A theorem on the absolute Cesàro summability of factored Fourier series.

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Summary. - *In this paper, the author proves a theorem on the absolute* CESÀRO *summability of factored* FOURIER *series. His theorem extends a theorem of* MATSUMOTO *and generalizes a theorem of* PRASAD *and* BHATT.

1.1. Let $f(t)$ be a periodic function with period 2π and integrable (L. over $(-\pi, \pi)$ and let its FOURIER series be given by

$$f(t) \sim \frac{1}{2} a_0 + \sum_1^\infty (a_n \operatorname{Cos} nt + b_n \operatorname{Sin} nt)$$

$$= \sum_0^\infty A_n(t).$$

We write

$$\varphi(t) = \frac{1}{2} \{ f(x+t) + f(x-t) \},$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u) du \qquad \alpha > 0,$$

$$\Phi_0(t) = \varphi(t),$$

and

$$\varphi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t), \qquad \alpha \geq 0$$

1.2. In 1948, CHENG proved the following theorem concerning the absolute summability factor of FOURIER series:

THEOREM A. - If $\varphi_\alpha(t)$ $(0 \leq \alpha \leq 1)$ is of bounded variation in $(0, \pi)$, then

$$\Sigma A_n(t) / \{ \log(n) \}^{1+\epsilon} \qquad \epsilon > 0,$$

at the point $t = x$, is summable $| C, \alpha |$.

Generalizing the above theorem of CHENG [1], MATSUMOTO [2] established the following theorem:

THEOREM B. - If

$$\int_0^{\pi} | d\{ t^{-\gamma} \Phi_\beta(t) \} | < \infty,$$

then the series

$$\sum_0^{\infty} n^{\gamma-\beta} A_n(t)/\{\log(n+2)\}^{1+\varepsilon},$$

at the point $t = x$, is summable $|C, \gamma|$, where $1 \geq \gamma \geq \beta \geq 0$ and $\varepsilon > 0$.

The object of the present paper is to find in place of this special type of factor $1/\{\log(n+2)\}^{1+\varepsilon}$ in the above theorem of MATSUMOTO [2], a general factor λ_n, where $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n/n$ is convergent.

2.1. In what follows we shall prove the following theorem:

THEOREM. - *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma \lambda_n/n < \infty$ *and if*

$$\int_0^{\pi} | d\{ t^{-\gamma} \Phi_\beta(t) \} | < \infty,$$

then the series

$$\Sigma\, n^{\gamma-\beta} \lambda_n A_n(t),$$

at the point $t = x$, *is summable* $|C, \gamma|$, *where* $1 \geq \gamma \geq \beta \geq 0$.

It may be remarked that this theorem also generalizes the following theorem of PRASAD and BHATT [3].

THEOREM C. - If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n/n < \infty$ and $\varphi_\alpha(t)$ is of bounded variation in $(0, \pi)$, where $0 \leq \alpha \leq 1$, then the series $\sum_{n=0}^{\infty} \lambda_n A_n(t)$, at the point $t = x$, is summable $|C, \alpha|$.

2.2. The following lemmas will be required for the proof of our theorem:

LEMMA 1. (MATSUMOTO [2]). - *Let*

$$S_k(n, t) = \sum_{\nu=0}^{k} A_{n-\nu}^{\gamma-1} \operatorname{Sin} \nu t, \qquad k \leq n \text{ and } 1 \geq \gamma > 0,$$

then

$$S_k(n, t) = \begin{cases} O\{k(n-k)^{\gamma-1}\} \\ O\{t^{-1}(n-k)^{\gamma-1}\} \end{cases} \qquad n > k,$$

and

$$S_n(n, t) = \begin{cases} O(n^{\gamma}) \\ O(t^{-\gamma}). \end{cases}$$

Lemma 2. (Matsumoto [2]). - *Let*

$$S_k^{\lambda}(n, t) = \left(\frac{d}{dt}\right)^{\lambda} S_k(n, t) \qquad \lambda \geq 1 \text{ and } k \leq n,$$

then we have

$$S_k^{\lambda}(n, t) = \begin{cases} O\{k^{\lambda+1}(n-k)^{\gamma-1}\} \\ O\{k^{\lambda}t^{-1}(n-k)^{\gamma-1}\} \end{cases} \qquad k < n,$$

and

$$S_n^{\lambda}(n, t) = \begin{cases} O(n^{\lambda+\gamma}) \\ O(n^{\lambda}t^{-\gamma}). \end{cases}$$

Lemma 3. - *Let*

$$F(n, \nu) = (n-\nu)^{\gamma-1} \mid \Delta\{\nu^{\delta}\lambda_{\nu}\} \mid, \qquad \delta = \gamma - \beta.$$

If

$$J(n, u) = \int_u^{\pi} (t-u)^{-\beta} \frac{d}{dt} H^{\gamma}(n, t) dt,$$

where

$$H^{\gamma}(n, t) = \frac{1}{A_n^{\gamma}} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \nu^{\delta}\lambda_{\nu} \operatorname{Sin} \nu t,$$

then

$$J(n, u) = \begin{cases} O\left[n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1} \nu F(n, \nu) + n^{\gamma+\delta}\lambda_n\right\}\right] \\ O\left[n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1} u^{-1} F(n, \nu) + n^{\delta}u^{-\gamma}\lambda_n\right\}\right]. \end{cases}$$

PROOF. - By partial summation we have

$$H^{\gamma}(n,\ t)=\frac{1}{A_n^{\gamma}}\left\{\sum_{\nu=0}^{n-1} s_\nu(n,\ t)\Delta\{\nu^{\delta}\lambda_\nu\}+s_n(n,\ t)n^{\delta}\lambda_n\right\}$$

$$(2.2.1)\qquad =\left\{\begin{array}{l} O\left[n^{-\gamma}\left\{\sum_{\nu=0}^{n-1}\nu F(n,\ \nu)+n^{\gamma+\delta}\lambda_n\right\}\right]\\ O\left[n^{-\gamma}\left\{\sum_{\nu=0}^{n-1}t^{-1}F(n,\ \nu)+t^{-\gamma}n^{\delta}\lambda_n\right\}\right]\end{array}\right.$$

by Lemma 1. Applying Lemma 2, we have

$$(2.2.2)\qquad \frac{d}{dt}H^{\gamma}(n,\ t)=\left\{\begin{array}{l} O\left[n^{-\gamma}\left\{\sum_{\nu=0}^{n-1}\nu^2F(n,\ \nu)+n^{\gamma+\delta+1}\lambda_n\right\}\right]\\ O\left[n^{-\gamma}\left\{\sum_{\nu=0}^{n-1}\nu t^{-1}F(n,\ \nu)+n^{\delta+1}t^{-\gamma}\lambda_n\right\}\right].\end{array}\right.$$

Now

$$J(n,\ u)=\left(\int_u^{u+n^{-1}}+\int_{u+n^{-1}}^{\pi}\right)(t-u)^{-\beta}\frac{d}{dt}H^{\gamma}(n,\ t)dt$$

$$=I_1+I_2,\ \text{say}.$$

It is easy to see that

$$I_1=\left\{\begin{array}{l} O\left[n^{\beta-1-\gamma}\left\{\sum_{\nu=0}^{n-1}\nu^2F(n,\ \nu)+n^{\gamma+\delta+1}\lambda_n\right\}\right]\\ O\left[n^{\beta-1-\gamma}\left\{\sum_{\nu=0}^{n-1}u^{-1}\nu F(n,\ \nu)+n^{\delta+1}u^{-\gamma}\lambda_n\right\}\right],\end{array}\right.$$

and from second mean value theorem, we have

$$I_2=n^{\beta}\int_{u+n^{-1}}^{\xi}\frac{d}{dt}H^{\gamma}(n,\ t)dt \qquad u+n^{-1}<\xi<\pi,$$

$$=\left\{\begin{array}{l} O\left[n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1}\nu F(n,\ \nu)+n^{\gamma+\delta}\lambda_n\right\}\right]\\ O\left[n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1}u^{-1}F(n,\ \nu)+u^{-\gamma}n^{\delta}\lambda_n\right\}\right].\end{array}\right.$$

LEMMA 4. - *Let*

$$I(n,\ u) = \int_0^u v^\gamma \frac{d}{dv} J(n,\ v) dv,$$

then

$$I(n,\ u) = O\left[u^\gamma n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1} \nu F(n,\ \nu) + n^{\gamma+\delta}\lambda_n\right\}\right].$$

PROOF.

$$I(n,\ u) = u^\gamma \int_\eta^u \frac{d}{dv} J(n,\ v) dv \qquad 0 < \eta < u,$$

$$= u^\gamma [J(n,\ v)]_\eta^u$$

$$= O\left[u^\gamma n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1} \nu F(n,\ \nu) + n^{\gamma+\delta}\lambda_n\right\}\right]$$

by using the first estimation for $J(n,\ u)$.

LEMMA 5. - *Let*

$$K(n,\ u) = \int_u^\pi v^\gamma \frac{d}{dv} J(n,\ v) dv$$

then, for $\beta > 0$, *we have*

$$K(n,\ u) = O\left[n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1} u^{\gamma-1} F(n,\ \nu) + n^{\delta}\lambda_n\right\}\right]$$

$$+ O\left[n^{-\gamma}\left\{\sum_{\nu=0}^{n-1} u^{\gamma-\beta-1} F(n,\ \nu) + u^{-\beta} n^{\delta}\lambda_n\right\}\right].$$

PROOF. - We have

$$K(n,\ u) = [v^\gamma J(n,\ v)]_u^\pi - \gamma \int_u^\pi v^{\gamma-1} J(n,\ v) dv = K_1 + K_2, \text{ say.}$$

Using second estimate of $J(n, u)$ of Lemma 3, we have

$$K_1 = \pi^\gamma J(n, \pi) - u^\gamma J(n, u)$$

$$= O\left[n^{\beta-\gamma}\left\{\sum_{\nu=0}^{n-1} u^{\gamma-1}F(n, \nu) + n^\delta \lambda_n\right\}\right].$$

Also

$$K_2 = -\gamma \int_u^\pi v^{\gamma-1} \int_v^\pi (t-v)^{-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt\, dv$$

$$= -\gamma \int_u^\pi \frac{d}{dt} H^\gamma(n, t) \int_u^t v^{\gamma-1}(t-v)^{-\beta} dv\, dt$$

$$= -\gamma \int_u^\pi \left(\frac{d}{dt}\right) H^\gamma(n, t) t^{\gamma-\beta} \int_{u/t}^1 z^{\gamma-1}(1-z)^{-\beta} dz\, dt$$

$$= -\gamma \int_{u/\pi}^1 z^{\gamma-1}(1-z)^{-\beta} dz \int_\xi^\pi t^{\gamma-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt \qquad u < \xi < \pi,$$

$$= O\left\{\int_\xi^\pi t^{\gamma-\beta} \frac{d}{dt} H^\gamma(n, t)\, dt\right\}.$$

Substituting the second estimate of $H^\gamma(n, t)$ from (2.2.1), we have

$$K_2 = O\left\{[H^\gamma(n, t) t^{\gamma-\beta}]_\xi^\pi - (\gamma - \beta) \int_\xi^\pi H^\gamma(n, t) t^{\gamma-\beta-1} dt\right\}$$

$$= O\left[n^{-\gamma}\left\{\sum_{\nu=0}^{n-1} u^{\gamma-\beta-1}F(n, \nu) + u^{-\beta} n^\delta \lambda_n\right\}\right].$$

This completes the proof of Lemma 5.

3.1. Proof of the theorem. - Case (*i*): $\gamma > \beta > 0$.
It is sufficient to show that

$$\sum_1^\infty |\zeta_n^\gamma|/n < \infty,$$

where ζ_n^γ is the n-th CESÀRO mean of order γ of the sequence $\{ n^{1+\delta}\lambda_n A_n(x) \}$ i. e.

$$\zeta_n^\gamma = \frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \nu \nu^\delta \lambda_\nu A_\nu(x) \qquad \delta = \gamma - \beta,$$

(3.1.1)
$$= \frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \nu \nu^\delta \lambda_\nu \frac{2}{\pi} \int_0^\pi \varphi(t) \operatorname{Cos} \nu t \, dt$$

$$= \frac{2}{\pi} \int_0^\pi \varphi(t) \frac{d}{dt} \left\{ \frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \nu^\delta \lambda_\nu \operatorname{Sin} \nu t \right\} dt$$

$$= \frac{2}{\pi} \int_0^\pi \varphi(t) \frac{d}{dt} H^\gamma(n, t) dt$$

$$= \frac{2}{\pi} \int_0^\pi \frac{d}{dt} H^\gamma(n, t) \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta} d\Phi_\beta(u)$$

$$= \frac{2}{\pi \Gamma(1-\beta)} \int_0^\pi d\Phi_\beta(u) \int_u^\pi (t-u)^{-\beta} \frac{d}{dt} H^\gamma(n, t) dt$$

$$= \frac{2}{\pi \Gamma(1-\beta)} \int_0^\pi d\Phi_\beta(u) J(n, u)$$

$$= - \frac{2}{\pi \Gamma(1-\beta)} \int_0^\pi \Phi_\beta(u) \frac{d}{du} J(n, u) du$$

(3.1.2)
$$= - \frac{2}{\pi \Gamma(1-\beta)} [u^{-\gamma} \Phi_\beta(u) I(n, u)]_0^\pi +$$

$$\frac{2}{\pi \Gamma(1-\beta)} \int_0^\pi I(n, u) d \{ \Phi_\beta(u) u^{-\gamma} \} .$$

Thus, it is sufficient to prove that

(3.1.3)
$$\sum_1^\infty \frac{| I(n, \pi) |}{n} < \infty$$

and

(3.1.4) $$\sum_{1}^{\infty} \frac{|I(n, u)|}{n} < \infty$$

uniformly with respect to u, for $0 < u < \pi$.

PROOF OF (3.1.3). - Taking $\varphi(t) = t^{\gamma-\beta}$, we have easily

$$\Phi_\beta(t) = \frac{\Gamma(\gamma - \beta + 1)t^\gamma}{\Gamma(\gamma + 1)}$$

therefore,

$$d\{u^{-\gamma}\Phi_\beta(u)\} = 0.$$

Also

$$\int_0^\pi \varphi(t) \operatorname{Cos} \nu t \, dt = \int_0^\pi t^{\gamma-\beta} \operatorname{Cos} \nu t \, dt$$

$$\simeq \nu^{-1-\gamma+\beta}\, \Gamma(\gamma - \beta + 1) \operatorname{Cos} \frac{\pi}{2}(\gamma - \beta + 1) \quad \text{(TITCHMARSH [4]).}$$

Hence from (3.1.1) and (3.1.2), we have

$$I(n, \pi) = = O\left\{\frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \lambda_\nu\right\},$$

so that

$$\sum \frac{|I(n, \pi)|}{n} = O\left\{\sum_{1}^{\infty} n^{-\gamma-1} n^{\gamma-1}\right\} + O\left\{\sum_{1}^{\infty} n^{-\gamma-1} \sum_{1}^{n} A_{n-\nu}^{\gamma-1} \lambda_\nu\right\}$$

(3.1.5) $$= O(1) + O\left\{\sum_{1}^{\infty} \lambda_\nu \sum_{\nu}^{\infty} (n - \nu + 1)^{\gamma-1} n^{-\gamma-1}\right\}$$

$$= O(1) + O\left\{\sum_{1}^{\infty} \lambda_\nu \int_\nu^\infty (x - \nu)^{\gamma-1} x^{-\gamma-1} dx\right\}$$

$$= O(1) + O\left\{\sum_{1}^{\infty} \frac{\lambda_\nu}{\nu}\right\} = O(1).$$

Proof of (3.1.4). - Let

$$\sum_{n=1}^{\infty} \frac{1}{n} |I(n, u)| = \sum_{1}^{\left[\frac{1}{u}\right]} + \sum_{\left[\frac{1}{u}\right]+1}^{\infty} = M_1 + M_2, \text{ say.}$$

Applying Lemma 4, we find that

$$M_1 = O\left[\sum_{1}^{\left[\frac{1}{u}\right]} u^{\gamma} n^{\beta-\gamma-1} \left\{ \sum_{0}^{n-1} \nu F(n, \nu) + n^{\gamma+\delta}\lambda_n \right\}\right]$$

$$= O\left[\sum_{1}^{\left[\frac{1}{u}\right]} u^{\gamma} n^{\beta-\gamma-1} \sum_{\nu=0}^{n-1} \nu(n-\nu)^{\gamma-1} \left| \Delta \{ \nu^{\delta}\lambda_\nu \} \right| \right] + O\left[\sum_{1}^{\left[\frac{1}{u}\right]} u^{\gamma} n^{\gamma-1}\lambda_n\right]$$

$$= M_{11} + M_{12}, \text{ say.}$$

Now

$$M_{12} = O\left(u^{\gamma}\left[\frac{1}{u}\right]^{\gamma}\right) = O(1)$$

and

$$M_{11} = O\left\{u^{\gamma} \sum_{1}^{\left[\frac{1}{u}\right]} n^{\beta-\gamma-1} \sum_{1}^{n} (n-\nu+1)^{\gamma-1}(\nu-1) \left| \Delta \{ (\nu-1)^{\delta}\lambda_{\nu-1} \} \right| \right\}$$

$$= O\left\{u^{\gamma} \sum_{1}^{\left[\frac{1}{u}\right]} (\nu-1) \left| \Delta \{ (\nu-1)^{\delta}\lambda_{\nu-1} \} \right| \times \sum_{n=\nu}^{\left[\frac{1}{u}\right]} n^{\beta-\gamma-1}(n-\nu+1)^{\gamma-1} \right\}$$

$$= O\left\{u^{\gamma} \sum_{1}^{\left[\frac{1}{u}\right]} (\nu-1) \left| \Delta \{ (\nu-1)^{\delta}\lambda_{\nu-1} \} \right| \nu^{\beta-1} \right\}$$

$$= O\left\{u^{\gamma} \sum_{1}^{\left[\frac{1}{u}\right]} (\nu-1)^{\delta+1} \Delta\lambda_{\nu-1} \nu^{\beta-1} \right\} + O\left\{u^{\gamma} \sum_{1}^{\left[\frac{1}{u}\right]} \nu^{\delta}\lambda_{\nu} \nu^{\beta-1} \right\}$$

$$= M_{111} + M_{112}, \text{ say.}$$

$$M_{111} = O\left\{u^{\gamma}\left[\frac{1}{u}\right]^{\gamma}\sum_{1}^{\left[\frac{1}{u}\right]}\Delta\lambda_{\nu-1}\right\} = O(1), \text{ since } \Delta\lambda_{\nu} \geq 0,$$

and

$$M_{112} = O\left\{u^{\gamma}\left[\frac{1}{u}\right]^{\gamma}\sum_{1}^{\left[\frac{1}{u}\right]}\frac{\lambda_{\nu}}{\nu}\right\} = O(1).$$

Therefore $M_1 = O(1)$.

As regards M_2, we have

$$I(n, u) = I(n, \pi) - k(n, u)$$

and since we have already proved in (3.1.5) that

$$\sum_{n=1}^{\infty}\frac{|I(n, \pi)|}{n} < \infty,$$

it is sufficient to prove that

$$\sum_{n=\left[\frac{1}{u}\right]+1}^{\infty}\frac{|k(n, u)|}{n} = O(1),$$

uniformly in u, for $0 < u < \pi$.

Now

$$\sum_{n=\left[\frac{1}{u}\right]+1}^{\infty}\frac{|k(n, u)|}{n} = O\left\{\sum_{n=\left[\frac{1}{u}\right]+1}^{\infty}\left[u^{\gamma-1}n^{\beta-\gamma-1}\sum_{\nu=0}^{n-1}(n-\nu)^{\gamma-1}\,|\,\Delta\{\nu^{\delta}\lambda_{\nu}\}\,|\right]\right.$$

$$\left. + \frac{\lambda_n}{n} + n^{-1-\beta}u^{-\beta}\lambda_n\right\}$$

$$= M_{21} + M_{22} + M_{23}, \text{ say.}$$

It is obvious that M_{22} and M_{23} are bounded and

$$M_{21} = O\left\{ u^{\gamma-1} \sum_{n=\left[\frac{1}{u}\right]+1}^{\infty} n^{\beta-\gamma-1} \sum_{\nu=0}^{n-1} (n-\nu)^{\gamma-1} \mid \Delta \{ \nu^{\delta}\lambda_{\nu} \} \mid \right\}$$

$$= O\left\{ u^{\gamma-1} \sum_{\nu=0}^{\left[\frac{1}{u}\right]} \mid \Delta(\nu^{\delta}\lambda\nu) \mid \sum_{n=\left[\frac{1}{u}\right]+1}^{\infty} n^{\beta-\gamma-1}(n-\nu)^{\gamma-1} \right\}$$

$$+ O\left\{ u^{\gamma-1} \sum_{\nu=\left[\frac{1}{u}\right]}^{\infty} \mid \Delta(\nu^{\delta}\lambda\nu) \mid \sum_{\nu+1}^{\infty} n^{\beta-\gamma-1}(n-\nu)^{\gamma-1} \right\}$$

$$= M_{211} + M_{212}, \text{ say.}$$

Now

$$M_{212} = O\left\{ u^{\gamma-1} \sum_{\nu=\left[\frac{1}{u}\right]}^{\infty} (\nu^{\delta}\Delta\lambda_{\nu} + \nu^{\delta-1}\lambda_{\nu+1})\nu^{\beta-1} \right\}$$

$$= O\left\{ u^{\gamma-1} \left[\frac{1}{u}\right]^{\gamma-1} \right\} = O(1), \text{ as } u \to 0.$$

Since, in the case of M_{211}, $\nu < \left[\frac{1}{u}\right] + 1$, we have

$$M_{211} = O\left\{ u^{\gamma-1} \sum_{n=\left[\frac{1}{u}\right]+1}^{\infty} n^{\beta-2} \right\} + O\left\{ u^{\gamma-1} \sum_{1}^{\left[\frac{1}{u}\right]} \mid \Delta \{ \nu^{\delta}\lambda_{\nu} \} \mid \right.$$

$$\left. \times \int_{\left[\frac{1}{u}\right]+1}^{\infty} x^{\beta-\gamma-1}\left(x - \left[\frac{1}{u}\right] - 1\right)^{\gamma-1} dx \right\}$$

$$= O\left\{ u^{\gamma-1} \left[\frac{1}{u}\right]^{\beta-1} \right\} + O\left\{ u^{\gamma-1} \left[\frac{1}{u}\right]^{\beta-1} \sum_{1}^{\left[\frac{1}{u}\right]} (\nu^{\delta}\Delta\lambda_{\nu} + \nu^{\delta-1}\lambda_{\nu+1}) \right\}$$

$$= O\left\{ u^{\gamma-1} \left[\frac{1}{u}\right]^{\gamma-1} \right\} = O(1).$$

Hence

$$M_2 = O(1).$$

This proves the theorem for $\beta > 0$.

Case (*ii*): *When* $\gamma > \beta = 0$. In this case, we are given that

$$\int_0^\pi | d\{t^{-\gamma}\varphi(t)\} | < \infty, \qquad 0 < \gamma \leq 1$$

and we have to show that $\sum_0^\infty n^\gamma \lambda_n A_n(x)$ is summable $| C, \gamma |$, that is, we have to establish the convergence of the series $\Sigma | \zeta_n^\gamma | / n$, where [1]

$$\zeta_n^\gamma = \frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \nu \nu^\gamma \lambda_\nu A_\nu(x)$$

$$= \frac{2}{\pi} \int_0^\pi \varphi(t) \frac{d}{dt} H^\gamma(n, t) dt$$

$$= \frac{2}{\pi} \int_0^\pi (t^{-\gamma}\varphi(t)) \left(t^\gamma \frac{d}{dt} H^\gamma(n, t) \right) dt$$

$$= O\left\{ \int_0^\pi v^\gamma \frac{d}{dv} H^\gamma(n, v) dv \right\}$$

$$+ O\left\{ \int_0^\pi d\{t^{-\gamma}\varphi(t)\} \int_0^t v^\gamma \frac{d}{dv} H^\gamma(n, v) dv \right\}.$$

Writing

$$\bar{I}(n, t) = \int_0^t v^\gamma \frac{d}{dv} H^\gamma(n, v) dv,$$

it is sufficient to show that

$$\sum_1^\infty \frac{| \bar{I}(n, t) |}{n} < \infty,$$

uniformly in t, for $0 < t \leq \pi$. Putting $\varphi(t) = t^\gamma$, we have, as in the case (*i*)

$$\bar{I}(n, \pi) = O\left\{ \frac{1}{A_n^\gamma} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \lambda_\nu \right\}.$$

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(1) From this place onward we shall be using the same notations as in the case (*i*) but with the obvious change that here $\delta = \gamma$.

Hence

$$\sum_{1}^{\infty} \frac{|\bar{I}(n, \pi)|}{n} = O\left\{\sum_{1}^{\infty} n^{-\gamma-1} \sum_{\nu=0}^{n} (n-\nu+1)^{\gamma-1}\lambda_\nu\right\} = O(1), \text{ as shown}$$

in (3.1.5). It, therefore, remains to show that

$$\sum_{1}^{\infty} \frac{|\bar{I}(n, t)|}{n} < \infty,$$

uniformly in t for $0 < t < \pi$.

Now

$$\sum_{1}^{\infty} \frac{|\bar{I}(n, t)|}{n} = \sum_{1}^{\left[\frac{1}{t}\right]} + \sum_{\left[\frac{1}{t}\right]+1}^{\infty} = N_1 + N_2, \text{ say.}$$

By the second mean value theorem and the first estimation of (2.2.1) we have

$$N_1 = O\left\{\sum_{1}^{\left[\frac{1}{t}\right]} \left[t^\gamma n^{-\gamma-1} \sum_{1}^{n-1} \nu(n-\nu)^{\gamma-1} |\Delta| \nu^\gamma \lambda_\nu || + t^\gamma n^{\gamma-1}\lambda_n\right]\right\}$$

$$= O\left\{t^\gamma \sum_{1}^{\left[\frac{1}{t}\right]} \nu |\Delta| \nu^\gamma \lambda_\nu || \sum_{\nu+1}^{\left[\frac{1}{t}\right]} n^{-\gamma-1}(n-\nu)^{\gamma-1}\right\} + O(1)$$

$$= O\left\{t^\gamma \sum_{1}^{\left[\frac{1}{t}\right]} |\Delta| \nu^\gamma \lambda_\nu ||\right\} + O(1) = O(1).$$

Also

$$N_2 = O\left\{\sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} \left[\frac{|\bar{I}(n, \pi)|}{n} + \frac{|\bar{K}(n, t)|}{n}\right]\right\},$$

where

$$\bar{K}(n, t) = \int_{t}^{\pi} v^\gamma \frac{d}{dv} H^\gamma(n, v) dv.$$

Now

$$\sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} \frac{|\bar{I}(n, \pi)|}{n} < \infty$$

and from the second relation for $H^{\gamma}(n, v)$, we have

$$\bar{K}(n, t) = [v^{\gamma} H_{-}^{\gamma}(n, v)]_t^{\pi} - \gamma \int_t^{\pi} v^{\gamma-1} H^{\gamma}(n, v) dv$$

$$= O\left\{ n^{-\gamma} t^{\gamma-1} \sum_{\nu=0}^{n-1} (n-\nu)^{\gamma-1} \left| \Delta \left| \nu^{\gamma} \lambda_{\nu} \right| \right| + \lambda_n \right\}$$

$$- \frac{\gamma}{A_n^{\gamma}} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} \nu^{\gamma} \lambda_{\nu} \int_t^{\pi} v^{\gamma-1} \operatorname{Sin} \nu v \, dv$$

$$= X_1 + X_2, \text{ say.}$$

But

$$\sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} \frac{|X_1|}{n} = O\left\{ \sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} \frac{\lambda_n}{n} \right\} + O\left\{ \sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} t^{\gamma-1} n^{-\gamma-1} \right.$$

$$\left. \times \sum_{\nu=0}^{n-1} (n-\nu)^{\gamma-1} \left| \Delta \left| \nu^{\gamma} \lambda_{\nu} \right| \right| \right\}$$

$$= O(1),$$

by proceeding as in the proof of the relation:

$$M_{21} = O(1).$$

Also

$$\sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} \frac{|X_2|}{n} = O\left\{ \sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} n^{-\gamma-1} \sum_{\nu=1}^{n} A_{n-\nu}^{\gamma-1} \nu^{\gamma-1} \lambda_\nu t^{\gamma-1} \right\}$$

$$= O\left\{ t^{\gamma-1} \sum_{\nu=1}^{\left[\frac{1}{t}\right]+1} \nu^{\gamma-1} \lambda_\nu \sum_{n=\left[\frac{1}{t}\right]+1}^{\infty} n^{-\gamma-1} A_{n-\nu}^{\gamma-1} \right\}$$

$$+ O\left\{ t^{\gamma-1} \sum_{\nu=\left[\frac{1}{t}\right]+1}^{\infty} \nu^{\gamma-1} \lambda_\nu \sum_{n=\nu}^{\infty} n^{-\gamma-1} A_{n-\nu}^{\gamma-1} \right\}$$

$$= O\left\{ t^{\gamma-1} \left[\frac{1}{t}\right]^{\gamma-1} \sum_{\nu=1}^{\left[\frac{1}{t}\right]+1} \frac{\lambda_\nu}{\nu} \right\} + O\left\{ t^{\gamma-1} \sum_{\nu=\left[\frac{1}{t}\right]+1}^{\infty} \nu^{\gamma-2} \lambda_\nu \right\}$$

$$= O\left\{ t^{\gamma-1} \left[\frac{1}{t}\right]^{\gamma-1} \right\} = O(1).$$

This completes the proof of the theorem for the case $\gamma > \beta = 0$.

Case *(iii)*: *When* $\gamma = \beta > 0$. The proof of the theorem for this case is similar to that of Case *(i)* and hence we omit it

Case *(iv)*: *When* $\gamma = \beta = 0$. In this case $A_n(x) = O\left(\frac{1}{n}\right)$, hence

$$\Sigma \lambda_n \mid A_n(x) \mid = \Sigma\, O \left\{ \frac{\lambda_n}{n} \right\} < \infty.$$

This completes the proof of the theorem.

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