An aspect of local property of $|C, 2|$ summability of the derived Fourier series

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Summary. - *In this paper the author has proved that the $|C, 2|$ summability of the first derived series of Fourier series is not the local property of the generating function and further he has investigated the condition under which it becomes the local property.*

1. Let $s_n^0 \equiv s_n$ denote the nth partial sum of the series Σa_n, and let $\sigma_n^{(\alpha)}$ and $t_n^{(\alpha)}$ denote the nth Cesàro means of the order $\alpha(\alpha > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series Σa_n is said to be absolutely summable (C, α), or summable $|C, \alpha|$, if the sequence $\{\sigma_n^{(\alpha)}\}$ is of bounded variation, that is to say, the infinite series

$$\Sigma \mid \sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)} \mid$$

is convergent [3], [7].

The following identities are well known for $\alpha > -1$.

$$t_n^{(\alpha+1)} = (\alpha + 1)(\sigma_n^{(\alpha)} - \sigma_n^{(\alpha+1)}), \text{ [4], [8].} \tag{1.1}$$

$$t_n^{(\alpha)} = n(\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}), \text{ [7], [8].} \tag{1.2}$$

2. Let $f(t)$ be a periodic function, integrable in the LEBESGUE sense over the interval $(0, 2\pi)$. Let the FOURIER series associated with the function $f(t)$ be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{2.1}$$

Then the first derived series of (2.1) is

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} nB_n(t). \tag{2.2}$$

We shall always write

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\},$$

$$g(t) = \frac{\{f(x+t) - P(t)\} + (-1)^r\{f(x-t) - P(-t)\}}{2t^r}.$$

where

$$P(t) = \sum_{\iota=0}^{r-1} \frac{\theta_\iota t^\iota}{L^\iota}.$$

Then

$$\varphi(t) \sim \sum_{n=0}^{\infty} A_n(x) \cos nt,$$

$$\psi(t) \sim \sum_{n=1}^{\infty} B_n(x) \sin nt.$$

The first CESÀRO mean of the function $g(u)$ for $r = 1$ is

$$g_1(t) = \frac{1}{t}\int_0^t g(u)du.$$

Regarding the absolute CESÀRO summability of the derived series of FOURIER series HYSLOP [5] has proved the following theorem.

THEOREM A. - If $g_1(t)$ is of bounded variation in $(0, \pi)$, then the derived series of FOURIER series of $f(t)$, at $t = x$, is summable $|C, 2 + \delta|$ where $\delta > 0$.

Since a LEBESGUE integral is absolutely continuous and therefore of bounded variation in any range (η, π) it is an immediate consequence of the above theorem that the summability $|C,2 + \delta|$ of the first derived series of FOURIER series is a local property of the generating function. The object of the present paper is to prove that the summability $|C, 2|$ of the same series is not a local property and to investigate the condition under which it becomes the local one. Analogous theorems for the case of FOURIER series have been proved by BOSANQUET and KESTELMAN [2], BHATT [1] and JURKAT and PEYERIMHOFF [6].

In what follows we shall prove the following theorems.

THEOREM 1. - *If* $f(t)$ *is a periodic function with period* 2π *and integrable in the Lebesgue sense over the interval* $(0, 2\pi)$, *then the summability* $|C, 2|$ *of the first derived series of the Fourier series of* $f(t)$ *is not a local property of the generating function.*

THEOREM 2. - *If*

$$\Sigma \frac{|B_n(x)|}{n} < \infty$$

then the $|C, 2|$ *summability of* $\Sigma\, nB_n(x)$ *depends only on the behaviour of the generating function* $f(t)$ *in the immediate neighbourhood of the point* $t = x$.

3. We require the following lemmas for the proof of our theorems.

LEMMA 1. ([2], Theorem 1). - *Suppose* $f_n(x)$ *to be measurable in* (a, b), *where* $b - a \leq \infty$, *for* $n = 1, 2, \ldots$, *then a necessary and sufficient condition, that for every function* $\varphi(x)$ *summable over* (a, b), *the functions* $f_n(x)\varphi(x)$ *should be summable over* (a, b) *and*

$$\sum_{n=1}^{\infty} \left| \int_a^b f_n(x)\varphi(x)dx \right| < \infty,$$

is that $\sum_{n=1}^{\infty} |f_n(x)|$ *is essentially bounded in* (a, b).

LEMMA 2. - *If the series*

$$\Sigma \frac{|\sigma_n^{(1)}|}{n+1}$$

is convergent, then the series Σa_n *is summable* $|C, 2|$.

PROOF OF THE LEMMA. - We have from the identity (1.1)

$$\begin{aligned} \Sigma \frac{|t_n^{(2)}|}{n} &= 2\,\Sigma\, \frac{|\sigma_n^{(1)} - \sigma_n^{(2)}|}{n} \\ &\leq 2\,\Sigma \frac{|\sigma_n^{(1)}|}{n} + 2\,\Sigma \frac{|\sigma_n^{(2)}|}{n} \\ &< \infty, \end{aligned}$$

by hypothesis and the consistency theorem for the absolute CESÀRO summability.

The lemma follows from the identity (1.2) and the definition of absolute CESÀRO summability.

4. PROOF OF THEOREM 1. - For proving the theorem it is essential to establish that if $0 < \alpha < \beta < 2\pi$, there is a function summable over (α, β) and zero in the remainder of $(0, 2\pi)$ whose first derived series of FOURIER series is not summable $|C, 2|$ at $t = 0$.

It is known that if $\Sigma n B_n$ is summable $|C, 2|$ then $\Sigma \frac{|B_n|}{n}$ is convergent [8].

It is, therefore, enough to show that there is a function $f(t)$, summable over (α, β), such that

$$\sum_{n=1}^{\infty} \left| \int_{\alpha}^{\beta} f(t) \frac{\sin nt}{n} dt \right| = \infty$$

But, if $0 < t < 2\pi$ $(t \neq \pi)$

$$\Sigma \frac{|\sin nt|}{n} \geq \Sigma \frac{\sin^2 nt}{n}$$

$$= \frac{1}{2} \Sigma \frac{1 - \cos 2nt}{n}$$

$$= \frac{1}{2} \Sigma \frac{1}{n} - \frac{1}{2} \Sigma \frac{\cos 2nt}{n}$$

$$= \infty.$$

The result therefore follows from Lemma 1.

PROOF OF THEOREM 2. - We have

$$S_n(x) = \sum_{\nu=1}^{n} \nu B_\nu(x) \tag{4.1}$$

$$= \frac{2}{\pi} \sum_{\nu=1}^{n} \int_0^{\pi} \psi(t) \nu \sin \nu t dt$$

$$= -\frac{2}{\pi} \int_0^{\pi} \psi(t) \frac{d}{dt} \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin t/2} dt$$

$$= -\frac{(2n+1)}{\pi}\int_0^{\pi} \psi(t)\,\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin t/2}\,dt$$

$$+\frac{2}{\pi}\int_0^{\pi} \frac{\psi(t)}{2\tan t/2}\,\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin t/2}\,dt$$

$$= I_1(n) + I_2(n), \text{ say,}$$

where

$$I_1(n) = -\frac{2n+1}{2\pi}\left[\int_0^{\delta} \frac{\psi(t)}{\sin\frac{1}{2}t}\left\{1-\left(\frac{\sin\frac{1}{2}t}{\sin\frac{1}{2}\delta}\right)^3\right\}\cos\left(n+\frac{1}{2}\right)t\,dt\right]$$

$$+\frac{1}{2\pi}\left[\int_0^{\delta} \frac{\psi(t)}{\tan\frac{1}{2}t\sin\frac{1}{2}t}\left\{1-\left(\frac{\sin\frac{1}{2}t}{\sin\frac{1}{2}\delta}\right)^3\right\}\sin\left(n+\frac{1}{2}\right)t\,dt\right]$$

and

$$I_2(n) = -\frac{2n+1}{2\pi}\left[\int_0^{\delta} \frac{\psi(t)}{\left(\sin\frac{1}{2}\delta\right)^3}\sin^2\frac{1}{2}t\cos\left(n+\frac{1}{2}\right)t\,dt\right.$$

$$\left.+\int_{\delta}^{\pi} \psi(t)\,\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt\right]$$

$$+\frac{1}{2\pi}\left[\int_0^{\delta} \frac{\psi(t)}{\tan\frac{1}{2}t}\,\frac{\sin^2\frac{1}{2}t}{\left(\sin\frac{1}{2}\delta\right)^3}\sin\left(n+\frac{1}{2}\right)t\,dt\right.$$

$$\left.+\int_{\delta}^{\pi} \frac{\psi(t)}{\tan\frac{1}{2}t\sin\frac{1}{2}t}\sin\left(n+\frac{1}{2}\right)t\,dt\right].$$

The sequence $\{S_n(x)\}$ will be summable $|C, 2|$, if the sequences $\{I_1(n)\}$ and $\{I_2(n)\}$ are summable $|C, 2|$. We observe that, for positive δ, however small, but fixed, the summability $|C, 2|$ of the sequence $\{I_1(n)\}$ depends on the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the point x defined by $x-\delta$, $x+\delta$. Hence the theorem will be established if we prove that under the hypothesis of the theorem the sequence $\{I_2(n)\}$ is summable $|C, 2|$.

Let G_n denote the arithmetic mean of the sequence $\{I_2(n)\}$. Then we have

$$G_n = -\frac{1}{(n+1)\pi}\left[\int_0^{\delta} \frac{\psi(t)}{\left(\sin\frac{1}{2}\delta\right)^3}\sin^2\frac{1}{2}t \sum_{\nu=0}^{n}\left(\nu+\frac{1}{2}\right)\cos\left(\nu+\frac{1}{2}\right)t\,dt\right.$$

$$\left.+\int_{\delta}^{\pi}\frac{\psi(t)}{\sin\frac{1}{2}t}\sum_{\nu=0}^{n}\left(\nu+\frac{1}{2}\right)\cos\left(\nu+\frac{1}{2}\right)t\,dt\right]$$

$$+\frac{1}{2(n+1)\pi}\left[\int_0^{\delta}\frac{\psi(t)\cos t/2}{\left(\sin\frac{1}{2}\delta\right)^3}\sin\frac{1}{2}t\sum_{\nu=0}^{n}\sin\left(\nu+\frac{1}{2}\right)tdt\right.$$

$$\left.+\int_{\delta}^{\pi}\frac{\psi(t)\cos t/2}{\sin^2\frac{1}{2}t}\sum_{\nu=0}^{n}\sin\left(\nu+\frac{1}{2}\right)tdt\right]$$

$$=-\frac{1}{(n+1)\pi}\left[\int_0^{\delta}\frac{\psi(t)}{\left(\sin\frac{1}{2}\delta\right)^3}\sin^2\frac{1}{2}t\,\frac{d}{dt}\left\{\frac{\sin^2(n+1)t/2}{\sin t/2}\right\}dt\right.$$

$$\left.+\int_{\delta}^{\pi}\frac{\psi(t)}{\sin\frac{1}{2}t}\frac{d}{dt}\left\{\frac{\sin^2(n+1)t/2}{\sin t/2}\right\}dt\right]$$

$$+\frac{1}{2(n+1)\pi}\left[\int_0^{\delta}\frac{\psi(t)\cos\frac{1}{2}t}{\left(\sin\frac{1}{2}\delta\right)^3}\sin^2(n+1)t/2\,dt\right.$$

$$\left.+\int_{\delta}^{\pi}\frac{\psi(t)\cos t/2}{\sin^3 t/2}\sin^2(n+1)t/2dt\right]$$

$$= -\frac{1}{2\pi}\left[\int_0^{\delta} \frac{\psi(t)}{\left(\sin\frac{1}{2}\delta\right)^3} \sin\frac{1}{2}t \sin(n+1)t\,dt \right.$$

$$\left. + \int_{\delta}^{\pi} \frac{\psi(t)}{\sin^2 t/2} \sin(n+1)t\,dt\right]$$

$$+ \frac{1}{(n+1)\pi}\left[\int_0^{\delta} \frac{\psi(t)\cos t/2}{\left(\sin\frac{1}{2}\delta\right)^3} \sin^2(n+1)t/2\,dt \right.$$

$$\left. + \int_{\delta}^{\pi} \frac{\psi(t)\cos t/2}{\left(\sin\frac{1}{2}t\right)^3} \sin^2(n+1)t/2\,dt\right]$$

$$= -\frac{1}{2\pi}\left[\int_0^{\delta} \frac{\psi(t)}{\left(\sin\frac{1}{2}\delta\right)^3} \sin\frac{1}{2}t \sin(n+1)t\,dt \right.$$

$$\left. + \int_{\delta}^{\pi} \frac{\psi(t)}{\sin^2 t/2} \sin(n+1)t\,dt\right] + O\left(\frac{1}{n}\right)$$

$$= -G'_n + O\left(\frac{1}{n}\right), \text{ say.}$$

Now let us define a function $\Omega(t)$ as follows:

$$\Omega(t) = \begin{cases} \left(\sin\frac{1}{2}\delta\right)^{-3} \sin\frac{1}{2}t & (0 \leq t \leq \delta) \\ \left(\sin\frac{1}{2}t\right)^{-2} & (\delta \leq t \leq \pi). \end{cases}$$

Then for $0 \leq t \leq \pi$, $\Omega(t)$ is of bounded variation and continuous, $\Omega'(t)$ is bounded and $\Omega''(t)$ is integrable (L).

Now since $\Omega(t)$ is of bounded variation in the range $(0, \pi)$ by a well known theorem we have on setting

$$B_{-\nu}(x) = - B_\nu(x) = - B_\nu$$

$$G'_n = \frac{1}{2\pi}\int_0^\pi \psi(t)\Omega(t) \sin (n+1)t dt$$

$$= \frac{1}{2\pi} \sum_{\nu=1}^{\infty} B_\nu \int_0^\pi \Omega(t) \sin \nu t \sin (n+1) t dt$$

$$= \frac{1}{4\pi} \sum_\nu{}' B_\nu \int_0^\pi \Omega(t) \cos (n-\nu+1) t dt$$

$$+ O(|B_n(x)|)$$

$$= -\frac{1}{4\pi} \sum_\nu{}' B_\nu(x) \int_0^\pi \Omega'(t) \frac{\sin (n-\nu+1)t}{n-\nu+1} dt$$

$$+ O(|B_n(x)|),$$

where Σ' denotes summations over $-\infty < \nu \leq -1$, $1 \leq \nu \leq n-1$ and $(n+1) \leq \nu < \infty$. Let

$$\mu = \min (|n-\nu|^{-1}, \delta).$$

Then we have

$$G'_n = -\frac{1}{4\pi} \Sigma' B_\nu(x) \left(\int_0^\mu + \int_\mu^\pi \right) \Omega'(t) \frac{\sin (n-\nu+1)}{n-\nu+1} dt$$

$$+ O(|B_n(x)|)$$

$$= - G'_{n,1} - G'_{n,2} + O(|B_n(x)|).$$

Therefore

$$G'_{n,1} = O(1) \sum_\nu{}' \frac{|B_\nu(x)|}{(n-\nu)^2}$$

and

$$G'_{n,2} = -\frac{1}{4\pi}\sum_{\nu}{}' B_\nu(x)\left[\Omega'(t)\,\frac{\cos(n-\nu+1)t}{(n-\nu+1)^2}\right]_{\mu,\,\delta+0}^{\delta-0,\,\pi}$$

$$+\frac{1}{4\pi}\sum_{\nu}{}' B_\nu(x)\int_{\mu}^{\pi}\Omega''(t)\,\frac{\cos(n-\nu+1)t}{(n-\nu+1)^2}\,dt$$

$$= O(1)\sum_{\nu}{}'\frac{|B_\nu(x)|}{(n-\nu)^2},$$

the integration by parts being taken separately over the ranges (μ, δ) and (δ, π). Collecting these estimates we have

$$G_n = O(1)\sum_{\nu}{}'\frac{|B_\nu(x)|}{(n-\nu)^2} + O(|B_n(x)|) + O\left(\frac{1}{n}\right) \tag{4.2}$$

$$= O(1)\left[\sum_{\nu=-\infty}^{-1} + \sum_{\nu=1}^{n-1} + \sum_{\nu=n+1}^{n+m} + \sum_{\nu=n+m+1}^{\infty}\right]\frac{|B_\nu(x)|}{(n-\nu)^2}$$

$$+ O(|B_n(x)|) + O\left(\frac{1}{n}\right).$$

$$= O(1)\left[\partial_1 + \partial_2 + \partial_3 + \partial_4 + |B_n(x)| + \frac{1}{n}\right], \text{ say.}$$

Thus by virtue of Lemma 2 and (4.2) the sequence $\{I_2(n)\}$ is summable $|C, 2|$, if we prove that

$$\sum\frac{\partial_r}{n} < \infty \qquad (r = 1, 2, 3, 4). \tag{4.3}$$

since by hypothesis

$$\sum\frac{|B_n(x)|}{n} < \infty;$$

and obviously

$$\Sigma \frac{1}{n^2} < \infty.$$

Now,

$$\sum_{n=1}^{m} n^{-1}\partial_1 \leq A \sum_{n=1}^{m} \frac{1}{n^2} = O(1);$$

$$\sum_{n=2}^{m} n^{-1}\partial_2 = \sum_{\nu=1}^{m-1} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|B_{n-\nu}|}{n}$$

$$\leq \sum_{\nu=1}^{m-1} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|B_{n-\nu}|}{n-\nu} = O(1);$$

$$\sum_{n=1}^{m} n^{-1}\partial_3 = \sum_{n=1}^{m} n^{-1} \sum_{\nu=1}^{m} \frac{|B_{\nu+n}|}{\nu^2}$$

$$= \sum_{\nu=1}^{m} \nu^{-2} \sum_{n=1}^{m} n^{-1} |B_{\nu+n}|$$

$$= \sum_{\nu=1}^{m} \nu^{-2} \sum_{n=1}^{\nu} n^{-1} |B_{\nu+n}| + \sum_{\nu=1}^{m} \nu^{-2} \sum_{n=\nu+1}^{m} \frac{|B_{n+\nu}|}{n+\nu} \left(\frac{n+\nu}{n}\right)$$

$$= O(1);$$

and

$$\sum_{n=1}^{m} n^{-1}\partial_4 = O(1) \left\{ \sum_{n=1}^{m} n^{-1} \sum_{\nu=n+m+1}^{\infty} \frac{1}{(\nu-n)^2} \right\}$$

$$= O\left\{ \frac{1}{m+1} \sum_{n=1}^{m} \frac{1}{n} \right\} = o(1),$$

as $m \to \infty$.

Thus we have established (4.3) and this completes the proof of Theorem 2.

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REFERENCES

[1] S. N. BHATT, *An aspect of local property of absolute summability* (C, 1) *of a Fourier series*, «Vijnana Parishad Anusandhan Patrika», vol. 2, 1959, pp. 73-78.

[2] L. S. BOSANQUET and H. KESTELMAN, *The absolute convergence of series of integrals*, «Proc. Lond. Math. Soc.», vol. 45, 1939, pp. 88-97.

[3] M. FEKETE, *Zur Theorie der divergenten Reihen*, «Mathematikai és termeszettudomanyi értesitö», (Budapest), vol. 29, 1911, pp. 719-726.

[4] G. H. HARDY, *Theorems relating to the summability and convergence of slowly oscillating series*, «Proc. Lond. Math. Soc.», (2), vol. 8, 1910, pp. 301-320.

[5] J. M. HYSLOP, *On the absolute summability of the successively derived series of a Fourier series and its allied series*, «Proc. Lond. Math. Soc.», vol. 46, 1940, pp. 55-80.

[6] W. JURKAT and A. PEYERIMHOFF, *Lokalisation bei absoluter Cesàro summierbarkiet von Potenzreihen und trigonometrischen Reihin II*, «Math. Zeit.», 64, 1956, pp. 151-158.

[7] ERVAND KOGBETLIANTZ, *Sur les séries absolument sommables par la méthode des moyennes arithmétiques*, «Bulletin des Sciences Mathematiques», (2), vol. 49, 1925, pp. 234-256.

[8] ERVAND KOGBETLIANTZ, *Sommation des séries et intégrales divergentes par les moyennes arithmétique et typiques*, «Mémorial des Sciences Mathématiques», no. 51, 1931.