The absolute summability (A) of ultraspherical series

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Summary. - *This paper is devoted to the discussion of $|A|$ summability of ultraspherical series on a sphere. The first theorem also gives a new result for LAPLACE series in case $\lambda = \frac{1}{2}$.*

1. Summability $|A|$ of FOURIER series was discussed by WHITTAKER [8] and PRASAD [6, 7]. STARTING with PRASAD's results, BOSANQUET [2] gave general conditions for the absolute ABEL summability of FOURIER series. BHATT [1] has studied conditions under which LAPLACE series is $|A|$ summable at a point on the surface of a sphere of unit radius. In the present paper I propose to discuss absolute summability (A) of the ultraspherical series at a point of the sphere. The ordinary ABEL summability along with FATOU's extensions for the ultraspherical series has been recently dealt with by BROUSSE [3].

2. If $F(\theta, \varphi)$ be a function defined for the range $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ on a sphere S , then the ultraspherical series associated with this function is,

$$(2.1) \quad F(\theta, \varphi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n + \lambda) \iint_S \frac{P_n^{(\lambda)}(\cos Y) F(\theta', \varphi') d\sigma'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2} - \lambda}}, \quad \lambda > 0,$$

where

$$\cos Y = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi')$$

and

$$d\sigma' = \sin \theta' d\theta' d\varphi'.$$

Assuming the LEBESGUE integrability of the function

$$F(\theta', \varphi') [\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\lambda - \frac{1}{2}}$$

on the surface of the sphere, KOGNETLIANTZ [4] (See also OBRECHKOFF [5]) has defined the spherical mean value of $F(\theta, \varphi)$ as follows :

$$(2.2) \quad f(Y) = \frac{1}{2\pi (\sin Y)^{2\lambda}} \int_{c_Y} \frac{F(\theta', \varphi') d\sigma'}{[\sin^2 \theta' \sin^2 (\varphi - \varphi')]^{\frac{1}{2} - \lambda}},$$

where the integral is taken along the small circle whose centre is (θ, φ) and whose curvilinear radius is Y .

We can now write (2.1) as

$$F(\theta, \varphi) \sim \sum_{n=0}^{\infty} (n + \lambda) \int_0^{\pi} f(Y) \sin^{2\lambda} Y P_n^{(\lambda)}(\cos Y) dY.$$

Now

$$\begin{aligned} (2.3) \quad S(x) &= \int_0^{\pi} f(Y) \sum_{n=0}^{\infty} (n + \lambda) P_n^{(\lambda)}(\cos Y) x^n \sin^{2\lambda} Y dY \\ &= \int_0^{\pi} f(Y) \frac{\lambda(1 - x^2)}{(1 - 2x \cos Y + x^2)^{\lambda+1}} \sin^{2\lambda} Y dY, \end{aligned}$$

since [See KOGNETLIANTZ [4], relation (7)]

$$\sum_{m=0}^{\infty} (m + \lambda) P_m^{(\lambda)}(\cos Y) x^m = \frac{\lambda(1 - x^2)}{(1 - 2x \cos Y + x^2)^{\lambda+1}}.$$

Therefore

$$Q(x) \equiv S(x) - F(P) = \int_0^{\pi} \left[f(Y) - f(0) \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right)} \right] \frac{\lambda(1 - x^2) \sin^{2\lambda} Y dY}{(1 - 2x \cos Y + x^2)^{\lambda+1}}.$$

We write

$$\left[f(Y) - \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \lambda\right)} f(0) \right] \sin^{2\lambda} Y = \varphi(Y);$$

$$\Phi_{\alpha}(Y) = \frac{1}{\Gamma(\alpha)} \int_0^Y (Y - u)^{\alpha-1} \varphi(u) du, \quad \alpha > 0;$$

$$\Phi(Y) = \varphi(Y);$$

$$\Phi_{\alpha}(Y) = \frac{d}{dY} \Phi_{\alpha+1}(Y), \quad (-1 < \alpha < 0);$$

and

$$\varphi_{\alpha}(Y) = \Gamma(\alpha + 1) Y^{-\alpha} \Phi_{\alpha}(Y).$$

We shall first prove the following theorem :

THEOREM 1. - *The ultraspherical series (2.1) is absolutely summable (A), if the integral*

$$\int_0^{\eta} \frac{|\varphi_{\alpha}(t)|}{t^{1+2\lambda}} dt$$

exists for some $\alpha \geq 0$, $\eta > 0$ and $0 < \lambda < 1$.

3. For the proof of the above theorem we shall require the following lemmas :

LEMMA 1. - *For $\beta > \alpha \geq 0$,*

$$\Gamma(\beta + 1 + 2\lambda) \int_0^{\eta} \frac{|\Phi_{\beta}(t)|}{t^{1+2\lambda+\beta}} dt \leq \Gamma(\alpha + 1 + 2\lambda) \int_0^{\eta} \frac{|\Phi_{\alpha}(t)|}{t^{1+2\lambda+\alpha}} dt.$$

PROOF. - The proof shall be exactly similar to that given by BOSANQUET [2, Lemma 2]. We have

$$\begin{aligned} \Gamma(\beta + 1 + 2\lambda) \int_0^{\eta} \frac{|\Phi_{\beta}(t)|}{t^{1+2\lambda+\beta}} dt &= \frac{\Gamma(\beta + 1 + 2\lambda)}{\Gamma(\beta - \alpha)} \int_0^{\eta} \frac{dt}{t^{1+2\lambda+\beta}} \left| \int_0^t (t-u)^{\beta-\alpha-1} \Phi_{\alpha}(u) du \right| \\ &\leq \frac{\Gamma(\beta + 1 + 2\lambda)}{\Gamma(\beta - \alpha)} \int_0^{\eta} \frac{dt}{t^{1+2\lambda+\beta}} \int_0^t (t-u)^{\beta-\alpha-1} |\Phi_{\alpha}(u)| du = \\ &= \frac{\Gamma(\beta + 1 + 2\lambda)}{\Gamma(\beta - \alpha)} \int_0^{\eta} |\Phi_{\alpha}(u)| du \int_u^{\eta} (t-u)^{\beta-\alpha-1} t^{-1-2\lambda-\beta} dt \\ &\leq \frac{\Gamma(\beta + 1 + 2\lambda)}{\Gamma(\beta - \alpha)} \int_0^{\eta} \frac{|\Phi_{\alpha}(u)| du}{u^{1+2\lambda+\alpha}} \int_1^{\infty} (v-1)^{\beta-\alpha-1} v^{-\beta-1-2\lambda} dv \\ &= \Gamma(\alpha + 2\lambda + 1) \int_0^{\eta} \frac{|\Phi_{\alpha}(u)|}{u^{1+2\lambda+\alpha}} du. \end{aligned}$$

LEMMA 2. - *If $0 < t < \frac{\pi}{2}$, $0 < x < 1$ and r is a non-negative integer,*

then

$$\left| \frac{\partial^{r+1} k(x, t)}{\partial x \partial t^r} \right| \begin{cases} \leq A t^{-r-2\lambda-2} \\ \leq A(1-x)^{-2} t^{-r-2\lambda}, \end{cases}$$

where A is independent of x and t , and

$$(3.1) \quad k(x, t) = \frac{1-x^2}{(1-2x \cos t + x^2)^{\lambda+1}}.$$

PROOF. - It is easily verified by induction that

$$\frac{\partial^r}{\partial t^r} k(x, t) = (1-x^2) \sum_{\mu=0}^r T_{\mu}(t) \frac{x^{r-\mu} (\sin t)^{r-2\mu}}{(1-2x \cos t + x^2)^{1+\lambda+r-\mu}},$$

where $T_{\mu}(t)$ is a trigonometrical polynomial.

Since $T_{\mu}(t)$, ($\mu = 0, 1, 2, \dots, r$) and $\frac{(1-x)(x-\cos t)}{1-2x \cos t + x^2}$ are bounded, we have

$$\frac{\partial^{r+1}}{\partial x \partial t^r} k(x, t) = \sum_{\mu=0}^r O \left\{ \frac{(\sin t)^{r-2\mu}}{(1-2x \cos t + x^2)^{1+\lambda+r-\mu}} \right\}.$$

Now

$$(3.2) \quad (1-2x \cos t + x^2) = (x - \cos t)^2 + \sin^2 t \geq \sin^2 t.$$

Therefore

$$\begin{aligned} \frac{\partial^{r+1}}{\partial x \partial t^r} k(x, t) &= \sum_{\mu=0}^r O \{ (\sin t)^{-2\mu+r-2(1+\lambda+r-\mu)} \} \\ &= O(t^{-r-2\lambda-2}). \end{aligned}$$

On the other hand

$$(3.3) \quad (1-2x \cos t + x^2) = (1-x)^2 + 4x \sin^2 \frac{t}{2} \geq (1-x)^2.$$

Consequently, by (3.2) and (3.3) we have

$$\begin{aligned} \frac{\partial^{r+1}}{\partial x \partial t^r} k(x, t) &= \sum_{\mu=0}^r O \{ (1-x)^{-2} (\sin t)^{r-2\mu-2(\lambda+r-\mu)} \} \\ &= O \{ (1-x)^{-2} \sum_{\mu=0}^r (\sin t)^{-r-2\lambda} \} \\ &= \{ (1-x)^{-2} t^{-r-2\lambda} \} \end{aligned}$$

which completes the proof.

4. PROOF OF THEOREM 1. - From (2.3) and (3.1) we obtain

$$\begin{aligned} Q(x) &= \int_0^\pi \varphi(t) k(x, t) dt \\ &= \left(\int_0^\eta + \int_\eta^\pi \right) \varphi(t) k(x, t) dt \\ &= Q_1(x) + Q_2(x), \quad \text{say.} \end{aligned}$$

Following the lines of argument of PRASAD [6, 7] it is sufficient to prove that

$$\int_0^1 |Q_1(x)| dx < \infty.$$

Without loss of generality we can suppose $0 < \eta < \frac{\pi}{2}$; and by Lemma 1, we can suppose that $\alpha = h$ is a positive integer ⁽¹⁾. Now, integrating by parts, we get

$$\begin{aligned} \int_0^1 |Q_1(x)| dx &\leq \int_0^1 dx \sum_{\rho=0}^h \left| \Phi_\rho(\eta) \frac{\partial^\rho}{\partial x \partial \eta^{\rho-1}} k(x, \eta) \right| + \\ &\quad \left| \int_0^1 dx \int_0^\eta \left| \Phi_h(u) \frac{\partial^{h+1}}{\partial x \partial u^h} k(x, u) du \right| \right| \\ &= \int_0^1 dx \sum_{\rho=0}^h \left| \Phi_\rho(\eta) \frac{\partial^\rho}{\partial x \partial \eta^{\rho-1}} k(x, \eta) \right| + \frac{1}{\Gamma(h+1)} \int_0^1 dx \left| \int_0^\eta u^h \varphi_h(u) \frac{\partial^{h+1}}{\partial x \partial u^h} k(x, u) du \right| \\ &= C + \frac{1}{\Gamma(h+1)} \int_0^\eta |\varphi_h(u)| du \int_0^1 u^h \frac{\partial^{h+1}}{\partial x \partial u^h} k(x, u) dx, \end{aligned}$$

where C is a constant.

Now

$$I = \int_0^1 u^h \frac{\partial^{h+1}}{\partial x \partial u^h} k(x, u) dx$$

⁽¹⁾ If α be not an integer, we can take $h = [\alpha] + 1$.

$$= \int_0^{1-u} + \int_{1-u}^0 = I_1 + I_2, \quad \text{say.}$$

From Lemma 2,

$$\begin{aligned} I_1 &= O \left[\int_0^{1-u} (1-x)^{-2} u^{-2\lambda} dx \right] \\ &= O(u^{-2\lambda-1}). \end{aligned}$$

And

$$I_2 = O \left[\int_{1-u}^1 u^{-2\lambda-2} dx \right] = O(u^{-2\lambda-1}).$$

Therefore

$$I = O(u^{-2\lambda-1}).$$

Thus we find that

$$\int_0^\eta |Q_1(x)| dx \leq C + \frac{1}{\Gamma(h+1)} \int_0^\eta \frac{|\varphi_h(u)|}{u^{1+2\lambda}} du,$$

which completes the proof.

5. We now write

$$(5.1) \quad \psi(\gamma) = \left[f(\gamma) - \frac{\Gamma(\lambda)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \lambda\right)} f(0) \right] (\sin \gamma)^{2\lambda-1}.$$

Using the corresponding RIEMANN-LIOUVILLE fractional integrals, we shall prove the following theorem which includes both the theorems of BHATT [1] as particular cases for $\lambda = \frac{1}{2}$.

THEOREM 2. - *If for some $\alpha \geq 0$, $0 < \lambda \leq \frac{1}{2}$, $\psi_\alpha(\gamma)$ is of bounded variation in $(0, \eta)$ and $\psi_\alpha(\gamma) \rightarrow 0$ as $\gamma \rightarrow 0$, then the ultraspherical series of $F(\theta, \varphi)$ is absolutely summable (A) at the point P.*

The general line of argument in the proof will be similar to that of BOSANQUET [2] in the case of FOURIER series. We need the following lemmas.

LEMMA 3. [4, Lemma 3] - If $\varphi_\alpha(t)$ is of bounded variation in $(0, \eta)$ for a given $\alpha \geq 0$, then $\varphi_\beta(t)$ is of bounded variation in $(0, \eta)$, for every $\beta > \alpha$.

LEMMA 4. - If $0 < t < \frac{\pi}{2}$, $0 < x < 1$ and r is a non-negative integer, then

$$\left| \frac{\partial^{r+1}}{\partial x \partial t^r} \xi(x, t) \right| \begin{cases} \leq A t^{-r-2\lambda-1} \\ \leq A(1-x)^{-2} t^{-r-2\lambda+1}, \end{cases}$$

where A is independent of x and t and

$$(5.2) \quad \xi(x, t) = \frac{(1-x^2) \sin t}{(1-2x \cos t + x^2)^{\lambda+1}}.$$

The proof is exactly similar to that of Lemma 2.

LEMMA 5. - If $0 < t < \frac{\pi}{4}$ and r is a non-negative integer, then

$$\left| \int_0^t u^r \frac{\partial^{r+1}}{\partial x \partial u^r} \xi(x, u) du \right| \begin{cases} \leq A t^{-2\lambda}, & \sigma \leq x < 1; \\ \leq A(1-x)^{-2} t^{2-2\lambda}, & \text{for } 0 < x < 1, \end{cases}$$

σ being a fixed positive constant less than unity.

PROOF. - When $r = 0$, the second inequality is evidently true since,

$$\int_0^t O \{ (1-x)^{-2} u^{-2\lambda+1} \} du = O[1-x)^{-2} t^{2-2\lambda}].$$

To prove the first inequality ⁽¹⁾ in the case $r = 0$, we write

$$\chi(x) = \int_0^t \xi(x, u) du.$$

Therefore

$$\frac{\partial \chi(x)}{\partial x} = \int_0^t \frac{\partial \xi(x, u)}{\partial x} du.$$

Putting

$$\Delta = 1 - 2x \cos t + x^2,$$

⁽¹⁾ I must thank DR. S. N. BHATT for his valuable suggestions in this connection.

$$\begin{aligned}\chi(x) &= \int_0^t \frac{(1-x^2) \sin u}{\Delta^{\lambda+1}} du. \\ &= - \left[\frac{(1-x^2)^t}{2\lambda x \Delta^\lambda} \right]_0^t = \frac{1+x}{2\lambda x(1-x)^{2\lambda-1}} - \frac{1-x^2}{2\lambda x \Delta^\lambda}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \chi(x)}{\partial x} &= \frac{\partial}{\partial x} \left[\frac{1+x}{2\lambda x(1-x)^{2\lambda-1}} - \frac{1-x^2}{2\lambda x \Delta^\lambda} \right] \\ &= O[x^{-2}(1-x)^{-2\lambda+1}\Delta^{-\lambda}],\end{aligned}$$

after simplification,

$$= O[\sin t (\sin t)^{-2\lambda}] = O(t^{-2\lambda}).$$

If $r > 0$, integration by parts gives

$$\int_0^t u^r \frac{\partial^{r+1}}{\partial x \partial u^r} \xi(x, u) du = t^r \frac{\partial^r}{\partial x \partial t^{r-1}} \xi(x, t) - r \int_0^t u^{r-1} \frac{\partial^r}{\partial x \partial u^{r-1}} \xi(x, u) du,$$

and the result of the lemma follows by induction.

6. PROOF OF THEOREM 2. - From (5.1) and (5.2), we have

$$\begin{aligned}Q(x) &= \int_0^\pi \psi(t) \xi(x, t) dt \\ &= \int_0^\delta + \int_\delta^\pi = Q_1(x) + Q_2(x), \quad \text{say.}\end{aligned}$$

We need now only prove that

$$(6.1) \quad \int_0^1 |Q_1(x)| dx < \infty.$$

Without loss generality we can suppose $0 < \delta < \frac{\pi}{4}$ and by Lemma 3 we may suppose $\alpha = h$ to be a positive integer.

Integration by parts gives as before

$$\begin{aligned}
 \int_0^1 |Q_1(x)| dx &\leq \int_0^1 dx \sum_{\rho=0}^h \left| \Psi_\rho(\delta) \frac{\partial^\rho}{\partial x \partial \delta^{\rho-1}} \xi(x, \delta) \right| + \\
 &\quad + \int_0^1 dx \left| \int_0^\delta \Psi_h(u) \frac{\partial^{h+1}}{\partial x \partial u^h} \xi(x, u) du \right| \\
 &\leq \int_0^1 dx \sum_{\rho=0}^h \left| \Psi_\rho(\delta) \frac{\partial^\rho}{\partial x \partial \delta^{\rho-1}} \xi(x, \delta) \right| + \\
 &\quad + \frac{1}{\Gamma(h+1)} \int_0^1 dx \left| \psi_h(\delta) \int_0^\delta u^h \frac{\partial^{h+1}}{\partial x \partial u^h} \xi(x, u) du \right| + \\
 &\quad + \frac{1}{\Gamma(h+1)} \int_0^1 dx \int_0^\delta |\psi'_h(\theta)| d\theta \left| \int_0^\theta u^h \frac{\partial^{h+1}}{\partial x \partial u^h} \xi(x, u) du \right| \\
 (6.2) \quad &\leq C + \frac{1}{\Gamma(h+1)} \int_0^\delta |\psi'_h(\theta)| d\theta \int_0^1 dx \left| \int_0^\theta u^h \frac{\partial^{h+1}}{\partial x \partial u^h} \xi(x, u) du \right|
 \end{aligned}$$

by virtue of the first inequalities of Lemmas 4 and 5 with $\theta = \delta$, C being a constant.

Also, Lemma 5 gives for $0 < \theta < \delta$,

$$\begin{aligned}
 &\int_0^1 dx \left| \int_0^\theta u^h \frac{\partial^{h+1}}{\partial x \partial u^h} \xi(x, u) du \right| \\
 &\quad = \int_0^{1-\theta} + \int_{1-\theta}^1 \\
 &\quad = \int_0^{1-\theta} O[(1-x)^{-2\theta-2\lambda}] dx + \int_{1-\theta}^1 \theta^{-2\lambda} O(1) dx \\
 (6.3) \quad &= O(1).
 \end{aligned}$$

Since

$$\int_0^{\delta} |\psi'_h(\theta)| d\theta < \infty,$$

(6.1) now follows from (6.2) and (6.3).

It now remains to show that

$$\lim_{x \rightarrow 1-0} Q(x) = 0.$$

This can be easily established by following the line of argument usually employed for FOURIER series.

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