The Cauchy Problem at Simply Characteristic Points and P-Convexity (*).

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Summary. – *Let $\Omega \subset \boldsymbol{R}^n$ be an open set and let P be a linear partial differential operator with constant coefficients in $\boldsymbol{R}^n$. Then Ω is said to be P-convex if for each $f \in C^\infty(\Omega)$ there is a $u \in \mathcal{D}'(\Omega)$ such that $P(D)u = f$. A complete geometric characterization of P-convex sets in $\boldsymbol{R}^3$ is given when P is of principal type and when Ω has C^2-boundary. As a step in the proof one also obtains necessary and sufficient conditions for uniqueness in the local Cauchy problem at simply characteristic points in $\boldsymbol{R}^3$. The tools are a sophisticated use of the author's uniqueness cones on one hand and his semi-global nullsolutions on the other hand. Hints are given on the difficulties that may be encountered in $\boldsymbol{R}^n$ for the same problem.*

1. – Introduction.

In this paper we shall treat uniqueness in the local Cauchy problem when the linear partial differential equation $P(D)u = f$ has constant coefficients. We shall also treat the connected problem of global solvability of $P(D)u = f$ in an open set Ω when $f \in C^\infty(\Omega)$. MALGRANGE [15, Theorème 4, p. 295] showed that $P(D)u = f$ has a solution $u \in \mathcal{D}'(\Omega)$ for each $f \in C^\infty(\Omega)$ if and only if for each compact set $K \subset \Omega$ there is another compact set $K' \subset \Omega$ such that

$$(1.1)\qquad u \in \mathcal{E}'(\Omega)\,, \quad \operatorname{supp} P(-D)u \subset K \Rightarrow \operatorname{supp} u \subset K'\,.$$

With HÖRMANDER [8, p. 80] we call such a set P-convex.

We have here used the standard notation of [8] and we shall also stick to it in the following except that we shall use $D_j = \partial/\partial x_j$, $j = 1, \dots, n$.

Let it now be that $K \subset \Omega$ and that K is compact. Let it also be that to each $x \in \partial\Omega$ we can find a neighbourhood Ω'_x of x in $\complement K$ such that

$$(1.2)\qquad u \in \mathcal{D}'(\Omega'_x)\,, \quad P(-D)u = 0\,, \quad \operatorname{supp} u \subset \Omega'_x \cap \bar{\Omega} \Rightarrow u = 0\,.$$

If $u \in \mathcal{E}'(\Omega)$ then we extend it to $\mathcal{E}'(\boldsymbol{R}^n)$ by letting $u = 0$ outside Ω. Then u restricted to Ω'_x in (1.2) is in $\mathcal{D}'(\Omega'_x)$ so u is zero in Ω'_x. Let $\hat{K}$ be the convex hull of K. LIONS [14], see also [8, Lemma 3.4.3, p. 80] has proved that $u \in \mathcal{E}'(\boldsymbol{R}^n)$, $\operatorname{supp} P(-D)u \subset K$ implies that $\operatorname{supp} u \in \hat{K}$. Let $K' = \hat{K} \cap \Omega \cap \Big(\complement\Big(\bigcup_{x\in\partial\Omega} \Omega'_x\Big)\Big)$. Now

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K' is a compact subset of Ω and (1.1) is satisfied. This argument shows the connection between local and semilocal uniqueness in the Cauchy problem and P-convexity.

A natural problem is to give a geometric characterization of P-convexity. In $\mathbf{R}^2$ there is the following complete characterization.

THEOREM 1.1 (HÖRMANDER [8, Theorem 3.7.2, pp. 89-90]). – *An open connected set $\Omega \subset \mathbf{R}^2$ is P-convex if and only if every characteristic line intersects Ω in an open interval.*

In $\mathbf{R}^n$, $n \geqslant 3$, one knows that every open set Ω is P-convex if and only if P is elliptic [8, Cor. 3.7.1, p. 89], and that Ω is P-convex for all P if, MALGRANGE [15, Théorème 3, p. 294], and only if, TREVES [26, Prop. 6.1, p. 351], Ω is convex. These are general theorems. However, no general theorem of the Theorem 1.1 type is known. A natural first step forward is to look at operators $P(D)$ of principal type, that is operators such that if $P_m(\xi) = 0$, $0 \neq \xi \in \mathbf{R}^n$ then grad $P_m(\xi) \neq 0$. In order to simplify further this paper is restricted to the case $n = 3$ and to Ω with $\partial\Omega$ in C^2. Then one can give necessary and sufficient geometric conditions for P-convexity. This also applies to uniqueness in the local Cauchy problem.

Some rather lengthy definitions will now be given and with their help the theorems will be stated in a short form. Afterwards one discusses the rather restricted hypothesis in the theorems and what one may expect in more general cases in connection to already known results.

First we need a proper definition of local uniqueness in the Cauchy problem.

DEFINITION 1.1. – *Let $\Omega \subset \mathbf{R}^n$ be an open set and let F be a nonempty set which is closed in Ω. Let $P(x, D)$ be a linear partial differential operator in Ω with C^∞-coefficients. Let $x^0 \in \partial F \cap \Omega$. Then there is local uniqueness in the Cauchy problem at x^0 (in relation to F) if to each neighbourhood Ω' of x^0 there is another neighbourhood Ω'' of x^0 such that $\Omega'' \subset \Omega'$ and such that if $u \in \mathcal{D}'(\Omega'')$, $P(x, D)u = 0$ in Ω'',* supp *$u \subset F \cap \Omega''$ implies that $u = 0$. When there is not local uniqueness at x^0 then we say that there is local nonuniqueness in the Cauchy problem at x^0 (in relation to F).*

REMARK. – Local nonuniqueness means that there exists a neighbourhood Ω' of x^0 such that for each neighbourhood Ω'' of x^0 with $\Omega'' \subset \Omega'$ there is a $u \in \mathcal{D}'(\Omega'')$, $u \neq 0$, with supp $u \subset F \cap \Omega''$ and $P(x, D)u = 0$.

DEFINITION 1.2. – *Let $P(\xi)$ be a polynomial of order m in $\mathbf{R}^n$ with constant coefficients. Let P_m be its principal part. If $P_m(\xi) \neq 0$ then ξ is said to be noncharacteristic. If $P_m(\xi) = 0$ then ξ is characteristic. If $P_m(\xi) = 0$ and* $\operatorname{grad}_\xi P_m(\xi) \neq 0$ *then ξ is called simply characteristic. Let $n = 3$. Let $\xi = (1, 0, 0)$ be simply characteristic and assume that* Re grad $P_m(1, 0, 0) = (0, 1, 0)$. *If $P_m(1, \xi_2, \xi_3) \neq 0$ for all small real $(\xi_2, \xi_3) \neq (0, 0)$ then we say that $(1, 0, 0)$ is simply characteristic of type 1. If $P_m(1, 0, \xi_3) = 0$ for all $\xi_3 \in \mathbf{R}$ then $(1, 0, 0)$ is said to be simply characteristic of* *type* ∞. *If* $\operatorname{Re} P_m(1, 0, \xi_3) = \sum_{j=k}^{m} a_j \xi_3^j$ *with* $a_k \neq m$, $2 \leqslant k \leqslant m$ *and if* $(1, 0, 0)$ *is not of type* 1 *then* $(1, 0, 0)$ *is said to be of type* k.

REMARK. – Let $(1, 0, 0)$ be a simply characteristic point. One can always rotate the x_2, x_3-coordinates and then multiply the operator P by a proper complex constant such that with P denoting the new operator $\operatorname{grad} \operatorname{Re} P_m(1, 0, 0) = (0, 1, 0)$. Then there is a unique analytic function $s(t)$, defined for small t such that $\operatorname{Re} P_m(1, s(t), t) = 0$ and $s(0) = 0$ since $\operatorname{Re} \operatorname{grad} P_m(1, 0, 0) = (0, 1, 0) \neq 0$. It follows that $t \to \operatorname{Im} P_m(1, s(t), t)$ is an analytic function with $\operatorname{Im} P_m(1, s(0), 0) = 0$. So there are two possibilities. $\operatorname{Im} P_m(1, s(t), t) \equiv 0$ for small t or $\operatorname{Im} P_m(1, s(t), t) \neq 0$, $0 < |t| < \delta$, for some $\delta > 0$. That proves that the type of a simply characteristic point $(1, 0, 0)$ is a well defined number. One also sees how to give a definition invariant under orthogonal transformations. Since we shall use the above definition in our computations we stick to it all the time.

In the following a function denoted by ψ will have real values.

We can now formulate a theorem on local uniqueness.

THEOREM 1.2. – *Let* Ω *be an open set,* $0 \in \Omega$, *and let* $\psi \in C^2(\Omega)$ *with* $\operatorname{grad} \psi \neq 0$ *in* Ω *and* $\psi(0) = 0$. *Let* $F = \{x; \psi(x) \geqslant 0\}$. *Let* $P(D)$ *be a linear partial differential operator with constant coefficients. Let* $\operatorname{grad} \psi(0) = (1, 0, 0)$ *and let* $(1, 0, 0)$ *be a simply characteristic point of type* k.

If $k = 1$ *then there is local nonuniqueness in the Cauchy problem at* $x = 0$ *if and and only if for some* $\delta > 0$

$$\{x; \psi(x) \geqslant 0, |x| < \delta\} \supset \{x; x_1 \geqslant 0, |x| < \delta\}. \tag{1.3}$$

If $2 \leqslant k < \infty$ *then there is nonuniqueness in the local Cauchy problem if and only if there is a* $K > 0$ *and* $\delta > 0$ *such that*

$$\{x; \psi(x) \geqslant 0, |x| < \delta\} \supset \{x; x_1 \geqslant K |x_3|^{k/(k-1)}, |x| < \delta\}. \tag{1.4}$$

If $k = \infty$ *and* P *is of principal type i.e. if* $\xi \neq 0$ $\xi \in \boldsymbol{R}^3$, $P_m(\xi) = 0$, *implies that* $\operatorname{grad} P_m(\xi) \neq 0$, *then there is local nonuniqueness in the Cauchy problem if and only if there is a* $K > 0$ *and* $\delta > 0$ *such that* (1.4) *is true when one reads* $k/(k-1) = 1$ *for* $k = \infty$.

The theorem will be proved in the following sections.

DEFINITION 1.3. *Let* Ω *be an open set in* $\boldsymbol{R}^3$ *with boundary in* C^2. *Let* $x_0 \in \partial\Omega$. *Let* $\delta > 0$ *and* $\psi \in C^2(\{x; |x - x^0| < \delta\})$ *such that* $\operatorname{grad} \psi \neq 0$, $|x - x^0| < \delta$, $\operatorname{grad} \psi(x^0) = (1, 0, 0)$, $\Omega \cap \{x; |x - x^0| < \delta\} = \{x; |x - x_0| < \delta, \psi(x) \geqslant 0\}$. *Let* $P(D)$ *be a linear operator of order* m *with constant coefficients. Let* P_m *be its principal part.*

If $P_m(1, 0, 0) = 0$ *then* x^0 *is said to be a characteristic point of* $\partial\Omega$.

If $(1, 0, 0)$ *is simply characteristic of type* 1 *for* P_m *then* x^0 *is said to be a semiglobal nonuniqueness point of type* 1 *if the component containing* x^0 *of*

$$H = \{x;\, x_1 = 0,\, x \in \partial\Omega\} \tag{1.5}$$

is compact in $\mathring{H}'$ *(in the topology of* $\{x;\, x_1 = 0\}$*) when*

$$H' = \{x;\, x_1 = 0,\, x \in \bar{\Omega}\}\,. \tag{1.6}$$

If H *in* (1.5) *is not a compact subset of* $\mathring{H}'$ *then* x^0 *is said to be a point of semiglobal uniqueness point of type* 1.

Let $(1, 0, 0)$ *be simply characteristic of type* ∞ *for* P_m *and let*

$$P_m(\xi) = \xi_1^{m-1}\xi_2 + \xi_2 \sum_{\substack{|\alpha| = m-1 \\ \alpha_1 < m-1}} a_\alpha \xi^\alpha\,. \tag{1.7}$$

Let

$$L = \{x;\, x^0 + t(0, 1, 0),\, t \in \boldsymbol{R}\}\,.$$

Let L' *be the interval of* $L \cap \bar{\Omega}$ *which contains* x^0 *and let* L'' *be the interval of* $L \cap \partial\Omega$ *which contains* x^0. *If* L' *and* L'' *have no common endpoints, finite or not finite, then* x^0 *is said to be a point of semiglobal nonuniqueness of type* ∞. *If* L' *and* L'' *have at least one common endpoint then* x^0 *is called a point of semiglobal uniqueness of type* ∞.

Let $(1, 0, 0)$ *be simply characteristic of type* k, $2 \leqslant k \leqslant m$. *Let* L, L' *and* L'' *be defined as above. Let* x'' *be the centre of* L'' *and assume that*

$$L'' = \{(0, t, 0);\, |t| \leqslant a'\}$$

for some $a' \geqslant 0$. *Let* $\operatorname{Re} P_m$ *have the form*

$$\operatorname{Re} P_m = \xi_1^{m-1}\xi_2 + \xi_2\Big(\sum_{\substack{|\alpha| = m-1 \\ \alpha_1 < m-1}} a_\alpha \xi^\alpha\Big) + \sum_{j=k}^{m} b_j \xi_1^{m-j}\xi_3^j \tag{1.8}$$

with $b_k \neq 0$. *Let* $a' > 0$. *Let the normal of* $\partial\Omega$ *along* L'' *be proportional to* $(1, 0, 0)$. *Let*

$$M = (k-1)|b_k a'|^{1/(1-k)} k^{k/(1-k)}\,.$$

Along L'' $\partial\Omega$ *is given by*

$$x_1 = N(x_2)x_3^2 + o(x_3^2)\,, \qquad |x_2| \leqslant a'\,.$$

Here $N(x_2)$ *is continuous on* $|x_2| \leqslant a'$. *Then* x^0 *is called a point of semiglobal nonuniqueness of type* k *if one of the following three conditions are fulfilled*

$$k = 2\,;\quad b_2 > 0\,;\quad N(x_2) < M(1 - x_2/a')^{-1}\,,\quad -a' \leqslant x_2 < a'\,, \tag{1.9}$$

$$k = 2\,;\quad b_2 < 0\,;\quad N(x_2) < M(1 + x_2/a')^{-1}\,,\quad -a' < x_2 \leqslant a'\,, \tag{1.10}$$

$$k > 2\,. \tag{1.11}$$

If $a' = 0$ and if x^0 is a point of local nonuniqueness of type k in the Cauchy problem with F replaced by $\bar{\Omega}$ and Ω replaced by $\boldsymbol{R}^n$ then x^0 is also called a point of semiglobal nonuniqueness of type k. If x^0 is not a point of semiglobal nonuniqueness of type k then it is called a point of semiglobal uniqueness of type k.

REMARK. – Let $(1, 0, 0)$ be normal to $\partial\Omega$ at x^0 and let x^0 be a simply characteristic point of type k, $2 \leqslant k \leqslant m$. Let $a' > 0$. If the normal to $\partial\Omega$ at some point of L'' is not proportional to $(1, 0, 0)$ then x^0 is a point of semiglobal uniqueness.

All those heavy definitions are made to suit the formulation of the following theorem.

THEOREM 1.3. – *Let Ω be an open set in $\boldsymbol{R}^3$ with C^2 boundary. Let P be a linear partial differential operator of principal type in $\boldsymbol{R}^3$ with constant coefficients. Then Ω is P-convex if and only if $\partial\Omega$ contains no point of semiglobal nonuniqueness for the operator P.*

REMARK. – In the conclusion of the theorem one should expect $\check{P}$-convex to be used instead of P-convex. The proof will show that uniqueness and nonuniqueness depends on the principal part of the operator in such a way that it does not matter if we look at $P(-D)$ or $P(D)$. So for the sake of convenience we just treat $P(D)$ where one should look at $P(-D)$.

The first one to treat nonuniqueness in the characteristic Cauchy problem seems to be Goursat. Let Ω be open in $\boldsymbol{R}^2$ and let

$$P(\xi) = \xi_1\xi_2 + \sum_{|\alpha| \leqslant 1} a_\alpha \xi^\alpha .$$

If $0 \in \Omega$ and $F = \{x; x_1 \geqslant 0, x \in \Omega\}$ then GOURSAT [5, pp. 303-308] shows that 0 is a point of local nonuniqueness for P. Now let P be an operator in $\boldsymbol{R}^n$ with constant coefficient and let $N = (1, 0, \ldots, 0) \in \boldsymbol{R}^n$ be a simple characteristic point. Let also P_m have real coefficients, let $\operatorname{grad} P_m(N) = (0, 1, 0, \ldots, 0) = N'$, and let $x = (x_1, x_2, x'') \in \boldsymbol{R}^n$, $x'' \in \boldsymbol{R}^{n-2}$. Let $\Omega' \subset \boldsymbol{R}^n$ be open and let $F = \bar{\Omega}'$. Let $0 \in F$, and let $F \cap \{x; |x| < \delta\} = \{x; \psi(x) \geqslant 0, |x| < \delta\}$ for some $\psi \in C^2$ with $\operatorname{grad} \psi(x) \neq 0$, $\operatorname{grad} \psi(0) = N$. Let $\Omega = \boldsymbol{R}^n$. Then MALGRANGE [16] shows that there is local nonuniqueness at $x = 0$ if for some $M > 0$ and $\delta' > 0$, one has

$$\psi(M|x''|^2, x_2, x_3) \geqslant 0 , \qquad |x'| < \delta' . \tag{1.12}$$

Malgrange's results are somewhat weaker but it is pointed out by HÖRMANDER [11, p. 108] that Malgrange's proof gives precisely this. It was pointed out by PERSSON [22, Theorem 5] that (1.12) is also sufficient for nonuniqueness in the analytic coefficient case. Malgrange's proof consists of a construction of a simply characteristic analytic hypersurface lying above $\psi(x) = 0$. Then a nullsolution is constructed by the Goursat technique. Other formulations to garantee (1.12) are given by TREVES [26, Theorem 6.10, pp. 377-378], ZACHMANOGLOU [35] and also by PERSSON [22]. The two last references refer to operators with analytic coefficients. Possible ways of extending Malgrange's result to operators with linearly independent $\operatorname{Re} P_m(N)$ and $\operatorname{Im} P_m(N)$ are indicated by HÖRMANDER [11, Theorem 1.3.5]. In our language we then have a point of type $k = 1$ in the $\boldsymbol{R}^3$ case. Here the only possible characteristic surface will be the plane.

If we have a hyperplane H as initial hyperplane then we can use Hörmander's nullsolution if H is simply characteristic or not or [7, Theorem 3.2] [8, Theorem 3.7.1, p. 89]. For variable analytic coefficients this corresponds to the local result by PERSSON [18] or the semiglobal result by PERSSON [23], [24, Theorem 4′]. The results in [23], [24] combined with an approximation technique are cornerstones in our proofs of nonuniqueness in this paper.

In our eagerness to get conditions of the necessary and sufficient type we treat operators in $\boldsymbol{R}^3$ with constant coefficients. The surfaces involved are supposed to be in C^2 and the characteristic points of the surfaces are supposed to be simple. We even suppose that the operator is of principal type in Theorem 1.3 and partly in Theorem 1.2. What happens when one looks at $\boldsymbol{R}^n$, $n > 3$. The type k may be defined for every $\xi'' \in R^{n-2}$, $|\xi''| = 1$, as k in $P_m(1, 0, t\xi'') = \sum_{j=k}^{m} b_j t^j$, $b_k \neq 0$, if $2 \leqslant k \leqslant m$. But already $k = 1$ seems to give trouble. And how can one combine it to give a necessary and sufficient condition. As to the regularity of ∂F in the Cauchy problem one sees that it does not matter for local nonuniqueness as long as one can get a simply characteristic analytic surface inside F close to x^0. As is proved by PERSSON [22] this also applies to surfaces of higher constant multiplicity and under certain additional conditions also to the semiglobal case [23], [24]. For the Cauchy problem with analytic coefficients other types of null solutions have been constructed by KOMATSU [13]. That one may have uniqueness in the class of C^m-functions for certain characteristic problems is proved by BAOUENDI and GOULAOUIC [1].

The origin of the uniqueness theorems which are used here is the well known theorem by HOLMGREEN [6] on the local noncharacteristic linear Cauchy problem for equations with analytic coefficients. HÖRMANDER [8, Theorem 5.3.1, p. 125] has generalized it to distribution solutions. In Section 2 after Theorem 2.1 there are some comments on the history of uniqueness theorems. There we also define what is meant by uniqueness cones. They were introduced by PERSSON in [21] but already Hörmander proved Theorem 2.1. The main idea with uniqueness cones is expressed by Theorem 2.2 and Theorem 2.3. These theorems are other cornerstones in the proof of local uniqueness and semiglobal uniqueness.

The local uniqueness at a simple characteristic point was first treated by HÖRMANDER [8, Theorem 5.3.2, p. 126] for a C^2 initial hypersurface. It was generalized by TREVES [26, Theorem 6.8, pp. 368-369] and ZACHMANOGLOU [33], [34] for higher order of contact between L from definition 1.3 and ∂F. In [8] and [34] also analytic coefficients are allowed. Anyhow x^0 is a common endpoint of L' and L''. In [9, Theorem 8.1] HÖRMANDER gives a very sharp uniqueness theorem of global character.

In its local version for constant coefficients it says that if ∂F is in C^1 and if x^0 is a common endpoint of L' and L'' then there is local uniqueness at x^0. The same result for ∂F in C^2 was proved by uniqueness cones in PERSSON [21]. Hörmander's result is written out for P_m with real coefficients. The result in [21] also applies to the case with complex valued coefficients in P_m. One may say that the treatment of points of type 1 in the present paper goes back to this observation.

The first example of uniqueness when $L = L' = L''$ was given by PERSSON [21, p. 79]. In [22] PERSSON gave some examples on uniqueness when L' and L'' have no common endpoint. Another simple example of local uniqueness is given by $\Omega = \boldsymbol{R}^3$, $F = \{x;\, x_1 \geqslant x_2 x_3\}$, $P(D) = D_1 D_2 + D_3^2$, $x^0 = 0$. Here $L = L' = L''$ and there is uniqueness at x^0 since the normal of ∂F at $(0, t, 0)$, $t \neq 0$, is noncharacteristic. Holmgren's theorem says that this point is a local uniqueness point. This zero is then transported to 0 by uniqueness cones or by Hörmander's result [9, Theorem 8.1].

In [22] it is said that the first examples above show that L' and L'' alone cannot decide on uniqueness or not uniqueness. The simple conjecture is that the existence of certain simply characteristic surfaces does. TREVES [29, Theorem I] has given a uniqueness theorem for a characteristic Cauchy problem at $x = 0$ in $\boldsymbol{R}^2$ when the multiplicity of the characteristic line at $x = 0$ is two but equal to one outside $x = 0$. It has been generalized to distribution solutions by BIRKELAND and PERSSON [2, Theorem 1.3]. This shows that the conjecture in [21] that one can use uniqueness cones to decide on uniqueness is not true in general although it is true in the case covered by Theorem 1.2.

Generalization to higher dimensions of uniqueness results and results on P-convexity will probably meet many difficulties. We have not written down some obvious ones to keep this paper at a moderate length. This also applies to local results when the coefficients are variable. One way to get sharper results seems to be to combine the philosophy of uniqueness cones with Hörmander's result [9, Theorem 8.1]. We did not manage in some details. Therefore we have C^2 surfaces everywhere.

To the already cited results we like to add the results on P-convexity by ZACHMANOGLOU [39], PERSSON [19] and [21, Theorem 9.1]. Local uniqueness is treated by ZACHMANOGLOU in [36], [37], and [38]. He gives a necessary and sufficient condition for uniqueness in the Cauchy problem for first order equation with analytic coefficients [36]. This condition specialized down to the Theorem 1.2 case is just the same as the condition of this theorem. In [37], [38] he gives one necessary and one sufficient condition for uniqueness for special higher order equations.

Other uniqueness theorems have been proved by BONY [3], HÖRMANDER [10] and BONY [4]. Further references are given in Section 2. The references are by no means complete. They include the sources which in the author's opinion, are most relevant for the problem of this paper.

The paper is organized as follows. Section 2 gives the definition of uniqueness cones and general theorems of Holmgren type. Section 3 treats the geometry of uniqueness cones connected with the proofs of Theorem 1.2 and Theorem 1.3. The uniqueness part of Theorem 1.2 is proved in Section 4 with the uniqueness cones of Section 3. In Section 5 local and semiglobal uonuniqueness results are proved. They complete the proof of Theorem 1.2. The remaining semiglobal uniqueness part of Theorem 1.3 is then proved in Section 6.

2. – Uniqueness cone.

Let $P(D)$ be a linear partial differential operator of order $m > 0$ in $\boldsymbol{R}^n$. Let P_m be its principal part. Let $M \subset \boldsymbol{R}^n$ be an open convex set contained in a half space of $\boldsymbol{R}^n$. For $N \in M$, $x^0 \in \boldsymbol{R}^n$, $\varrho > 0$ define

$$K(N, M, x^0, \varrho) = \{x;\ x \in \boldsymbol{R}^n,\ \langle x - x_0, N\rangle \geqslant -\varrho,\ \langle x - x^0, \delta\rangle \leqslant 0,\ \delta \in M\}\,.$$

We also allow $\varrho = \infty$ here. If $M \subset \{\xi; P_m(\xi) \neq 0\}$ then $K(N, M, x^0, \varrho)$ is said to be a uniqueness cone for P at x^0 in the direction N. We use this language in the formulation of the following theorem due to Hörmander.

THEOREM 2.1 (HÖRMANDER [8, Cor. 5.3.3, p. 130]). – *Let $P(D)$ be a linear partial differential operator in $\boldsymbol{R}^n$ with constant coefficients. Let $\Omega \subset R^n$ be an open set. Let $K(N, M, x^0, \varrho) \subset \Omega$ be a uniqueness cone for P. Let $u \in \mathcal{D}'(\Omega)$ be such that*

$$(\operatorname{supp} u) \cap K(N, M, x^0, \varrho) \cap \{x; \langle x - x^0, N\rangle = -\varrho\}$$

is empty. If $P(D)u = 0$ in the inner points of $K(N, M, x^0, \varrho)$ then also $u = 0$ there.

REMARK. – Hörmander's result is a generalization to distribution solutions of results by F. JOHN [12] for function solutions of partial differential equations. However Hörmander has not made much use of this result in his book or otherwise. He has mostly used various deformations of noncharacteristic hypersurfaces to show uniqueness as is already done in his proof of Holmgren's uniqueness theorem.

In [28, Th. 12.1, p. 60] TREVES gave a new proof of Holmgren's uniqueness theorem for a C^2 initial hypersurface. By a nonlinear analytic change of coordinates he reduced the Cauchy problem to a special case of his dual Cauchy-Kovalevskij theorem for functions in one variable with values in the space of analutic functionals. See also PERSSON [25]. TREVES calls the abstract theorem behind the dual Cauchy-Kovalevskij theorem Ovsjannikov's theorem [17] although he himself independently found it [28, p. 2]. If the abstract theorem should be labelled it seems appropriate to call it Yamanaka's theorem since YAMANAKA [30] proved it already in 1961.

Let us call the direction normal to the initial hyperplane of a noncharacteristic Cauchy problem for the time direction. The other variables are the space variables. The deformation in the proof of Holmgren's uniqueness theorem leads to a Cauchy problem with solutions having compact support in the space variables for a fixed time. In [20] Persson showed that cut off functions in the space variables of the original problems combined with a direct application of the dual Cauchy-Kovalevskij theorem gave a finite velocity of propagation of zeros. Here the coefficients of the operator are assumed to be analytic functions or even continuous functions analytic in the space variables. This implies Holmgren's uniqueness theorem. The author used this in [21] to prove Theorem 2.1. This shows that one can reach this theorem for constant coefficients without the use of nonlinear transformations. The trick with cut off functions has also been used by BAOUENDI and GOULAOUIC [1] for other generalizations of Holmgrens's uniqueness theorem. We also like to point out a simplification applicable to the proof in [20] due to YAMANAKA and PERSSON [32]. In [32] there is a beautiful procedure due to Yamanaka by which one avoids the reduction of the differential equation $P(D)u = 0$ to a first order system gaining additional information on the velocities of propagation of zeros of the solution of the equation. See also YAMANAKA [31].

The real idea behind [21] is the following immediate reformulation of Theorem 2.1.

THEOREM 2.2 (PERSSON [21, Theorem 4.2, p. 74]). – *Let Ω, $P(D)$, N, M, ϱ and x^0 be as in the hypothesis of Theorem* 2.1. *If $u \in \mathcal{D}'(\Omega)$ and if $P(D)u = 0$ in Ω and if*

$$(\operatorname{supp} u) \cap K(N, M, x^0, \varrho) \cap \{x; \langle x - x^0, N\rangle = -\varrho\}$$

is empty then there is an $\varepsilon > 0$ independent of u such that u is zero in the inner points of $K(N, M, x^0 + \varepsilon N, \varrho + \varepsilon|N|^2)$.

REMARK. – We notice that $x^0 \in \Omega' = \mathring{K}(N, M, x^0 + \varepsilon N, \varrho + \varepsilon\langle N, N\rangle)$, if $\Omega' \neq \emptyset$. Indeed we need a sharper form of Theorem 2.2.

THEOREM 2.3. – *Let Ω, P, N, M and x^0 be as in the hypothesis of Theorem* 2.1. *Let $u \in \mathcal{D}'(\Omega)$ be such that $P(D)u = 0$* in *Ω. Let F be a non-empty closed convex set in Ω such that x^0 is an inner point of F and such that $K(N, M, x^0, \infty) \cap F$ is a compact subset of Ω. Let $\mathring{K}(N, M, x^0, \infty)$ be non-empty. If*

$$\overline{\mathring{K}(N, M, x^0, \infty) \cap \partial F} \subset \complement \operatorname{supp} u \tag{2.1}$$

then u is zero in some neighbourhood Ω' of $K(N, M, x^0, \infty) \cap F$ where Ω' is independent of u.

PROOF. – We see that (2.1) shows that $u = 0$ in a neighbourhood of the compact set $\overline{\mathring{K}(N, M, x^0, \infty) \cap \partial F}$. The translation giving Theorem 2.2 obviously also gives Theorem 2.3.

3. – The geometry of uniqueness cones.

In this section we shall study the connection between real simple zeroes of polynomials in $\boldsymbol{R}^3$ and associated uniqueness cones in $\boldsymbol{R}^3$. Let $P(D)$ be a linear partial differential operator in $\boldsymbol{R}^3$ of order m with constant coefficients such that

$P_m(1, 0, 0) = 0$ and

$$\operatorname{Re} \operatorname{grad} P_m(1, 0, 0) = (0, 1, 0) . \tag{3.1}$$

At first we assume that P_m has real coefficients. Let

$$P_m(\xi) = \xi_1^{m-1}\xi_2 + \sum_{\substack{\alpha_1 < m-1 \\ \alpha_2 \leqslant 1 \\ |\alpha| = m}} a_\alpha \xi^\alpha + \sum_{j=k}^{m} b_j \xi_1^{m-j} \xi_3^j . \tag{3.2}$$

Now (3.1) implies that $k \geqslant 2$ if $b_k \neq 0$. Because of (3.2) there is a real valued analytic solution $s(t)$ of

$$P_m(1, s, t) = 0 \tag{3.3}$$

for small s and t. One sees that

$$s(t) = -b_k t^k + O(t^{k+1}) , \tag{3.4}$$

if $b_k \neq 0$. If $b_k = 0$ and $k = m$ then

$$s(t) = 0 \tag{3.5}$$

solves (3.3) for all t.

It is noticed in [21, Lemma 5.1] that if $b_k \neq 0$ then there are constants $K > 0$ and $c > 0$ such that

$$M = \{\delta;\ \delta = (1, s, t),\ 0 < -s < c,\ |t|^k < K|s|\} . \tag{3.6}$$

is an open convex set of noncharacteristic directions of P_m. Let $N = (1, -c/2, 0)$ and let $x^0 \in \boldsymbol{R}^n$. Then $K(N, M, x^0, \infty)$ is a uniqueness cone for P. Also $K(N', M', x^0, \infty)$ is a uniqueness cone of P if $N' = (1, c'/2, 0)$,

$$M' = \{\delta;\ \delta = (1, s, t),\ 0 < s < c',\ |t|^k < K's\} , \tag{3.7}$$

and c' and K' small.

Let $x^0 = (0, x_2', 0)$. In [21, p. 77] it is pointed out that for $x_2'' < x_2'$ there is a $d > 0$ and $K'' > 0$ such that

$$\begin{aligned} K(N', M', x^0, \infty) \cap \{x;\ x_2 = x_2'',\ |x_1| < d,\ |x_3| < d\} \subset \\ \subset \{x;\ x_1 \leqslant -K''|x_3|^{k/(k-1)}(x_2' - x_2'')^{1/(1-k)}\} . \end{aligned} \tag{3.8}$$

If $k = m$ and $b_m = 0$ then for some $c > 0$

$$M'' = \{\delta;\ \delta = (1, s, t);\ 0 < s < c,\ |t| < c\} \tag{3.9}$$

is a set of noncharacteristic directions of P_m: Let $N' = (1, c/2, 0)$, $x^0 = (0, x_2, 0)$. Then $K(N', M'', x^0, \infty)$ is a uniqueness cone of P. We also notice that for $x_2'' < x_2'$, there is a $d > 0$ such that

$$K(N', M'', x^0, \infty) \cap \{x; x_2 = x_2'', |x_1| < d, |x_3| < d\} \subset \{x; x_1 \leqslant -c|x_3|\}. \tag{3.10}$$

This is immediate from (3.9) and the definition of $K(N', M'', x^0, \infty)$.

We notice that (3.10) shows that $K(N', M'', x^0, \infty)$ is indeed a useful uniqueness cone. Then we let $b_k > 0$, k even. A look at P_m shows that also in this case there is a $c > 0$ such that every $\delta \in M''$ in (3.9) is noncharacteristic for P_m. So also in this case (3.10) is true. If $b_k > 0$, k odd, then there is a $c > 0$ such that $P_m(\delta) \neq 0$, $\delta \in M_2$ if we define

$$M_2 = \{\delta; \delta = (1, s, t)\}, \quad 0 < s < c, \quad -c < -t < c|s|^{1/k}. \tag{3.11}$$

It is obvious that with $N = (1, c/2, 0)$, $x^0 = (0, x_2', 0)$ $K(N, M_2, x^0, \infty)$ is a uniqueness cone of P and that for $x_2'' < x_2'$ and some $d > 0$, $K' > 0$,

$$\begin{aligned} K(N, M_2, x^0, \infty) \cap \{x; x_2 = x_2'', |x_1| < d, |x_3| < d\} \subset \\ \subset \{x; x_1 < cx_3, x_3 \leqslant 0 \text{ or } x_1 < -K'\big(x_3^k/(x_2' - x_2'')\big)^{1/(k-1)}, x_3 \geqslant 0\}. \end{aligned} \tag{3.12}$$

How the cases $b_k < 0$, k even or k odd, should be treated is obvious and we do not write down the expression (3.9)-(3.12) modified to these cases.

Before we start looking at operators with complex coefficients we notice that

$$\begin{aligned} \operatorname{grad}_\xi P_m(1, \xi_2, \xi_3) = \big((m-1)\xi_2 + b_k(m-k)\xi_3^k + O(\xi_2^2 + |\xi_2\xi_3| + |\xi_3|^{k+1}), \\ 1 + O(|\xi_2| + |\xi_3|),\ kb_k\xi_3^{k-1} + O(|\xi_2| + |\xi_3|^k)\big). \end{aligned} \tag{3.13}$$

Now we allow complex coefficients in P_m. Let $\xi \in \mathbf{R}^3$, $P_m(\xi) = 0$ and grad $P_m(\xi) \neq 0$. After a rotation of coordinates and a proper multiplication of P with a complex constant we can always get $\xi' = \xi/|\xi| = (1, 0, 0)$ and Re grad $P_m(\xi') = (0, 1, 0)$. So our assumption at the beginning of this section is no restriction. We solve Re $P_m(1, s, t) = 0$ for small s and t and get (3.4) if Re P_m is given by (3.2) with P_m replaced by Re P_m and $b_k \neq 0$. If $k = m$ and $b_k = 0$ then we get (3.5). We notice that

$$g(r) = \operatorname{Im} P_m(1, s(r), r)$$

is an analytic function in r for small r.

If $g \neq 0$ then $g(r) \neq 0$ for $0 < |r| < d'$, some $d' > 0$. That implies

$$P_m(1, s, r) \neq 0, \qquad (s, r) \neq (0, 0), \qquad (s, r) \text{ small}. \tag{3.14}$$

In this case for all $(a, b) \in \mathbf{R}^2$ with $a^2 + b^2$ small

$$M_3 = \{(1, s, t);\ (s - a)^2 + (t - b)^2 < a^2 + b^2\} \tag{3.15}$$

is an open convex set of noncharacteristic directions for P. Let $N'' = (1, a, b)$. Then $K(N'', M_3, x^0, \infty)$ is a uniqueness cone of P. We notice that

$$K(N'', M_3, 0, \infty) \cap \{x;\ x_1 = 0\} = \{x;\ x = t(0, a, b), -\infty < t \leqslant 0\}. \tag{3.16}$$

If $g = 0$ then $\operatorname{Re} P_m(1, s(r), r) = P_m(1, s(r), r) = 0$ for small r. In this case we get uniqueness cones just by looking at $\operatorname{Re} P_m$ and we are back in the case which we already have treated.

In the proof of local uniqueness we shall also look at uniqueness cones of the following type. Let P_m be given by (3.2) with $b_k \neq 0$. It follows that for all small $a > 0$

$$M = \{\delta;\ \delta = (1, s, t),\ |s| < |b_k| a^k/2,\ a < -t < 2a\} \tag{3.17}$$

is a convex set of noncharacteristic directions of P_m. Let $N = (1, 0, -3a/2)$. Then we see that

$$\begin{aligned} &K(N, M, 0, \infty) = \\ &= \{x;\ x_1 \leqslant -|b_k| a^k |x_2|/2 + 2a x_3,\ x_3 \leqslant 0, \text{ or } x_1 \leqslant -|b_k| a^k |x_2|/2 + a x_3,\ x_3 > 0\}. \end{aligned} \tag{3.18}$$

4. – Proof of local uniqueness in Theorem 1.2.

Let $k = 1$ and let (1.3) be false for all $\delta > 0$. We notice that $\psi \in C^2$. Let M_3 be defined by (3.15) and let $N'' = (1, a, b)$. Then there is an $r > 0$ such that for all (a, b) with $a^2 + b^2 = r^2$

$$K(N'', M_3, (0, a, b), r^2) \cap \{x;\ \langle x - (0, a, b), N''\rangle = -r^2\} \subset \left(\complement \operatorname{supp} u\right) \cup \{0\}.$$

Theorem 2.1 then says that for some $\delta' > 0$

$$(\operatorname{supp} u) \cap \{x;\ |x_1| < \delta',\ x_2^2 + x_3^2 < r^2/2\} \subset \{x;\ x_1 \geqslant 0\}. \tag{4.1}$$

To a given $\delta' > 0$ there is (a', b'), $a'^2 + b'^2 = \delta^2 < \delta'^2$ such that $\psi(0, a', b') < 0$. Let $a = -a'$, $b = -b'$, $N = (1, a, b)$. Then (4.1), δ' small, $\delta' < r/2$ and $\psi(0, a', b') < 0$ give that

$$K(N, M_3, 0, \delta^2) \cap \{x;\ \langle x, N\rangle = -\delta^2\} \subset \complement \operatorname{supp} u.$$

Theorem 2.2 shows that u is zero in some neighbourhood of $x = 0$ independent of u.

Let $2 \leqslant k \leqslant \infty$: We can always use uniqueness cones with M from (3.6) or with M' from (3.7) to clear up along $t \to (0, t, 0)$ such that for some $K'' > 0$, $d > 0$

$$\{x; x \in \operatorname{supp} u, |x_j| < d, j = 1, 2, 3\} \subset \{x; x_1 \geqslant -K''|x_3|^2\}. \tag{4.2}$$

Here it is important that $\psi \in C^2$. See (3.8) and see also [21, pp. 76-77]. If there is a sequence $(0, t_j, 0)$ with say $t_j > 0$, $t_j \to 0$ such that $\psi(0, t_j, 0) < 0$ then we can use the same uniqueness cones $K(N, M, 0, \varrho)$ with vertex at 0 and $\varrho = -\langle (0, t_j, 0), N\rangle$. Theorem 2.2 gives $u = 0$ around $x = 0$. At last we assume that $\psi(0, t, 0) \geqslant 0$, $|t| \leqslant \delta$ for some $\delta > 0$ but that (1.4) still is not true for any (K, δ). Then $2 \leqslant k \leqslant m$ since (1.2) must be true for some $K > 0$ and some $\delta > 0$ when $k = \infty$.

We may now assume that $\psi(x) = 0$ is given by $x_1 - \varphi(x_2, x_3) = 0$ where φ is in C^2 near $(x_2, x_3) = 0$, $\varphi(0, 0) = 0$ and $x_1 - \varphi(x_2, x_3) \geqslant 0 \Leftrightarrow \psi(x) \geqslant 0$ for small x. If (1.12) is not true then there is a sequence $(x_{2j}, x_{3j}) \to (0, 0)$ such that $\varphi(x_{2j}, x_{3j}) > > jx_{3j}^{k/(k-1)}$ for all j. We notice that we may choose all $x_{3j} > 0$ possibly after shifting $x_3' = -x_3$ and then deleting the primes. We assume that this is done. Then we notice that we can always choose $t \to \varphi(x_{2j}, t)$ increasing at x_{3j} by choosing a new smaller x_{3j}. We always keep in mind that $\varphi(x_2, 0) \leqslant 0$.

Now let $b_k > 0$, k odd. Then we take M_2 from (3.11). Let $x^0 = (jx_{3j}^{k/(k-1)}, x_2', x_{3j})$ and let $N = (1, c/2, 0)$. Here $x_2' \neq 0$ is some fixed number to be specified below such that $x_{2j} < x_2'$ for all j. From (3.12) one gets with

$$\begin{aligned} K(N, M_2, x^0, \infty) \cap \{x; x_2 = x_{2j}, |x_1 - jx_{3j}^{k/(k-1)}| < d, |x_3 - x_{3j}| < d\} \subset \\ \subset \{x; x_1 - jx_{1j}^{k/(k-1)} < c(x_3 - x_{3j}), \; x_3 - x_{3j} \leqslant 0, \text{ or} \\ x_1 - jx_{3j}^{k/(k-1)} \leqslant -K'\big((x_3 - x_{3j})^k/(x_2' - x_{2j})\big)^{1/(k-1)}, \; x_3 - x_{3j} \geqslant 0\}. \end{aligned} \tag{4.3}$$

Let $t > x_{3j}$ be the solution of

$$-K''t^2 = jx_{3j}^{k/(k-1)} - K'\big((t - x_{3j})^k/(x_2' - x_{2j})\big)^{1/(k-1)}$$

where K' is taken from (4.3) and K'' from (4.2). Let $x_{1j} = -2K''t^2$. We choose $x_2' > 0$ such that with d from (4.2) $|x_2'| < d$ and also such that $K'' < K'(x_2' - x_{2j})^{-1/(k-1)}$ for all j sufficiently big. Let $F_j = \{x; x_2 \geqslant x_{2j}, x_1 \geqslant x_{1j}\}$. We notice that x_{3j} is choosen such that $t \to \varphi(x_{2j}, t)$ is increasing at $t = x_{3j}$. From (4.3) one sees that

$$\overline{\mathring{K}(N, M_2, x^0, \infty)} \cap \partial F_j \subset \complement \operatorname{supp} u.$$

Theorem 2.3 gives that u is zero in a neighbourhood of $K(N, M_2, x^0, \infty) \cap F_j$. Since $x_{2j} \to 0$ when $j \to \infty$ and since (4.2) is true one now knows that u is zero in a neighbourhood of

$$\{x; x_2'/2 \leqslant x_2 \leqslant x_2', -d < x_1 < jx_{3j}^{k/(k-1)}, x_3 = x_{3j}\} \tag{4.4}$$

for some $d > 0$ and all big j.

Now one lets M be defined by (3.17) where $2a = jx_{3j}^{1/(k-1)}$. One chooses $x^0 = (jx_{3j}^{k/(k-l)}, 3x_2'/4, x_{3j})$, $N = (1, -3a/2, 0)$. From (3.18) one gets

$$(4.5)\qquad K(N, M, x^0, \infty) =$$
$$= \{x;\, x_1 - jx_{3j}^{k/(k-1)} \leqslant b_k\, a^k\, |x_2 - 3x_2'/4|/2 + a(x_3 - x_{3j}),\, x_3 - x_{3j} \geqslant 0 \text{ or}$$
$$x_1 - jx_{3j}^{k/(k-1)} \leqslant b_k\, a^k |x_2 - 3x_2'/4|/2 + 2a(x_3 - x_{3j}),\; x_3 - x_{3j} < 0\}\,.$$

Let $x_1 = -K'' x_{3j}^2$ and let

$$x_1 - jx_{3j}^{k/(k-1)} = -\, b_k\, a^k |x_2 - 3x_2'/4|/2\,.$$

Since $k \geqslant 2$ and $a = jx_{3j}^{1/(k-1)}/2$ one sees that

$$|x_2 - 3x_2'/4| \leqslant 2(j + K'')\, x_{3j}^{k/(k-1)} (b_k j^k x_{3j}^{k/(k-1)}/2^k)^{-1} \to 0$$

when $j \to \infty$.

It is now obvious from (4.2) and from the fact that u is zero in a neighbourhood of the set defined by (4.4) that for some big fixed j there is $\varepsilon > 0$, and $\varepsilon' > 0$ such that for

$$F = \{x;\, x_3 < x_{3j} + \varepsilon,\, x_1 \geqslant -\varepsilon'\}\,, \qquad x^0 = (jx_{3j}^{k/(k-1)}, 3x_2'/4, x_{3j})$$

one has

$$\overline{\mathring{K}(N, M, x^0, \infty)} \cap \partial F \subset \complement \operatorname{supp} u\,.$$

Theorem 2.3 shows that u is zero in a neighbourhood of $K(N, M, x^0, \infty) \cap F$. Since $0 \in K(N, M, x^0, \infty) \cap F$ this completes the proof of uniqueness in Theorem 1.3 when $b_k > 0$, k odd. It is obvious how one modifies this for general k and b_k. These proofs will not be written out here. This completes the proof of the uniqueness part of Theorem 1.2. The cases when (1.3) or (1.4) are satisfied will be treated in the next section.

5. – Characteristic surfaces and nullsolutions.

In order to prove nonuniqueness we shall use nullsolutions with an analytic simply characteristic initial plane. The problem will be to find the characteristic surfaces. In the local case with $x^0 \in \partial F$, F closed, to every sufficiently small neighbourhood Ω' of x^0 we seek a simply characteristic surface $S = \{x;\, \varphi(x) = 0\}$ for some real valued analytic φ with grad $\varphi \neq 0$ and $\{x;\, \varphi(x) \geqslant 0,\, x \in \Omega'\} \subset F$. Then we seek a $u \in C^\infty(\Omega')$ such that $P(D)u = 0$, $\operatorname{supp} u \subset \{x;\, \varphi(x) \geqslant 0\}$, $\operatorname{supp} u \neq \emptyset$.

In order to prove that a set Ω is not P-convex we shall find a simply characteristic analytic surface S such that there is one nonempty compact component H of $S \cap \partial\Omega$

with a positive distance to $\partial'(S \cap \bar{\Omega})$ the boundary of $S \cap \bar{\Omega}$ in the topology of S. First we construct a solution u of $P(D)u = 0$ in a neighbourhood Ω' of $S \cap \partial\Omega$ in $\boldsymbol{R}^3$ such that $(\operatorname{supp} u) \cap \partial\Omega \neq \emptyset$ and $\operatorname{supp} u \subset \Omega' \cap \bar{\Omega}$, then a $v \in C_0^\infty(\Omega')$ such that $v = 1$ in a neighbourhood of $S \cap \partial\Omega$.

We also choose v such that

$$\overline{\{x;\, v(x) \neq 1\}} \cap (\operatorname{supp} u) \subset \Omega$$

Let $w = vu$. Then

$$d(\operatorname{supp} P(D)w,\ \complement\Omega) > d(\operatorname{supp} w,\ \complement\Omega) = 0\,. \tag{5.1}$$

This or rather the variant shown below shows that there is a compact set K such that (1.1) is not true for any compact set $K' \subset \Omega$. One uses that $P(D)$ is invariant under translations and finds a w such that the inequality of (5.1) is fulfilled. If the equality is not true then translations give the result above. If we have (5.1) the construction admits a translation back to the other case. All these facts will be used in the following without explicit reference to them.

Let P be such that P_m is given by (3.2). Let P_m have real coefficients. Let $b_k \neq 0$ in (3.2). We seek a function ψ such that for a given $M > 0$.

$$P_m(\operatorname{grad} \psi) = 0\,;\ \psi(x) = x_1 - M|x_3|^{k/(k-1)}\,, \qquad x_2 = 0\,;\ \operatorname{grad} \psi(0) = (1, 0, 0)\,. \tag{5.2}$$

We let

$$g(r) = -\,Mk(k-1)^{-1}|r|^{1/(k-1)} \operatorname{sign} r\,. \tag{5.3}$$

Then with $s(t)$ from (3.4) we let

$$h(r) = s\big(g(r)\big)\,. \tag{5.4}$$

We notice that $P_m\big(1, h(r), g(r)\big) = 0$ for small r. The surface $\psi(x) = 0$ is now given by

$$x = \big(M|r|^{k/(k-1)}, 0, r\big) + t \operatorname{grad} P_m\big(1, h(r), g(r)\big)\,. \tag{5.5}$$

From (4.3), (5.4), (3.4) and (3.13) one gets

$$\partial P_m/\partial\xi_1(1, h, g) = -\,(k-1)b_k g^k + O(g^{k+1})\,,$$

$$\partial P_m/\partial\xi_2(1, h, g) = 1 + O(g)\,,$$

and

$$\partial P_m/\partial\xi_3(1, h, g) = kb_k g^{k-1} + O(g^k)\,.$$

We shall now make the coordinate transformation

$$x = \big(x_1' + M|r|^{k/(k-1)}, 0, r\big) + t \operatorname{grad}_\xi P_m\big(1, h(r), g(r)\big)\,. \tag{5.6}$$

We get

$$x_1 = x_1' + M|r|^{k/(k-1)} + t\big(-(k-1)b_k(g(r))^k + O((g(r))^{k+1}\big), \tag{5.7}$$

$$x_2 = t\big(1 + O(g(r))\big), \tag{5.8}$$

$$x_3 = r + tkb_k\big((g(r))^{k-1}\big) + O\big((g(r))^k\big). \tag{5.9}$$

We shall prove that this is injective in a certain domain. Now $(x_1', t, r) \to (x_1', t, g)$ is injective for small r by (5.3). We notice that x_2 and x_3 are analytic in t and g. We first look at $(x_1', t, g) \to (x_1', x_2, g)$ where (5.8) defines x_2. It is obviously injective for small g. Let $r = f(g)$ for small g. Let

$$x_3 = f(g) + tkb_k g^{k-1} + O(g^k).$$

Let

$$a = kb_k\big(-kM \operatorname{sign} r/(k-1)\big)^{k-1} \operatorname{sign} r.$$

We get

$$\begin{aligned} x_3 &= r + atr + O\big(|r|^{k/(k-1)}\big) = \\ &= r\big(1 + ax_2(1 + O(g(r))\big) + \big(1 + O(g(r)\big)x_2 O\big(|r|^{k/(k-1)}\big) = \\ &= f(g)(1 + ax_2) + O(g^k). \end{aligned} \tag{5.10}$$

If $1 + ax_2 \geqslant d > 0$ and g small we see that x_3 is strictly monotone in g so $(x_4', x_2, g) \to$ $\to (x_1', x_2, x_3)$ is injective for $|g| \leqslant d'$ some d' depending on d. Thus $(x_1', t, r) \to$ $\to (x_1', x_2, x_3)$ is injective for $|r| \leqslant d''$ and $1 + ax_2 \geqslant d$. We notice that a may depend on the sign of r. At last $(x_1', x_2, x_3) \to x$ with $x_1 = x_1' + M|r|^{k/(k-1)}$ where r is a well defined function of (x_2, x_3), is injective. That means that $(x_1', t, r) \to x$ given by (5.6) is injective in

$$H = \{(x_1', t, r);\, 1 + ta \geqslant d > 0,\ |r| < d'',\ x_1' \in \boldsymbol{R}\} \tag{5.11}$$

where $d'' > 0$ depends on d. It follows from (5.5) and (5.8) that

$$\begin{aligned} x_1 = {}& M\big(|x_3|/(1 + ax_2)\big)^{k/(k-1)} + \\ &+ x_2\big(-(k-1)b_k(-Mk \operatorname{sign} x_3/(k-1))^k\big)\big(|x_3|/(1 + ax_2)\big)^{k/(k-1)} + \\ &O\big(|x_3|^{(k+1)/(k-1)}\big) = M(1 + ax_2)^{1/(1-k)}|x_3|^{k/(k-1)} + O\big(|x_3|^{(k+1)/(k-1)}\big). \end{aligned} \tag{5.12}$$

Here $O\big(|x_3|^{(k+1)/(k-1)}\big)$ is uniform in $|x_2| \leqslant (1 - d)/|a|$ with $d > 0$.

We shall need an approximation lemma.

LEMMA 5.1. *Let k be an integer, $k \geqslant 2$, and let $G(t) = M|t|^{k/(k-1)}$, where $M > 0$ is a constant. Then to each $c > 0$ there is a real even polynomial P such that $P(0) = c$, $P(t) \geqslant G(t)$, $|t| \leqslant 1$, $|P'(t)| \leqslant \big(kM/(k-1)\big)|t|^{1/(k-1)}$ with P' increasing in $|t| \leqslant 1$.*

PROOF. – Let $c < 0$. Let $\varphi \in C^\infty(\mathbf{R})$, $\varphi(t) \geqslant 0$, be an even function around $t = 0$ with $\operatorname{supp} \varphi \subset \{t;\ |t| < 1\}$ and $\int \varphi = 1$. Let $\varepsilon > 0$ and let $\varphi_\varepsilon(t) = \varepsilon^{-1}\varphi(t/\varepsilon)$. Let

$$h(t) = \sup\{G(t), c/2\}\,.$$

Let $h_\varepsilon(t) = h * \varphi_\varepsilon$. It is now obvious that there is an $\varepsilon' > 0$ such that

$$(5.13)\qquad |h_\varepsilon'(t)| < |G'(t)|\,, \quad G(t) \leqslant h(t) < h_\varepsilon(t) < G(t) + 3c/4\,, \quad 0 < |t| \leqslant 1,\ 0 < \varepsilon < \varepsilon'\,.$$

We notice that h_ε is a convex C^∞ function in $|t| \leqslant 1$ with h_ε'' even. Let $\delta > 0$. Then it is always possible to choose an even polynomial $p(t) > 0$ such that $|p(t) - h_\varepsilon''(t)| < \delta$, $|t| \leqslant 1$. We get

$$(5.14)\qquad |h_\varepsilon'(t) - \int_0^t p(s)\,ds| \leqslant \delta|t|\,, \qquad |t| \leqslant 1\,.$$

Let

$$(5.15)\qquad P(t) = c + \int_0^t\Big(\int_0^s p(u)\,du\Big)\,ds\,.$$

It is obvious from our choice of p that P' is increasing. There is an $\varepsilon' > 0$ such that for some $d > 0$ $h_\varepsilon' = 0$ for $|t| \leqslant d$, $0 < \varepsilon \leqslant \varepsilon'$. For $|t| \leqslant d$ (5.14) gives

$$(5.16)\qquad |P'(t)| \leqslant \delta|t| \leqslant \big(kM/(k-1)\big)|t|^{1/(k-1)} = |G'(t)|\,, \qquad |t| \leqslant d$$

for $\delta \leqslant \big(kM/(k-1)\big)$. The continuity of h_ε' and G' on $d \leqslant |t| \leqslant 1$ and (5.13) says that for some $d' > 0$

$$(5.17)\qquad |G'(t)| \geqslant |h_\varepsilon'(t)| + d'\,, \qquad d \leqslant |t| \leqslant 1\,.$$

Let $\delta = \min\big(d', kM/(k-1)\big)$ then (5.14) and (5.17) give

$$(5.18)\qquad |G'(t)| - |P'(t)| \geqslant d' - \delta|t| \geqslant 0\,, \qquad d \leqslant |t| \leqslant 1\,.$$

Now (5.18) and (5.16) give $|P'(t)| \leqslant |G'(t)|$, $|t| \leqslant 1$. It is obvious from (5.13)-(5.15) that we can choose δ so small that $P(t) > G(t)$, $|t| \leqslant 1$. The lemma is proved. We also get

LEMMA 5.1$'$. – *If for* $k = \infty$, *one lets* $k/(k-1) = 1$, *and* $1/(k-1) = 0$ *then Lemma* 5.1 *also covers this case.*

REMARK. – In the case $k = 2$ we may allow $c = 0$ and choose $P(t) = Mt^2$.

Let $2 \leqslant k \leqslant m$. Let $c > 0$. Choose P from Lemma 5.1. Let $P' = p$. Let then P denote the differential operator of Theorem 1.3. Let

$$x = \left(x_1' + c + \int_0^{x_3'} p(t)\,dt,\ 0,\ x_3'\right) + x_2' \operatorname{grad} P_m\left(1,\ s(p(x_3')),\ p(x_3')\right). \tag{5.19}$$

With $r = x_3'$ and with a defined above (5.10), (5.19) is an analytic transformation of coordinates in $|x_2'| < (1-d)/|a|$, $|x_3'| < d''$, $d > 0$ and $d'' > 0$ depending on d. Here we use (5.11) and the fact that $p(t)$ is increasing in $|t| \leqslant 1$ and that

$$|p(t)| \leqslant \bigl(kM/(k-1)\bigr)|t|^{1/(k-1)}.$$

An easy argument using the properties of p given by Lemma 5.1 shows that the bicharacteristic lines

$$t \to \left(c + \int_0^r p(s)\,ds,\ 0,\ r\right) + t \operatorname{grad} P_m\bigl(1,\ s(p(r)),\ p(r)\bigr)$$

and

$$t' \to \left(M|r'|^{k/(k-1)},\ 0,\ r'\right) + t' \operatorname{grad} P_m\bigl(1,\ s(g(r')),\ g(r')\bigr)$$

for $|t| \leqslant (1-d)/|a|$, $|t'| \leqslant (1-d)/|a|$, $|r| < d''$, $|r'| < d''$ do not have any common point. That means that the analytic surface $x_1' = 0$ given by (5.19) lies above the surface defined by (5.2) for the corresponding (x_2, x_3).

We now take the inverse of (5.19). After an eventual multiplication of P by a nonvanishing analytic function P_m has the form (3.2) with analytic coefficients in a neighbourhood of $|x_2'| \leqslant (1-d)/|a|$ and $|x_3'| \leqslant d''$, $|x_1'| \leqslant d''$. Now we use PERSSON [23], [24, Theorem 4'] to get a solution u of $P(x', D')u = 0$ in $|x_2'| < (1-d)/|a|$, $|x_3'| < d''$, $|x_1'| < d'$, for some d', $0 < d' \leqslant d''$. Here

$$0 \in \operatorname{supp} u \subset \{x';\ x_1' \geqslant 0\}.$$

In fact the proof shows that

$$\{x';\ x_1' = 0,\ |x_2'| < (1-d)/|a|,\ |x_3'| < d'\} \subset \operatorname{supp} u. \tag{5.20}$$

Let $\varepsilon > 0$. Choose $v \in C_0^\infty$ equal to 1 in

$$\{x';\ |x_1'| < d' - \varepsilon,\ |x_2'| < \bigl((1-d)/|a|\bigr) - \varepsilon,\ |x_3'| < d'' - \varepsilon\}$$

and with compact support in

$$\{x';\ |x_3'| < d',\ |x_2'| < (1-d)/|a|,\ |x_3'| < d''\}. \tag{5.21}$$

Let $w = uv$. It is now obvious that in the original coordinates supp w is above the surface defined by (5.12).

Let x^0 be a point of global nonuniqueness of type k, $2 \leqslant k \leqslant m$. If $M = (k-1) \times \times |a' b_k k^k|^{1/(1-k)}$ then one sees that $a = a'^{-1}(-1)^{k-1}(\operatorname{sign} r)^k \operatorname{sign} b_k$. We notice that for any $M' > 0$ there is a $\delta > 0$ such that

$$N(x_2) x_3^2 < M' |x_3|^{k/(k-1)}, \qquad |x_2| \leqslant a', \ 0 < |x_3| < \delta, \ 2 < k \leqslant m. \tag{5.22}$$

The continuity of $N(x_2)$ on $|x_2| \leqslant a'$, the fact that L'' and L' have no common endpoints, (1.9), (1.10) and (5.22) shows that we can choose $M > 0$ somewhat smaller than before such that

$$M = (k-1)|a'' b_k k^k|^{1/(1-k)}$$

with $a'' > a'$, $(0, a' + t, 0) \in \Omega$, $(0, -a' - t, 0) \in \Omega$, $0 < t \leqslant a'' - a'$ and such that $x \in \Omega$ if x is defined by (5.12) with $a = (a'')^{-1}(-1)^{k-1}(\operatorname{sign} x_3')^k \operatorname{sign} b_k$, $0 < |x_3'| < d''$, $|x_2'| \leqslant \leqslant (1-d)/|a|$. Here $x_3' = r$, $x_2' = t$, $x_1' = 0$ in (5.6). If we now assume that $(1-d)/|a| - \varepsilon > a'$, $\varepsilon > 0$, then the construction above shows that when we choose $c > 0$ small enough in the approximation then we get

$$d\Big(\operatorname{supp} P(D) w, \ \complement \Omega\Big) > d(\operatorname{supp} u, \ \complement \Omega).$$

This implies that Ω is not P-convex.

If $a' = 0$ and x^0 is a point of local nonuniqueness of type k, $2 \leqslant k \leqslant m$, then (1.4) is true and

$$M(1 + a x_2)^{-1} > K$$

for $M = 4K$ and $|x_2| \leqslant |a|^{-1}/2$. See (5.12). The construction above applies to this case equally well.

Let x^0 be a point of semiglobal nonuniqueness of type ∞. We assume that $x^0 = (0, x_2', 0)$ and that the normal of $\partial\Omega$ at x^0 is $(1, 0, 0)$ and that the normal along L'' of Definition 1.3 is proportional to $(1, 0, 0)$. We also assume that P_m has the form (1.7). Let 0 be the centre of L'', $L'' = \{(0, t, 0);\ |t| \leqslant a'\}$. Then there is an $M > 0$, $d' > 0$ and $\varepsilon > 0$ such that

$$\{x;\ x_1 = M|x_3|,\ |x_2| < a' + \varepsilon,\ |x_1| < d',\ 0 < |x_3| < d'\}$$

is a subset of Ω. We choose P from Lemma 5.1′ let $P'(t) = p(t)$ and denote the operator by P as before. We make the transformation (5.19) with $s = 0$. Since P is of principal type the analytic surface we get by letting $x_1' = 0$ in (5.19) is simply characteristic. The argument of the case $2 \leqslant k \leqslant m$ applies equally well here. Thus Ω is not P-convex.

If x^0 still has type ∞ but the normal along L'' of $\partial\Omega$ is not proportional to $(1, 0, 0)$ we use the fact that $\partial\Omega$ is in C^1 because it is in C^2. This gives that arbitrarily close to an endpoint of L'' there is a point of semiglobal nonuniqueness of type ∞ of the kind treated above. Thus Ω is not P-convex.

This also covers the local case with $k = \infty$. We notice that we may allow complex coefficients in P_m. If the type is k, $2 \leqslant k \leqslant \infty$, then a surface characteristic for $\operatorname{Re} P_m$ must certainly be characteristic for $\operatorname{Im} P_m$ too. The null solutions exist independent of whether the coefficients are real valued or not.

Let x^0 be a point of semiglobal nonuniqueness of type $k = 1$. Then we just use [24, Theorem 4′], without any change of coordinates, and a proper cut off function. That shows that Ω is not P-convex. See also HÖRMANDER [8, Theorem 3.7.1, p. 89].

By this we have completed the proof of Theorem 1.2. In the next section we shall complete the proof of Theorem 1.3.

6. – Semiglobal uniqueness.

Let x^0 be a point of semiglobal uniqueness of type k, $2 \leqslant k \leqslant \infty$. If L' and L'' have a common endpoint, then the zero at this endpoint or outside a given compact set K if the endpoint is not finite, can be transported to x^0 by uniqueness cones of the Section 3 type. If L' and L'' have no common endpoint, and $2 \leqslant k \leqslant m$, and the normal to $\partial\Omega$ along L'' is not proportional to $(1, 0, 0)$ at some point then there must be one point with noncharacteristic normal $(1, 0, c)$, $c \neq 0$. Holmgren's uniqueness theorem or Theorem 2.2 gives a zero that can be transported to x^0 by uniqueness cones of Section 3 type or general Section 2 type. If the normal along L'' is proportional to $(1, 0, 0)$ and $a' > 0$ then $k = 2$ and (1.9) or (1.10) is not true depending on the sign of b_2. Assume that $b_2 > 0$ and that (1.9) is not true. Then

$$N(x_2') \geqslant M(1 - x_2'/a')^{-1} \tag{6.1}$$

for some x_2', $-a' \leqslant x_2' < a'$. Then one uses uniqueness cones of the type treated in (3.8) to prove that for some $\varepsilon > 0$, $d > 0$

$$(\operatorname{supp} u) \cap \{x;\ |x_2| < a' + 2\varepsilon,\ |x_1| < d,\ |x_3| < d\} \subset \{x;\ x_1 > -K'' x_3^2\} \tag{6.2}$$

for some $K'' > 0$. Then we let $M'' = (k-1)\big(b_k(a' + \varepsilon)\big)^{1/(1-k)} k^{k/(1-k)}$. Now one sees from (6.1) and the definition of M'' that

$$N(x_2') > M''\big(1 - x_2'/(a' + \varepsilon)\big)^{-1} = M' . \tag{6.3}$$

We solve $P_m\big(1, s(-2M'r), -2M'r\big) = 0$. Let $h(r) = s(-2M'r)$ and let $g(r) =$

$= -2M'r$. New coordinates are chosen for a fixed small r.

$$(6.4)\quad \begin{cases} x_1' = x_1 + h(r)x_2 + g(r)x_3\,, \\ x_2' = \partial P_m/\partial\xi_1(1, h, g)x_1 + \partial P_m/\partial\xi_2(1, h, g)x_2 + \partial P_m/\partial\xi_3(1, h, g)x_3\,, \\ x_3' = (h\,\partial P_m/\partial\xi_3 - g\,\partial P_m/\partial\xi_2)x_1 + (g\,\partial P_m/\partial\xi_1 - \partial P_m/\partial\xi_3)x_2 + \\ \qquad\qquad + (\partial P_m/\partial\xi_2 - h\,\partial P_m/\partial\xi_1)x_3\,. \end{cases}$$

This is an orthogonal transformation. It is close to the identity for small r. After deleting the primes and after multiplication of P by a proper constant one sees that P_m has the form (3.2) with a b_2 which depends continuously on r.

We go back to the original coordinates. We shall follow the bicharacteristic line through $((M'+\varepsilon')r^2, x_2', r)$ given by $\xi(0) = (1, h(r), g(r))$. Then (5.7)-(5.9) for $x_1' = 0$ gives

$$(6.5)\quad \begin{cases} x_3 = r - t((a'+\varepsilon) - x_2')^{-1}r + O(r^2)\,, \\ x_2 = x_2' + t(1 + O(r))\,, \\ x_1 = (M'+\varepsilon')r^2 + t(-b_2 4M'^2r^2) + O(r^3) = \\ \quad = (M'+\varepsilon')r^2 + M'r^2t((a'+\varepsilon) - x_2')^{-1} + O(r^3)\,. \end{cases}$$

For fixed r solve $x_3 = 0$ for t. Then $t = (a'+\varepsilon) - x_2' + O(r)$. One gets $x_2 = a' + \varepsilon + O(r)$, and $x_1 = \varepsilon'r^2 + O(r^3)$. Let $\bar{x} = (x_1, x_2, 0) = (\varepsilon'r^2 + O(r^3), a' + \varepsilon + O(r), 0)$. For ε' small it is obvious from (6.3) that $((M'+\varepsilon')r^2, x_2', r) \in \complement\Omega$. Now one chooses a set M'' from (3.9) in the new coordinates given by (6.4). Here one can choose c independent of r since b_2 is continuous in r. Let $N' = (1, c/2, 0)$ in the (6.4) coordinates. Let $F = \{x; x_1 \geqslant -d, x_2 \geqslant x_2'\}$ in the original coordinates with $d > 0$ from (6.2). It follows from (6.4), (3.10) and (6.2) that

$$(6.6)\qquad \overline{\mathring{K}(N', M'', \bar{x}, \infty) \cap \partial F} \subset \complement \operatorname{supp} u$$

for some small $r > 0$. One also realizes that for some $\bar{\varepsilon} > 0$ close to ε by choosing r small one gets $(0, a' + \bar{\varepsilon}, 0) \in \mathring{K}(N, M, \bar{x}, \infty) \cap F$. Theorem 2.3 then says that u is zero around $(0, a' + \bar{\varepsilon}, 0)$. This zero can then be transported back to x^0 by uniqueness cones of Section 3 type.

For local uniqueness points of type k, $2 \leqslant k \leqslant m$ when L' and L'' have no common endpoint we can use Theorem 1.2 to get a zero around x^0. We also notice that there is no complication in the argument above when P_m has complex coefficients as long as $2 \leqslant k \leqslant \infty$.

Let $x^0 = 0$ be a point of semiglobal uniqueness of type 1. Let $K \subset \Omega$, K compact, $\operatorname{supp} P(D)u \subset K$, $\operatorname{supp} u \subset \Omega$, $u \in \mathcal{E}'$. Let $\hat{K}$ be the convex hull of K. Let $R > 0$ be such that $|x| < R$ if $x \in \hat{K}$. If $x^0 \notin \hat{K}$ then u is zero around x^0. See LIONS [14] or [8, Lemma 3.4.3, p. 80]. Let $\Omega' = \Omega \cap \{x;\ |x| < R\}$. Let $x^0 \in \hat{K}$. Let $(1, 0, 0)$ be normal to $\partial\Omega$ at x^0. Let H' be the component of $\{x; x \in \partial\Omega', x_1 = 0\}$ which contains x^0. Let $\partial' H'$ be the boundary of H' in $\{x; x_1 = 0\}$. Let $\partial'(\complement\bar{\Omega}')$ be the boundary of $\complement\bar{\Omega}' \cap \{x; x_1 = 0\}$ in $\{x; x_1 = 0\}$. Since x^0 is a point of semiglobal uniqueness we must have

$$\partial' H' \cap \partial'(\complement\bar{\Omega}') \neq \emptyset . \tag{6.7}$$

First assume that $(1, 0, 0)$ is normal to $\partial\Omega$ at every point of H'. We know that H' is compact. By the use of uniqueness cones of the (3.16) type we can get a finite covering $(O_j)_{j=1}^N$ of H' in $\{x; x_1 = 0\}$. Here O_j are discs of a constant radius r with centre at $x^{(j)} = (0, x_2^{(j)}, x_3^{(j)})$ such that for some $d > 0$

$$(\operatorname{supp} u) \cap \{x;\ (x_2 - x_2^{(j)})^2 + (x_3 - x_3^{(j)})^2 < r^2,\ -d < x_1 < 0\}$$

is empty, $j = 1, \ldots, N$. Since $\partial' H' \cap \partial'(\complement\bar{\Omega}')$ is not empty, u is zero around some point in $\bigcup O_j$. This point can be connected to x^0 by a finite polygon in $\bigcup O_j$. Then we use cones of the (3.16) type along this polygon to transport the zero to x^0.

Let H'' be the maximal connected subset of H' containing x^0 and which has normal $(1, 0, 0)$ to $\partial\Omega$ at every point. If $H'' \cap \partial'(\complement\bar{\Omega}') \neq \emptyset$ the proof goes exactly as above for H'. If $\partial' H'' \cap \partial'(\complement\bar{\Omega}')$ is empty then H'' must be equal to H' and that is impossible because of (6.7).

Since any noncharacteristic point at $\partial\Omega$ is a point of local uniqueness the argument in the introduction shows that the proof of Theorem 1.3 is completed.

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