. It is probably safe to say that most of what we prove is classical, in the sense that special cases of our theorems have been known for many years. In fact our interest in the subject has been motivated by the importance of the cone construction in the intersection theory of cycles, where it seems to play an essential role (see, for instance, Chow [2], p. 458). This theory can be traced to SEVERI [7], VAN DER WAERDEN [9], HODGE and PEDOE [4], SAMUEL [6], and CHOW [2], among others.

Special cases of many of our theorems occur in these paperrs; but often the theorems are either stated without proof or dismissed as easily seen. Moreover, when a theorem is proved, the proof frequently involves a tedious use of GRASSMANN varieties (for example [1], p. 3-03). We have attempted here to remedy this situation. In keeping with the foundational nature of the subject, we have succeeded in employing only elementary tools, with the exception of the criterion for unit multiplicity and its converse ([11], theorem 6, p. 152). CHOW has shown, however, that the criterion itself may be proved without intersection multiplicity [3], so it is only in the use of the converse to this criterion at the end of section 2 (in theorem 2.10) that these results depend on intersection theory.

Our language is that of WEIL [11]. Projective n-space will be denoted by P^n and affine n-space by S^n. In section 1 we prove a theorem (the dimension theorem) relating the dimension of a cone having a linear vertex to that of the base variety. In section 2 we prove a theorem relating the degree of the cone to that of the base variety (the degree theorem) and a theorem (the simple point thorem) connecting the simplicity of a point on the base variety with the simplicity of the corresponding fiber on the cone.

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Section 3 is devoted to proving that under certain conditions a cone with a generic vertex intersects another variety in components which occur with multiplicity 1 and which do not intersect in a point rational over the ground field, and section 4 discusses some examples in the characteristic $p \neq 0$ case.

In conclusion, we are indebted to PROF. A. SEIDENBERG for reading a preliminary version of the manuscript and for making many helpful suggestions which have been incorporated in the present version.

1. The dimension theorem.

1.1 Definition. Let V, W be varieties in S^n defined over a field k and having independent generic points (x), (y) respectively over k. If λ is a transcendental quantity over $k(x, y)$, the point $(x) + \lambda((y) - (x))$ has a locus over k which we shall denote by $|VW|$. Furthermore, if $W = |VW|$, we shall say that W is a cone with V in its vertex.

It is then clear that $|VW| = |WV|$ and that $|VW|$ is independent of the choice of (x), (y), λ and k. We also find it convenient to make the convention that $\dim \Phi = -1$ and $|\Phi V| = |V\Phi| = V$.

1.2 LEMMA - *Let V be a variety in S^n having a generic point (x) over a field k, let (α) be a point in S^n with coordinates in k, and let V' be the variety in S^n having generic point $(x') = (\alpha) - (x)$ over k. Then V is a cone with (α) in its vertex if and only if V' is a cone with O (the origin) in its vertex. Moreover, if L is the linear variety attached to V at a point (y) in V and L' the linear variety attached to V' at the point $(y') = (\alpha) - (y)$, then (z) is a generic point over $k(y)$ for L if and only if $(z') = (\alpha) - (z)$ is a generic point over $k(y)$ for L'.*

PROOF. - The first assertion is immediate from the definitions. Suppose then that (z) is a generic point over $k(y)$ for L. Let $F_\mu(X) = 0$ be a set of equations for V over k so that $G_\mu(X) = 0$ is a set of equations for V' over k, where $G_\mu(X) = F_\mu(\alpha - X)$. Then $\partial G_\mu/\partial y'_j = -\partial F_\mu/\partial y_j$, so L and L' have the same dimension. Moreover, $\dim (z)/k(y) = \dim (z')/k(y')$; and $\Delta_{y'} G_\mu(z' - y') = \Delta_y F_\mu(z - y) = 0$ since $(z) \varepsilon L'$ and hence is generic over $k(y')$ for L'. The converse follows by symmetry.

1.3 PROPOSITION. - *Let V be a cone in S^n with a point P in its vertex, let $Q \neq P$ be a point of V and $R \neq P$ a point of $|PQ|$. Then the linear variety attached to V at Q is also the linear variety attached to V at R.*

PROOF. - Suppose $P = O$, so that V is defined over a field of definition k by a set of homogeneous equations $F_\mu(X) = 0$. Since $|OQ| = |OR| \subset V$ and hence contained in the attached linear varieties L_Q, L_R to V at Q, R respectively, $O \varepsilon L_Q$ and L_R. Therefore, if $Q = (x)$ and $R = (y)$, $\Delta_x F_\mu(X) = 0$ is a set of equations for L_Q and $\Delta_y F_\mu(X) = 0$ a set of equations for L_R. But there exists a $\lambda \neq 0$ such that $(y) = \lambda(x)$; so $\lambda^{\rho(\mu)}\Delta_x F\mu(X) = \Delta_y F\mu(X)$ due to the homogeneity. Therefore $L_Q = L_R$.

If $P \neq O$, let V' be the variety of lemma 1.2, and let $K = k(P, Q, R)$. Using the notation of that lemma, by the above case we have $L'_{Q'} = L'_{R'}$. Therefore, if (z') is a generic point for $L'_{Q'} = L'_{R'}$ over K, (z) is generic for L_Q and L_R over K; so $L_Q = L_R$.

1.4 COROLLARY. - *Let V and W be varieties in S^n, and let L_P, L_Q be the linear varieties attached to V, W at distinct points P, Q respectively. Let R be a point of $|PQ|$ different from P and Q, and let L' be the linear variety attached to $|VW|$ at R. Then $|L_P L_Q| \subset L'$* (2).

PROOF. - $|PW|$ is a cone with P in its vertex. Let L'_Q, L'_R denote the linear varieties attached to $|PW|$ at Q, R respectively. Then $L'_Q = L'_R$ by 1.3. But $W \subset |PW|$ implies $L_Q \subset L'_Q$, and $|PW| \subset |VW|$ implies $L'_R \subset L'$. Therefore $L_Q \subset L'_Q = L_R \subset L'$. Similarly, $L_P \subset L'$; so the corollary follows from the linearity of L'.

1.5 LEMMA - *Let k be a field of characteristic 0, let $F(X) \neq 0$ be in $k[X_1, \ldots, X_n]$ and let $G(X) = \partial F/\partial X_1 . X_1 + \ldots + \partial F/\partial X_n . X_n$. Then $G(X) = \gamma F(X)$ for some $\gamma \in k$ (if* (3) *and) only if γ is an integer ≥ 0 and F is homogeneous of deg γ.*

PROOF. - If $n = 0$, the lemma is immediate; so assume the lemma is true when $n > \beta$ and let $n = \beta > 0$. We can write $F(X) = \sum_{i=0}^{m} \alpha_i X_1^{m-i}$, where $\alpha_i \in k[X_2, \ldots, X_\beta]$ and not all $\alpha_i = 0$ Then

$$G(X) = \sum_{i=0}^{m} (m-i)\alpha_i X_1^{m-i} + \sum_{j=2}^{\beta} \left(\sum_{i=0}^{m} \partial \alpha_i/\partial X_j X_1^{m-i} \right) X_j$$

$$= \sum_{i=0}^{m} \left[(m-i)\alpha_i + \sum_{j=2}^{\beta} (\partial\alpha_i/\partial X_j) X_j \right] X_1^{m-i}.$$

Equating coefficients of powers of X_1 in the relation $G(X) = \gamma F(X)$, we get

$$(m-i)\alpha_i + \sum_{j=2}^{\beta} \partial\alpha_i/\partial X_j X_j = \gamma \alpha_i \qquad i = 0, \ldots, m;$$

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(2) It is easily seen from examples that $|L_P L_Q|$ does not always equal L'. For instance, let W be a non-linear curve in S^3, let Q be a simple point of W, and let $V = P$ be a point of L_Q different from Q and $\notin W$. Then $L_P = P$ and $|L_P L_Q| = L_Q$ but $|PW|$ has dim 2 and therefore $L_Q \neq L'$.

(3) See, for instance [10], p. 27.

or

$$\sum_{j=2}^{\beta} \partial\alpha_i/\partial X_j X_j = [\gamma-(m-i)]\ \alpha_i .$$

Therefore by the induction hypothesis, either $\alpha_i = 0$ or α_i is homogeneous of deg γ-$(m-i)$. Hence F is homogeneous of deg γ.

1.6 PROPOSITION. - *Let V^r be a variety in S^n defined over a field k of characteristic* 0, *let P be a generic point of V over k, and let T be the tangent linear variety to V at P. If W is a variety defined over k and contained in T, then V is a cone with W in its vertex.*

PROOF. - Suppose first $W = O$, and let $P = (x)$. Choose a transcendence basis for $k(x)$ over k, say $x_1, \ldots, x_r$, so the remaining $x_{r+1}, \ldots x_n$ are algebraic over $k(x_1, \ldots, x_r)$. Let $F_i(X_1, \ldots, X_r, X_{r+i})$ be the unique (up to a constant factor) irreducible polynomial in $k[X_1, \ldots, X_r, X_{r+i}]$ determined by $(x_1, \ldots, x_r, x_{r+i})$, so that if $\partial F_i/\partial x_{r+i} = 0$, then $\partial F_i/\partial X_{r+i} \varepsilon\, \vartheta(F_i(X))$ in $k[X_1, \ldots, X_r, X_{r+i}]$. But deg $\partial F_i/\partial X_{r+i}$ in X_{r+i} is $<$ deg F_i in X_{r+i}, so this is impossible. $\Delta_x F_i(X - x) = 0$ is a set of n-r linear equations such that $tr\, k \,\| \partial F_i/\partial x_j \| = n-r$, so these equations are a set of equations for T. Then $\Delta_x F_i(x) = 0$ since $O \varepsilon T$, so $\Delta_x F_i(X) = \gamma_i F_i(X)$ for some $\gamma_i \varepsilon \mathrm{k}[X_1, \ldots, X_r, X_{r+i}]$ (and hence in k). Therefore $F_i(X)$ is homogeneous of deg γ_i, by 1.5. The equations $F_i(X) = 0$ define n-r hypersurfaces, mutually orthogonal at (x). Hence by the criterion for unit multiplicity ([11], theorem 6, p. 152), their intersection contains a unique component through (x) of dim r, so V is this component. But if λ is transcendental over $k(x)$, $\lambda(x)$ is in this intersection and $\lambda(x) \to (x)$ over $\bar{k}$; so $\lambda(x) \varepsilon V$ also. Thus $|OV| = V$ and V is a cone with O in its vertex.

Suppose now $W \neq O$, and let (α) be a generic point of W over $k(x)$. Then (x) is generic for V over $K = k(\alpha)$. It follows from the previous case and 1.2 that V is a cone with (α) in its vertex, *i. e.* $(\alpha) + \lambda((x) - (\alpha)) \varepsilon V$. Therefore the locus of this point over k is contained in V. But this locus is by definition $|WV|$, so $|WV| = V$. Q. E. D.

Let S^n be an affine space obtained from P^n by choosing a hyperplane at ∞. If V, W are subvarieties of S^n, there exists uniquely determined representative cones V', W' in S^{n+1} for V, W. Moreover, one sees immediately from the definitions that the representative cone $|VW|'$ of $|VW|$ is the same as $|V'W'|$. Hence, we shall denote the uniquely determined subvariety of P^n having $|V'W'|$ as its representative cone also by $|V'W'|$.

1.7 THEOREM (DIM THEOREM). - *Let L^r and V^s be varieties in P^n with L linear, let k be a field of definition for L and V, and let T be the tangent linear variety to V at a generic point P of V over k. Then $r + s - dim\,(L \cap T) \leq$*

$\leq$ *dim* $|LV| \leq r + s + 1$ (*where dim* $\Phi = -1$). *Moreover, if the universal domain has characteristic* 0, *then* $r + s -$ *dim* $(L \cap T) =$ *dim* $|LV|$ [4].

PROOF. - Let Q be a generic point of L over $k(P)$, so that if (x), (y) are sets of independent homogeneous coordinates for P, Q, then the representative cone for $|LV|$ has $\mu(x) + \nu(y)$ as a generic point over k, where μ, ν are independent variables over $k(P, Q)$. Then dim $|LV| \leq r + s + 1$. On the other, hand, if R is the generic point of $|LV|$ having $\mu(x) + \nu(y)$ as homogeneous coordinates, then R is simple on $|LV|$ and is in $|QP|$; so by 1.4, $|LT| \subset T'$, where T' is the tangent linear variety to $|LV|$ at R. Therefore dim $T' \geq$ dim $|LT| = r + s -$ dim $(L \cap T)$. This proves the first assertion, and also the second when $L \cap T = \Phi$.

Suppose then $L_1 = L \cap T \neq \Phi$ and the universal domain has characteristic 0. If $L_1 = L$, then by 1.6 $V = |LV|$ and the theorem is proved; so we may assume $L_1 \subsetneq L$. By intersecting L with a generic linear variety over $k(P)$ of dim n-dim $L_1 - 1$, we obtain a subvariety L_2 of L such that (a) dim $L_2 = r -$ dim $L_1 - 1$, (b) $L_2 \cap L_1 = \Phi$, and (c) L_2 is defined over a purely transcendental extension $k(u)$ of k and P is generic for V over $k(u)$. Then $L_2 \cap T = \Phi$ implies, by the above case, that dim $|L_2V| =$ dim $L_2 + s + 1$. If Q_2 is a generic point of L_2 over $k(u, P)$, there exists a generic point R_2 for $|L_2V|$ over $k(u)$ such that $R_2 \varepsilon |Q_2P|$; so by 1.4, if T_2 is the tangent linear variety to $|L_2V|$ at R_2, then $|L_2T| \subset T_2$. Since $L = |L_2L_1| \subset |L_2T| \subset T_2$, by 1.6 $|L_2V|$ is a cone with L in its vertex. Hence, $|LV| = |L_2V|$, and therefore dim $|LV| = (r -$ dim $L_1 - 1) + s + 1 = r + s -$ dim $(L \cap T)$.

DEFINITION. - Let L^s and V^r be varietes in P^n with L linear, let k be a field of definition for L and V, and let T be the tangent linear variety to V at a generic point P of V over k. If dim $|LV| =$ dim $|LT|$, we shall say *V is an L-variety.*

The definition is independent of the choice of k and P and is equivalent to requiring that dim $|LV| = r + s -$ dim $(L \cap T)$. In some of our theorems we shall assume one of the varieties involved is an L-variety, and the dim theorem tells us then that in the case of characteristic 0 this hypothesis is superfluous. However, we shall give in section 4 examples which show that most of these statements are actually false in general without the added condition.

In the notation of the above definition, we list some instances of

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(4) This is a special case of [4], lemma 1, p. 144. However, it is not clear how one makes rigorous the proof suggested there, especially when $L \cap T \neq \Phi$.

L-varieties. V is an L variety if it satisfies any of the following:

i) V is linear

ii) $L \cap T = \Phi$

iii) $L \cap T$ is proper

iv) the universal domain has characteristic 0

v) $V = |LW|$ for some variety W.

(i) — (iv) are immediate from the definitions and the dimension theorem. (v) follows from the observation that $|LV| = V$ and from applying 1.4 to obtain $L \subset T$ and hence $|LT| = T$.

1.8 COROLLARY - *Let L^s and V^r be varieties in P^n such that L is linear and V is an L-variety, let k be a field of definition for L and V, and let T the tangent linear variety to V at a generic point P of V over k. Then dim $(L \cap T) \leq$ dim $(L \cap V)$.*

PROOF. - Suppose dim $(L \cap T) \geq$ dim $(L \cap V)$, let $t = \dim (L \cap T) \geq 0$, and let N^{n-t-1} be a generic linear variety over $k(P)$. Then $N \cap L = L_1$ is a linear variety of dim $s - t - 1$ (or else $L_1 = \Phi$), and there exists an extension field K of k such that L_1 is defined over K and P is generic for V over K. Moreover $N \cap (L \cap T) = \Phi$ implies $L_1 \cap T = \Phi$. $\therefore$ $\dim |L_1 V| = (s - t - 1) + r + 1$ by the dimension theorem. Since $\dim |LV| = s + r - t$, this implies $|L_1 V| = |LV|$ and hence $L \subset |L_1 V|$. But $L_1 \cap V = \Phi$ since $N \cap (L \cap T) = \Phi$ and dim $(L \cap V) \leq$ dim $(L \cap T)$, so every point of $|L_1 V|$ lies on a variety of the form $|L_1 Q|$ with $Q \varepsilon V$. In particular then, every point of L lies on such a variety; so $L \subset |L_1 V_1|$, where V_1 is a component of $L \cap V$. Therefore $|L_1 V_1| = L$. But dim $|L_1 V_1| \leq$ dim $L_1 +$ dim $V_1 + 1$ by the dimension theorem, so $s \leq (s - t - 1) + \dim V_1 + 1$ and $t \leq \dim V_1$.

1.9 PROPOSITION. - *Let L^r and V^s be varieties in P^n such that L is linear and $L \cap V = \Phi$. Then dim $|LV| = r + s + 1$.* (5)

PROOF. - Suppose dim $|LV| < r + s + 1$. If P is any point of L, dim $|PV| = s + 1$ since $P \notin V$. $\therefore$ if $L = L_r \supset L_{r-1} \supset \ldots \supset L_1 \supset L_0$ is a properly decreasing sequence of $r + 1$ linear varieties (obtained for instance by cutting down with generic hyperplanes), then $|L_{r-i+1} V| = |L_{r-i} V|$ for some $i = 1, \ldots, r$. For if not, $r + s + 1 > \dim |L_r V| > \dim |L_{r-1} V| > \ldots >$

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(5) This is proved also in [1], p. 3-03. Note that when V is an L-variety the proposition follows from 1.8 and the dim theorem.

dim $|L_1V| > \dim |L_0V| = s+1$ which is impossible. But if Q is a point of L_{r-i+1} such that $Q \notin L_{r-i}$, then $|L_{r-i}Q| = L_{r-i+1}$. Moreover, since $L_{r-i} \cap V = \Phi$ and $Q \varepsilon |L_{r-i}V|$, there exists a point $R \varepsilon V$ such that $Q \varepsilon |L_{r-i}R|$. $\therefore |L_{r-i}R| = |L_{r-i}Q| = L_{r-i+1}$, so $R \varepsilon L_{r-i+1} \cap V$, a contradiction to the hypothesis that $L \cap V = \Phi$.

2. The degree and simple point theorems.

2.1 PROPOSITION. - *Let V^r and W^t be varietes in S^n defined over a field k, and let L^s be a generic linear variety over k in S^n such that $L \cap V = \Phi$. If C is a component of $|LV| \cap W$ not contained in V, then C is proper and general over k for W(i.e. C contains a generic point of W over k).*

PROOF. - Suppose first that $(r+s+1)+t-n < 0$. Dim $|LV| = r+s+1$ by 1.9, so we must see in this case that $|LV| \cap W = \Phi$ outside V. Dim $|VW| \leq r+t+1 < n-s$, so $L \cap |VW|$ is proper implies $L \cap |VW| = \Phi$. But $L \cap V = \Phi$ by hypothesis; so if $|LV| \cap W$ contains a point $Q \in V$, there exists a point $P \varepsilon V$ such that $Q \varepsilon |LP|$. Then $|QP| \cap L \neq \Phi$, so $|VW| \cap L \neq \Phi$, a contradiction.

Consider now the case where $(r+s+1)+t-n \geq 0$, and suppose C is an improper component of $|LV| \cap W$ which is not contained in V. Then $\alpha = \dim C > (r+s+1)+t-n$; so if $N^{n-\alpha}$ is a generic linear variety over the field $K = k(u)$, where (u) is the set of coefficients in the defining equations of L, then $N \cap C$ has dim 0 and $N \cap W$ has dim $t-\alpha$. Let Q be a point of $N \cap C$ and W' a component of $N \cap W$ containing Q. $Q \varepsilon |LV| \cap W'$ and $Q \in V$ since Q is generic over $\bar{K}$ for C. But we can apply the previous case to conclude $|LV| \cap W' = \Phi$ outside V, a contradiction.

For the second assertion, let $\bar{C}$ denote the $\bar{k}$-closure of C. Then $C \subset \bar{C} \subset W$, so C is a proper component of $|LV| \cap W$ and of $|LV| \cap \bar{C}$. This implies dim $\bar{C} = \dim W$, and hence $\bar{C} = W$. Q. E. D.

Notice that the condition $L \cap V = \Phi$ in proposition 2.1 is equivalent to the requirement dim $|LV| = r+s+1$.

We take the liberty at this point to make a slight addition to the language for the sake of convenience: We shall say a *linear variety L is transversal to a variety V along a subvariety U of $L \cap V$* provided L is transversal to V at some point of U. If k is a field of definition for L, V, U and L is transversal to V along U, then L is transversal to V at any generic point of U over k.

The next proposition plays a crucial role in what follows. In the case of characteristic 0 it follows from [11], prop. 14, p. 131, via the converse criterion for unit multiplicity. We give here a proof using no intersection multiplicity.

2.2 PROPOSITION. - *Let k be a field of definition for a variety V^r and, a linear variety L^s in P^n, let V be an L-variety, and let M^t be a generic linear variety over k in P^n such that $t \leq n - s - 2$. Moreover, suppose there exists a component C^α of $|ML| \cap V$ such that $C \not\subset L$. Then $|ML|$ is transversal to V along C.*

PROOF. - By 2.1, C is proper and general over k for V. Therefore, there exists a point $P \varepsilon C$ such that P is generic over k for V. $P \notin L$ implies $|LP| \cap M \neq \Phi$ and hence $|LV| \cap M \neq \Phi$, so dim $|LV| \geq n$-t. Let T be the tangent linear variety to V at P, and consider two cases:

CASE 1. - $t = 0$. Dim $(T \cap L) =$ dim $L +$ dim $V -$ dim $|LV| \leq s + r - (n - t)$ by hypothesis. Since $t = 0$, dim $(T \cap L) \leq s + r - n$; so $T \cap L$ is proper if $\neq \Phi$. Therefore $T \cap |ML|$ is also proper if $T \cap L \neq \Phi$. On the other hand, if $T \cap L = \Phi$ and $T \cap |ML| \neq \Phi$, then dim $(T \cap |ML|) = 0$ since L is a divisor in $|ML|$. But if $T \cap |ML|$ is a point not in L, then $T \cap |ML|$ is improper only if $r + (s + 1) - n < 0$; and this is impossible since $\alpha = (s + 1) + r - n$.

CASE 2 - $t > 0$. M has a homogeneous generic point over $k(x^0, \ldots, x^t)$ of the form $\mu_0(x^0) + \ldots + \mu_t(x^t)$, where $(x^0, \ldots, x^t)$ are algebraically independent quantities over k and the μ_i are indeterminates over $k(x^0, \ldots, x^t)$. Let $Q = (x^0)$ and let M_1^{t-1} be the linear variety having homogeneous generic point $\mu_1(x^1) + \ldots + \mu_t(x^t)$ over $k(x^1, \ldots, x^t) = K$. Then M_1 is generic over k, so $L_1 = |M_1 L|$ has dim $s + (t - 1) + 1 = s + t$ by the dimension theorem. Applying 2.1, any component of $L_1 \cap V$ not contained in L has dim $(s + t) + r - n < \alpha$; so $C \not\subset L_1$.

Also, V is an L_1-variety. For, let R be a generic point of V over K and let T_1 be the tangent linear variety to V at R. Then M_1 is generic over $k(R)$; and therefore $L_1 \cap T_1$ is improper implies $L_1 \cap T \subset L$. Claim: $L_1 \cap T_1$ is proper. For, if not, dim $(L \cap T_1) \geq$ dim $(L_1 \cap T_1) > (s + t) + r - n$. But dim $(L \cap T) = s + r -$ dim $|LV|$ and dim $|LV| \geq n - t$, by hypothesis, so dim $(L \cap T_1) \leq s + r - (n - t)$, a contradiction. Therefore $L_1 \cap T_1$ is proper, so V is an L_1-variety.

Thus, we can apply case 1 to conclude $|QL_1|$ is transversal to V along C. But $|QL_1| = |ML|$. Q. E. D.

In what follows we use the concept of *degree* of a variety. To make it clear that we are keeping our resolve to use no intersection multiplicity other than the criterion for unit multiplicity and its converse we shall state here explicitly what is used. If V^r is a variety in P^n and k a field of definition for V, let L^{n-r} be a generic linear variety over k in P^n, and let d be the number of points in $V \cap L$. It follows from the criterion for unit multiplicity that if M^{n-r} is any other variety such that $M \cap V$ is proper and such that M is transversal to V at every point of $M \cap V$, then $d =$ number of points in $M \cap V$ also. By deg V we shall mean this number d.

We use one further fact, namely that a variety is of deg 1 if and only if it is linear. One sees this as follows: Let V be a variety in P^n, let k be a field of definition for V, and let P be a generic point of V over k. If then S^n is an affine space containing V at finite distance, let (x) be a set of coordinates for P in S^n. There exists coodinates $x_{i_1}, \ldots, x_{i_r}$ algebraically independent over k such that $x_{i_{r+1}}, \ldots, x_{i_n}$ are separably algebraic over $k(x_{i_1}, \ldots, x_{i_r}) = K$. Then $X_{i_j} - x_{i_j} = 0$ $(j = 1, \ldots, r)$ is a set of equations for a linear variety L of dim $n-r$ in S^n; and if $F^{\mu}(X_{i_1}, \ldots, X_{i_r}, X_{i_{r+\mu}})$ is the unique (up to a constant factor) irreducible polynominal in $k[X_{i_1}, \ldots, X_{i_r}, X_{i_{r+\mu}}]$ determined by $(x_{i_1}, \ldots, x_{i_r}, x_{i_{r+\mu}})$, then $\Delta_x F_{\mu}(X - x) = 0$ is a set of equations for the tangent linear variety T to V at (x), by reasoning similar to that in the proof of 1.6. Therefore L is transversal to V at (x); and since any other point of $L \cap V$ in S^n is a conjugate of (x) over K, we see that L is transversal to V at every point of $L \cap V$ in P^n; so if deg $V = 1$, $L \cap V = P$. Therefore F_{μ} has deg 1 in $X_{i_{r+\mu}}$, since otherwise P specializes over K to another point in $L \cap V$ (in P^n). By the fact L does not intersect V in the hyperplane at ∞, F_{μ} must then be linear. Therefore $T = V$ and V is linear. The converse statement that a linear variety has deg 1 follows from [11], corollary, p. 91, which asserts the intersection of two linear varieties is again a variety if it is $\neq \Phi$.

2.3. LEMMA. - *Let V^r be a cone in P^n which is not linear. Then every point of the vertex is singular on V.*

PROOF. - If P is in the vertex of V, $| PV | = V$; so whenever $Q \neq P$ is in V, then $| PQ | \subseteq V$. Suppose P is simple on V, and let L^{n-r-1} be a generic linear variety over $k(P)$. Then if T is the tangent linear variety to V at P $T \cap | PL |$ and $V \cap | PL |$ are both proper, by 2.1. Therefore $V \cap |PL| = P$ since if $Q \neq P$ is in $| PL | \cap V$, then $| PQ | \subset | PL | \cap V$ and the intersetion is improper. Thus, deg $V = 1$ and V is linear, a contradiction.

2.4. LEMMA. - *Let V be a variety in S^n defined over a field k; let (x) and (z) be points of V, and (x', x') a finite specialization of (x, z) over k with coordinates x'_i in k. Let t be any quantity, and let (y) be the point defined by*

$$y_i = z_i + t(x_i - z_i) \qquad (1 \leq i \leq n).$$

Then every finite specialization $(y) \longrightarrow (y')$ over $(x, z) \longrightarrow (x', x')$ over k is a point of the linear variety attached to V at (x').

PROOF. - The proof is an immediate generalization of [11], prop. 20 p. 98.

2.5. LEMMA. - *Let k be a field of definition for a variety V^r and a linear variety L^s in P^n, let P_1, P_2 be independent generic points of V over k, and suppose dim $(|LP_i| \cap V) = 0$. If then μ_i is the number of points of $|LP_i| \cap V$ which are not in L, we have $\mu_1 = \mu_2$.*

PROOF. - The specialization $P_1 <-> P_2$ over k extends to a generic specialization $(P_1, Q_1, \ldots, Q_{\mu_1}) <-> (P_2, \bar{Q}_1, \ldots, \bar{Q}_{\mu_1})$, where the Q_i are the points of $|LP_1| \cap V$ not in L. Then the $\bar{Q}_i$ are in $|LP_2| \cap V$ and are distinct and not in L because the specialization is generic. $\therefore \mu_2 \leq \mu_1$.

2.6. LEMMA. - *Let L^s and V^r be varieties in P^n with L linear and such that $L \cap V$ has dim ≤ 0; let P be any point of $|LV|$; and suppose whenever Q is a point of $L \cap V$, $|LP|$ intersects the linear variety attached to V at Q in just Q. Then there exists a point R in V such that $P \varepsilon |LR|$.*

PROOF. - If $P \varepsilon L$, the lemma is immediate; so assume $P \notin L$. Let k be a field of definition for L and V and the points of $L \cap V$; let R_1 be a generic point of $|LR_1|$ over $k(R_1)$ and hence also a generic point of $|LV|$ over k. Then $P_1 -> P$ over k extends to a specialization $(P_1, R_1, |LR_1|) -> (P, R, L')$, where L' is a linear variety of dim $s+1$ containing P, R, and L. Also, $L' = |LR|$ if $R \notin L$, and in this case we are done. Suppose then $R \varepsilon L \cap V$. $(R_1, |LR_1|, |RR_1|) -> (R, L', l')$ over k, where l' is a line contained in the linear variety T attached to V at R, by [11], prop. 20, p. 98. But then $l' \subset L' \cap T = |LP| \cap T$. This is a contradiction to the hypothesis that $|LP| \cap T$ has dim 0.

2.7 LEMMA - *Let k be a field of definition for a linear variety L^s and a variety V^r in P^n, let P be a generic point of V over k, let $L_1 = |LP|$, and suppose V is an L-variety and $|L_1V| \neq |LV|$. Then both V and $|LV|$ are L_1-varieties.*

PROOF. - Let Q be a generic point of V over $k(P)$, so that P is then generic for V over $k(Q)$; and let T be the tangent linear variety to V at Q. Since dim $|L_1V| =$ dim $|LV| + 1$, to see V is an L_1-variety we must verify that dim $(T \cap L_1) =$ dim $(T \cap L)$. If $T \cap L \subset T \cap L_1$, then $|LT| = |L_1T|$ and therefore $P \varepsilon |LT|$. Since $|LT|$ is defined over $k(Q)$, this implies $V \subset |LT|$; so $|L_1V| \subset |LT|$. But then dim $|L_1V| \leq$ dim $|LT| =$ dim $|LV|$ by the hypothesis that V is an L-variety. This is impossible because $|L_1V| \neq |LV|$, so V is an L_1-variety.

We now apply the result to conclude $|LV|$ is also an L_1-variety. For, $|LV|$ is an L-variety; $L_1 = |LR|$, where R is a generic point for $|LV|$ over k since there exists such an R in $|LP|$; and $|L_1|LV|| \neq |L|LV||$ because $|L_1|LV|| = |L_1V|$ and $|L|LV|| = |LV|$.

2.8 THEOREM (DEGREE THEOREM). *Let k be a field of definition for a variety V^r and a linear variety L^s in P^n, let P be a generic point of V over k, and suppose $|LP| \cap V$ consists of a finite set of points $P_1, \ldots, P_{\mu+\nu}$* such that each P_i is simple on V and $|LP|$ intersects the tangent linear variety to V at P_i in just P_i. Then $\nu . \deg |LV| = \deg V - \mu$.

PROOF. - To begin, observe that $P \varepsilon L$ imples $r = 0$ and the theorem is trivially true, so we may assume $P \notin L$ and hence $\nu > 0$. Also, since P is generic over k for V and the tangent T to V at P intersects $|LP|$ in just P, $T \cap L = \Phi$, so by the dim theorem, dim $|LV| = r + s + 1$. Let $L_1 = |LP|$. If the P_i are simple on $|LV|$, then $L_1 \cap V$ is proper in $|LV|$; and if in particular $|LV|$ is linear, then L_1 is transversal to V at P_i in $|LV|$ (where $|LV|$ is considered as P^{r+s+1}). Therefore in this case we have deg $V = \mu + \nu$, which agrees with the statement of the theorem. Thus, we may further assume $|LV|$ is not linear.

Let now N^t be a generic linear variety over $k(P)$ with $t = n - (r+s+2)$. $t \geq 0$ since $n - (r + s + 1) > 0$ by hypothesis. L_1 has dim $s + 1$ and is defined over $k(P)$, so $N \cap L_1 = \Phi$; and therefore dim $|NL_1| = n - r$ by the dim theorem. By 2.1, any component of $|NL_1| \cap V$ not in L_1 is proper and general over $k(P)$ for V and hence has dim 0. Therefore $|NL_1| \cap V = P_1 \cup \ldots \cup P_{\mu+\nu} \cup Q_1 \cup \ldots \cup Q_\tau$, with $Q_i \notin L_1$. Also, if T_i is the tangent linear variety to V at P_i, $|NL_1| \cap T_i$ is proper. For, T_i is defined over $\overline{k(P)}$; and therefore by 2.1, either the intersection is proper or is contained in L_1. If the intersection is contained in L_1, it must be of dim 0 by our hypothesis. Thus, $|NL_1|$ is transversal to V at P_i. Claim: $|NL_1|$ is also transversal to V at Q_i. For, $k(P)$ is a field of definition for V and L_1; N^t is generic over $k(P)$ with $t \leq n - (s + 1) - 2$; Q_i is a component of $|NL_1| \cap V$ not contained in L_1; and by 2.7 V is an L_1-variety except when $|L_1 V| = |LV|$. Thus, 2.2 applies when $|L_1 V| \neq |LV|$, and we can conclude in this case that $|NL_1|$ is transversal to V at Q_i. But $|L_1 V| = |LV|$ implies $|LV|$ is a cone with P in its vertex; so by 2.3 and the fact we have assumed $|LV|$ is not linear, P is in the singular locus of $|LV|$. But if Q is a generic point of L over $k(P)$, $|QP|$ contains a point which is generic over k for $|LV|$ and hence which is simple on $|LV|$. Therefore, by 1.3, P must be simple on $|LV|$; so $|L_1 V| \neq |LV|$ and 2.2 always applies. Thus, deg $V = \mu + \nu + \tau$.

On the other hand, by 2.1 any component of $|NL_1| \cap |LV|$ not contained in L_1 is proper, so $L_1, |LQ_1|, \ldots, |LQ_\tau|$ are components of $|NL_1| \cap |LV|$. Suppose R is a point of $|NL_1| \cap |LV|$ not in L_1. $|LR| \subset |NL_1|$ and $|NL_1|$ intersects the tangent linear variety to V at any point of $L \cap V$ in just that point, so $|LR|$ also has this property. Therefore we can apply 2.6 to conclude there exists a point $R_1 \varepsilon V$ such that $|LR| = |LR_1|$. Then $R_1 \varepsilon |NL_1|$ also, so $R_1 \varepsilon |NL_1| \cap V$. Thus R_1 is some Q_i and $R \varepsilon L_1 \cup |LQ_1| \cup \ldots \cup |LQ_\tau|$.

Hence $|NL_1| \cap |LV| = L_1 \cup |LQ_1| \cup \dots \cup |LQ_\tau|$. We have already observed that Q_i is generic over $k(P)$ for V, so we can apply 2.5 to conclude $|LQ_1|$ intersects V in the same number of points outside L as L_1. Therefore $|LQ_1| \cap V$ contains exactly ν points not in L. But these are points of $|NL_1| \cap V$, so $|LQ_{i_1}| = \dots = |LQ_{i_\nu}|$ for some $Q_{i_1}, \dots, Q_{i_\nu}$. Thus, exactly $\frac{\tau}{\nu}$ of of the varieties $|LQ_1|, \dots, |LQ_\tau|$ are distinct. Therefore $|NL_1| \cap |LV|$ contains exactly $\frac{\tau}{\nu} + 1$ distinct components.

Furthermore, L_1 is simple on $|LV|$ because it contains a generic point of $|LV|$ over k. Therefore there exists a point $A \varepsilon L_1$ such that A is rational over $\overline{k(P)}$ and simple on $|LV|$. If T_A is the tangent linear variety to $|LV|$ at A, $|NL_1| \cap T_A$ is proper by 2.1; so $|NL_1|$ is transversal to $|LV|$ along L_1. But also $\dim |L_1|LV|| > \dim |LV|$ since $|LV|$ is not a cone with P in its vertex, as we have seen previously; so, since by 2.7 $|LV|$ is an L_1-variety, 2.2 applies and $|NL_1|$ is transverval to $|LV|$ along each $|LQ_i|$. Thus, $\deg |LV| = \frac{\tau}{\nu} + 1$. This combined with the above relation for $\deg V$ is the desired result. Q. E. D.

In the next theorem we make use for the first time of the converse criterion for unit multiplicity ([11], theorem 6, p. 152). We use the criterion in the following form: Let V^r be a variety, and let L^{n-r} be a linear variety such that $L \cap V = P_1 \cap \dots \cap P_\mu$. Suppose moreover that L is transversal to V at $P_2, \dots, P_\mu$ and $\deg V = \mu$. By the criterion for unit multiplicity and its converse, we can then conclude that L is transversal to V at P_1 More generally, suppose M is a linear variety such that $M \cap V$ is proper and $= M_1 \cup \dots \cup M_\mu$ with the M_i all linear; and suppose M is transversal to V along $M_2, \dots, M_\mu$ and $\deg V = \mu$. Then by cutting down with an appropriate linear variety and reducing to the previous case, we see that M is also transversal to V along M_1. By the converse criterion it now follows that if P is any point of M_1 which is not in any other M_i, then P is simple on V.

2.9 LEMMA. - *Let V be a variety in P^n, let L and L_1 be linear varieties in P^n such that $L \subset L_1$, and suppose V is an L-variety. Then V is an L_1-variety if and only if $|LV|$ is an L_1-variety.*

PROOF. - Let P, Q be independent generic points of L, V respectively over a field k of definition for both L and V; and let T be the tangent linear variety to V at Q and T' the tangent linear variety to $|LV|$ at a generic point of $|PQ|$ over $k(P, Q)$. Then since V is an L-variety and $|LT| \subset T'$ by 1.4, $|LT| = T'$. $\therefore |L_1T'| = |L_1|LT|| = |L_1T|$, and $|L_1|LV|| = |L_1V|$, so $\dim |L_1T| = \dim |L_1|LV||$ if and only if $\dim |L_1T| = \dim |L_1V|$.

2.10 THEOREM (SIMPLE POINT THEOREM). - *Let k be a field of definition for a variety V^r and a linear variety L^s in P^n, let P be a point of V not in the vertex of $|LV|$ and such that V is an $|LP|$-variety, and let T be the linear variety attached to V at P. Suppose moreover that $|LP| \cap V = P$, and that $|LQ| \cap V = Q$ whenever Q is a generic point of V over k.*

Then if $|LP| \cap T = P$ and P is simple on V, every point of $|LP|$ not in L is simple on $|LV|$; and conversely, if $|LP|$ is simple on $|LV|$, then $|LP| \cap T = P$ and P is simple on V.

PROOF. - Let $L_1 = |LP|$, and let N^t be a generic linear variety over $k(P)$ with $t = n - (r + s + 2)$. Since $|LV|$ is a cone with L in its vertex and by hypothesis $P \notin$ the vertex of $|LV|$, $P \notin L$. Therefore $\dim |NL_1| = n - r$ and by 1.9 $\dim |LV| = r + s + 1$. By 2.1 every component of $|NL_1| \cap V$ not contained in L_1 is proper and general over k for V. Since by hypothesis $L_1 \cap V = P$, this implies $|NL_1| \cap V$ is proper and $= P \cup P_2 \cup ... \cup P_\mu$ with $P_i \notin L_1$ and generic over k for V. Similarly, $|NL_1| \cap |LV|$ is proper and contains $L_1, |LP_2|, ..., |LP_\mu|$ as components. Suppose R is a point of $|NL_1| \cap |LV|$ not in L_1. $|LR| \subset |NL_1| \cap |LV|$ and $L \cap V = \Phi$; so by 2.6, there exists a point $R_1 \varepsilon V$ such that $|LR| = |LR_1|$. Then $R_1 \varepsilon |NL_1| \cap V$. Therefore R_1 is some P_i and $R \varepsilon L_1 \cup ... \cup |LP_\mu|$. Hence $|NL_1| \cap |LV| = L_1 \cup |LP_2| \cup ... \cup |LP_\mu|$. Moreover, since P_i is generic over k for V, $|LP_i| \cap V = P_i$ by hypothesis; so $|LP_i| \neq |LP_j|$ for $i \neq j$.

To apply 2.2, we must verify that $t \leq n - (s + 1) - 2$ and $\dim |L_1 V| \leq n - t = r + s + 2$. The first inequality follows if we assume $r > 0$, which we may do because the theorem is otherwise trivially true. The second inequality follows from the observation that $\dim |LV| = r + s + 1$ and the fact that $|LV| \subset |L_1 V|$ because P is not in the vertex of $|LV|$. Therefore by 2.2, we can conclude that $|NL_1|$ is transversal to $|LV|$ along $|LP_1|$; for, the hypothesis that V is an L_1-variety along with the inequality $\dim |L_1 V| \geq r + s + 2$ imply V is an L-variety, and then by 2.9 $|LV|$ is an L_1-variety. Finally, observe that $\deg V = \deg |LV|$ by the degree theorem. For, if Q is generic for V over k, $|LQ| \cap V = Q$ by hypothesis; and if T_Q is the tangent linear variety to V at Q, $|LQ| \cap T_Q = Q$ since $L \cap T_Q = \Phi$ by the condition $\dim |LV| = r + s + 1$ and the above observation that V is an L-variety.

Suppose then that $|LV| \cap T = P$ and P is simple on V. It follows from 2.1 that $|NL_1| \cap T$ is proper; so $|NL_1|$ is transversal to V at P also. Therefore $\deg V = \mu = \deg |LV|$. Hence, by the converse criterion for unit multiplicity, every point of L_1 not in L is simple on $|LV|$.

Conversely, suppose L_1 is simple on $|LV|$. Then $|NL_1|$ is transversal to $|LV|$ along L_1; for $|NL_1|$ intersects properly the tangent linear variety to $|LV|$ at any point of L_1 which is rational over $\overline{k(P)}$ and simple on $|LV|$, by 2·1. Hence $\deg |LV| = \mu = \deg V$. Therefore by the converse criterion for unit multiplicity, $|NL_1|$ is transversal to V at P, *i. e.* P is simple on V and $|NL_1| \cap T = P$. Then a fortiori, $L_1 \cap T = P$.

2.11 Corollary. - *Let k be a field of definition for a variety V^r in P^n, let L^s be a generic linear variety over k in P^n with $s \leq n - r - 2$, let P be a point of V rational over k and simple on V and such that V is a P-variety, and let P' be any point of $|LP|$ not in L. Then P' is simple on $|LV|$; and if T is the tangent linear variety to V at P and T' the tangent linear variety to $|LV|$ at P', then $T' = |LT|$.*

Proof. - We may assume V is not linear since then the corollary is trivial. Let $k(u)$ be a purely transcendental extension of k over which L is defined, and let Q be a generic point of V over $k(u)$. Then $|LP| \cap V = P$, $|LP| \cap T = P$, and $|LQ| \cap V = Q$, by 2.1. Moreover, if T_Q is the tangent linear variety to V at Q, $|LP| \cap T_Q = \Phi$. For by 2.1, if the intersection is $\neq \Phi$, it must be P. But then V is a cone with P in its vertex, by the hypothesis that V is a P-variety. Since P is simple on V by hypothesis, 2.3 implies V must be linear, a contradiction to our assumption. Therefore $|LP| \cap T_Q = \Phi$, so V is an $|LP|$-variety and dim $||LP|V| >$ dim $|LV|$ by the dim theorem. Thus P is not in the vertex of $|LV|$, and we can apply the simple point theorem to conclude P' is simple on $|LV|$. Finally by the dim theorem we have dim $|LV|$ = dim $|LT|$ since $L \cap T_Q = L \cap T = \Phi$. Therefore dim T' = dim $|LT|$, so the second assertion follows from 1.4 when $P' \neq P$ and 1.3 when $P' = P$.

3. Intersection properties.

3.1 Proposition. - *Let V^r be a variety and L^s a linear variety in P^n such that $L \cap V = \Phi$, and let W^t be a variety in P^n such that $t \geq n - s - 1$ and $L \cap W$ is proper. Then (i) every component W_1 of $|LV| \cap W$ is proper; and (ii) if k is a field of definition for V, L, W and P is a point of V, then $|LP|$ contains a generic point of W_1 over $\bar{k}$ if and only if P is generic over k for V.*

Proof. - By 1.9 dim $|LV| = r + s + 1$, and then $W_1 \not\subset L$ since $L \cap W$ is proper and W_1 is a component of $|LV| \cap W$. Let P_1 be a generic point of W_1 over $\bar{k}$ and let P be a point of V such that $P_1 \varepsilon |LP|$. There exists at least one such P since $V \cap L = \Phi$. Then the locus U of P_1 over $\overline{k(P)}$ is contained in $|LP|$ and not contained in L. Therefore $U \cap L$ is proper in $|LP|$, since L is a divisor in $|LP|$; so dim $(U \cap L)$ = dim $U - 1$ (where dim $\Phi = -1$).

Thus, dim $U - 1$ = dim $(U \cap V) \leq$ dim $(W \cap L) = t + s - n$, where the inequality follows from the fact $U \subset W$ if $U \cap L \neq \Phi$ and from the condition $t \geq n - s - 1$ when $U \cap L = \Phi$. But dim $P/k \geq$ dim P_1/k — dim $P_1/k(P) =$ $=$ dim W_1 — dim $U \geq (r + s + 1 + t - n) - (t + s - n + 1) = r$. Therefore P

is generic over k for V. Also, since the equalities now hold, we have $\dim W_1 = \dim U + r \leq (t + s - n + 1) + r$; so W_1 is proper in $|LV| \cap W$.

Conversely suppose P is generic over k for V. Let Q_1 be a generic point of W_1 over $\bar{k}$, and let Q be a point of V such that $Q_1 \varepsilon |LQ|$. Then Q is generic over k for V by the above result, so $P <--> Q$ over $\bar{k}$. This specialization extends to a generic specialization $(|LQ|, Q_1) <--> (|LP|, P_1)$ over $\bar{k}$, where P_1 is then evidently a point of $|LP|$ which is generic over $\bar{k}$ for W_1.

3.2 LEMMA. - *Let V be a variety in S^n defined over a field k, let $\mathfrak{L}$ denote the set of points which are in every linear variety attached to V, let π be the set of points P of S^n such that V is not a P-variety, and let $\mathfrak{S}$ be the singular locus of V. Then $\mathfrak{L}$ is a $\bar{k}$-closed set, and either V is linear or $\mathfrak{L} \cap \mathfrak{S} = \pi \cap \mathfrak{S}$.*

PROOF. - Let $F_\mu(X) = 0$ be a set of equations for V over k. Then $\Delta_X F_\mu(Y - X) = 0$ define a k-closed set $\mathfrak{F}$ in $S^n X S^n$. If $\mathfrak{L}'$ is the set of points which are in every linear variety attached to V at those points of V which are rational over $\bar{k}$, then $\mathfrak{L}'$ is a $\bar{k}$-closed set also. Then $V X \mathfrak{L}'$ is a $\bar{k}$-closed set such that every point of it which is rational over $\bar{k}$ is in $\mathfrak{F}$, so $VX\mathfrak{L}' \subset \mathfrak{F}$. But this means $\mathfrak{L}' \subset \mathfrak{L}$, and hence $\mathfrak{L}' = \mathfrak{L}$.

Suppose now V is not linear. If $Q \varepsilon \pi$, then either $|QV| = V$ or Q is in the tangent linear variety T to V at a generic point of V over $k(Q)$. If $|QV| = V$, $Q \varepsilon \mathfrak{S}$ by 2.3. In either case then, $Q \varepsilon \mathfrak{L} \cap \mathfrak{S}$. Conversely, if $Q \varepsilon \mathfrak{L}$ and $\notin \mathfrak{S}$, then $Q \varepsilon T$ and $|QV| \neq V$, so $Q \varepsilon \pi$.

3.3 THEOREM - *Let k be a field of definition for varieties V^r and W^t in P^n, and let L^s be a generic linear variely over k in P^n such that $s \leq n - r - 2$ and $t \geq n - s - 1$.(6) If there exists a point $P \varepsilon V$ such that W is a P-variety and P is not in the singular locus of W, then $|LV|$ is transversal to W along every component of $|LV| \cap W$.* (7)

PROOF. - Let $k(u)$ be a purely transcendental extension of k over which L is defined, let P be a generic point of V over $\overline{k(u)} = K$, let W_1 be a component of $|LV| \cap W$ and hence not contained in L, and suppose V is not contained in the singular locus of W. Then there exists a point $Q \varepsilon V$ such that Q is rational over k and Q is simple on V and is not in the

(6) One sees from simple examples that $t \geq n$-s-1 is needed in order for the components of $|LV| \cap W$ to be always proper; for instance, consider $r = 0$, $t = 1$, $s = 0$, $n = 3$.

(7) Chow mentions in [2], p. 459, that this is true when $V \subset W$ and $t = n$-s-1; but he proves there a more special result. Chow however makes no mention of an extra hypothesis in the case of characteristc $p \neq 0$, and we shall give in section 4 an example where $V \subset W$ and $t = n$-s-1 but for which the theorem is not true without the condition that V contains a point P such that W is a P-variety.

singular locus of W; and furthermore, by 3.2, Q may be chosen so that both V and W are Q-varietes. Moreover, $|LP| \cap W$ and $|LQ| \cap W$ are proper and $|LP|$ contains a point P_1 which is generic over k for W_1 by 3.1. Therefore if C is a component of $|LP| \cap W$ containing P_1, then $C \subset W_1$. But W_1 is defined over k; so $(|LP|, C) \longrightarrow (|LQ|, X')$ over K, where X' is a K-closed set contained in W_1 and such that any component C' of X' is a component of $|LQ| \cap W$ ([8], p. 147, prop. 18).

$|LQ|$ is transversal to W along C' by 2.2. Also since both $L \cap W$ and $|LQ| \cap W$ are proper and hence C' cannot be a component of both, $C' \not\subset V$. Thus, if Q' is a generic point of C' over K, then Q' is simple on W and the tangent linear variety M to W at Q' intersects $|LQ|$ properly. Moreover, by 2.11, Q' is simple on $|LV|$, and if T' is the tangent linear variety to $|LV|$ at Q' and T the tangent linear variety to V at Q', then $T' = |LT|$. But $M \cap T'$ is proper whenever $M \cap |LQ|$ is proper, since $|LQ| \subset T'$ and $M \cap |LQ| \neq \Phi$; so $M \cap T'$ is also proper. Therefore $|LV|$ is transversal to W at Q'.

3.4 PROPOSITION - *Let V^r be a subvariety of a variety W^t in P^n, let P be a point of V simple on W, and let T be the tangent linear variety to W at P. If L^s is a linear variety in P^n such that $L \cap T = \Phi$ and $|LP| \cap V = P$, then P is in a unique component of $|LV| \cap W$.*

PROOF. - Suppose P is in two components W_1, W_2 of $|LV| \cap W$. Since $V \subset |LV| \cap W$, at least one of these, say W_1, is not contained in V. Let k be a field of definition for L, V, W, W_1, P; *and let* P_1 be a generic point of W_1 over k. Since $L \cap V = \Phi$, there exists a point $Q \varepsilon V$ such that $P_1 \varepsilon |LQ|$; and then $P_1 \neq Q$. Moreover, $P_1 \notin L$; for $P_1 \varepsilon L$ implies $W_1 \subset L$ and hence $P \varepsilon L \cap T$, a contradiction to our hypothesis. Therefore dim $|LP_1| = s+1$, so $(P_1, |LP_1|) \longrightarrow (P, |LP|)$ over k. This specialization then extends to $(P_1, Q, |LP_1|, |P_1Q|) \longrightarrow (P, Q', |LP|, l)$, where $Q' \varepsilon |LP| \cap V = P$. Therefore by 2.4, $l \subset T$. But $|P_1Q| \cap L \neq \Phi$, so $l \cap L \neq \Phi$; and consequently $T \cap L \neq \Phi$, a contradiction.

3.5. COROLLARY - *Let V^r be a subvariety of a variety W^t in P^n, let P be a point of V simple on W, and let k be a field of definition for V, W, P. If L^s is a generic linear variety over k in P^n such that $t \leq n-s-1$, then P is in a unique component of $|LV| \cap W$.* (8)

PROOF. - Apply 3.4.

3.6 THEOREM - *Let k be a field of definition for V^r and W^t in P^n, let P be a point of $V \cap W$ rational over k and simple on W, and L^s be a generic*

(8) Van der Waerden proves a special case of this in [9], p. 635.

linear variety over k such that $t \geq n - s - 1$ and $r \leq n - s - 2$. If W_1 is a component of $|LV| \cap W$ through P such that every component of $LP| \cap W_1$ different from P is proper, then W_1 is the only component of $|LV| \cap W$ through P.

PROOF. CASE 1. - $t = n - s - 1$. If $V \subset W$, the theorem follows from 3.5; so assume $V \not\subset W$. By 3.1, W_1 is a proper component of $|LV| \cap W$ and hence has dim r. Also dim $|LW_1| = r + s + 1$ by 1.9; so $|LW_1| = |LW|$ Now apply 3.4 to conclude $|LW_1| \cap W$, and hence $|LV| \cap W$, has a unique component through P.

CASE 2. - $t > n - s - 1$. Let $k(u)$ be a purely transcendental extension of k over which L is defined, let $K = \overline{k(u)}$, and suppose P is in a second component W_2 of $|LV| \cap W$. Let N be a generic linear variety over K of dim $n - [t - (n - s - 1)] - 1 < n - 1$. $|NP|$ is transversal to W at P by 2.1; so by the criterion for unit multiplicity, there exists a unique component W' of $|NP| \cap W$ through P, W' has dim $n - s - 1$, and P is simple on W'. But $|LV| \cap W$ and $|LV| \cap W'$ are proper by 3.1, and $W_i \cap |NP|$ is proper, and every component is general over k for W by 2.1; so we can conclude $W_i \cap |NP|$ contains a component W'_i through P such that W'_i is general over K for W_i and is also a component of $|LV| \cap W'$. Hence W'_1, W'_2 are distinct components of $|LV| \cap W'$ through P. But

$$|LP| \cap W'_1 \subset |LP| \cap (W_1 \cap |NP|) = (|LP| \cap W_1) \cap |NP|,$$

and by hypothesis either

$$\dim (LP| \cap W_1) = s + 1 + (r + s + 1 + t - n) - n \text{ or } |LP| \cap W_1 = P.$$

In either case, by 2.1,

$$(|LP| \cap W_1) \cap |NP| = P; \text{ so } |LP| \cap W_1' = P.$$

Hence we can apply case 1 to conclude W'_1 is the only component of $|LW| \cap W'$ through P, a contradiction Q. E. D.

REMARK - It is not clear in theorem 3.6 just when there exists a component W_1 satisfying the condition that $|LP| \cap W_1$ be proper outside P. In the case $t = n - s - 1$ and $V \subseteq W$, there exists such a W_1, namely V.

Question: If $t > n - s - 1$ and $V \subset W$, does there always exist such a W_1?

(9) This example was pointed out to me by A. Seidenberg.

4. Some Counter examples.

Let k be a field of characteristic $p \neq 0$, let x be a transcendental quantity over k, and let A be the locus of $P = (x, -x - x^{p+1})$ over k in S^2. Then A is defined by $Y + X + X^{p+1} = 0$ over k, and the tangent T to A at P is given by

$$(1 + x^p)\,(X - x) + (Y - y) = 0\,.$$

Hence $O = (0, 0) \varepsilon T$, but A is not a cone with O in its vertex. $\therefore$ A is not an O-variety. Moreover, the example shows 1.6 and 1.7 are not true without the characteristic O hypothesis. It also shows 2.2 is false without the assumption that V is an L-variety; for, (in the notation of 2.2) taking $V = A$, $M = a$ generic point of T over k, and $C = (x, y)$, then $|\,ML\,| = T$ is now tangent to V at C. For the same reason 3.3 is false without the assumption W is a P-variety for some $P \varepsilon V$, taking there $V = O$, $W = A$, $L = a$ generic point of T over k.(9)

The following example shows 1.8 is false without the hypothesis that V is an L-variety: let A be the curve in S^2 defined by

$$X^{p-1}\,Y + Y^{p-1}\,X - 1 = 0$$

over an appropriate field k of characteristic $p > 2$. The tangent V to A at (x, y) is given by

$$(y^{p-1} - x^{p-2}y)X + (x^{p-1} - y^{p-2}x)\,Y = 0 \quad \text{so} \quad O \varepsilon T \quad \text{but} \quad O \neq A.$$

As for the simple point theorem, it is not clear if the theorem remains true when one omits the hypothesis that V be an $|\,LP\,|$-variety. At any rate, the possibility of omitting this condition seems to depend on being able to apply [11], prop. 14, p. 131.

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