An Estimate on Convergence of Approximation by Iterations of a Solution to a Quasi-Variational Inequality and Some Consequences on Continuous Dependence and G-Convergence (*).

MARCO BIROLI (Milano)

Summary. – *Si dà una stima per l'approssimazione mediante iterazione della soluzione di una disequazione quasi-variazionale e se ne deducono risultati circa la dipendenza continua della soluzione dall'ostacolo e dal termine noto e circa la G-convergenza.*

1. – Introduction and results.

Let be $\Omega \subset \boldsymbol{R}^N$ a bounded open set with smooth boundary $\partial\Omega$ and $a_{ij} \in \mathcal{L}^\infty(\Omega)$, $i, j = 1, \ldots, N$, such that

$$\sum_{ij=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2 \qquad \text{a.e. in } \Omega\,. \tag{1.1}$$

$\forall \xi \in \boldsymbol{R}^N$; we suppose $0 \in \Omega$.

We indicate, by $H^1_\Gamma(\Omega)$ the subspace of $H^1(\Omega)$

$$H^1_\Gamma(\Omega) = \{u \in H^1(\Omega) \ \ u|_\Gamma = 0\}$$

where $\Gamma \subset \partial\Omega$ is closed and regular (for the hypothesis on regularity of Γ cf. [16]); we define $A\colon H^1_\Gamma(\Omega) \to (H^1_\Gamma(\Omega))^*$ $((H^1_\Gamma(\Omega))^*$ is the dual of $H^1_\Gamma(\Omega))$ by

$$\langle Au, v\rangle = \int_\Omega \sum_{ij=1}^{n} a_{ij}(x)\, u_{x_i}(x)\, v_{x_j}(x)\, dx + \lambda \int_\Omega u(x)\, v(x)\, dx \tag{1.2}$$

(where $\lambda > 0$ if $\Gamma \neq \partial\Omega$, $\lambda \geqslant 0$ if $\Gamma = \partial\Omega$) and $M\colon \mathcal{L}^\infty(\Omega) \to \mathcal{L}^\infty(\Omega)$ by

$$M\varphi(\mathrm{T}) = \operatorname*{Inf}_{\substack{\xi \geqslant 0\\ x+\xi\in\Omega}} \big(\varphi(x+\xi) + K(x, \xi)\big) \tag{1.3}$$

$\forall \varphi \in \mathcal{L}^\infty(\Omega)$, where $K(x, \xi) \in \mathcal{L}^\infty(\Omega \times \boldsymbol{R}^n_+)$ $(\boldsymbol{R}^n_+ = \{\xi \in \boldsymbol{R}^n,\ \xi \geqslant 0\})$ $K(x, \xi) \geqslant K_0 \geqslant 0$.

(*) Entrata in Redazione il 29 marzo 1978.

Let be $f \in \mathfrak{L}^r(\Omega)$, $r > N$, with $f(x) \geq 0$ a.e. in Ω; we consider the problem

$$(1.4) \qquad \begin{cases} \langle Au, v-u \rangle \geq \langle f, v-u \rangle , \\ \forall v \in H^1_\Gamma(\Omega) \quad v(x) \leq Mu(x) \quad \text{a.e. in } \Omega , \\ u \in H^1_\Gamma(\Omega) \quad u(x) \leq Mu(x) \quad \text{a.e. in } \Omega . \end{cases}$$

The problem (1.4) has been introduced by A. BENSOUSSAN, J. L. LIONS [1], [2], in relation to some problems in stochastic control; A. BENSOUSSAN, M. GOURSAT, J. L. LIONS [4], have shown the following existence result:

TH. 1. – *Let be u^0 the solution to the problem*

$$Au^0 = f$$

and u^n defined by

$$(1.4_n) \qquad \begin{cases} \langle Au^n, v-u^n \rangle \geq \langle f, v-u^n \rangle , \\ \forall v \in H^1_\Gamma(\Omega); \quad v(x) \leq Mu^{n-1}(x) \quad \text{a.e. in } \Omega , \\ u^n \in H^1_\Gamma(\Omega); \quad u^n(x) \leq Mu^{n-1}(x) \quad \text{a.e. in } \Omega . \end{cases}$$

We have

$$u_n \downarrow u \qquad in\ \mathfrak{L}^2(\Omega)$$

where u is the maximum solution of the problem (1.4).

Th. Laestch has shown the following uniqueness result.

TH. 2. – *Let be $K_0 > 0$; the solution to the problem* (1.4) *is unique.*

In the framework of Th. 1,2 some results on continuous dependence can be obtained from Th. 4 [4]; these results have, however, some monotonicity hypothesis, which reduce their applicability.

In the framewock of Th. 1,2 some results on G-convergence for problems of type (1.4) can be obtained from [6].

These results are valid in the following hypothesis: let be u_0 a subsolution (cf. [20]) of the limit problem we define u_n by

$$\begin{cases} \langle Au_n, v-u_n \rangle \geq \langle f, v-u_n \rangle \\ \forall v \in H^1_\Gamma(\Omega); \quad v(x) \leq Mu_{n-1}(x) \quad \text{a.e. in } \Omega \\ u_n \in H^1_\Gamma(\Omega); \quad u_n(x) \leq Mu_{n-1}(x) \quad \text{a.e. in } \Omega \end{cases}$$

the $u_n \uparrow u$, where u is the solution of the limit problem.

This hypothesis is difficult to verify and is related to the problem of the regularity of a solution of (1.4).

We specify, however, that the operators M considered in [4], [6] are more general than the operator given by (1.3).

The aim of this paper is to obtain a more careful result on convergence of u^n to u than this of Th. 1 and to deduce from it some results on continuous dependence and G-convergence, which improve those of [4], [6].

We obtain:

TH. 3. – *Let be* $K_0 > 0$; *we have*

$$0 \leqslant u^n - u \leqslant K\theta_0^n , \qquad 0 < \theta_0 < 1$$

where K, θ_0 *are constants dependent only on* K_0 *and* $\operatorname{Sup} u^0(x)$.

From Th. 3 we have a result shown by other methods by C. TROIANIELLO [22]:

COROLL. 1. – *Let be* $K_0 > 0$, $K(x, \xi) = K(\xi) \in C(\boldsymbol{R}_+^N)$; *the solution to the problem* (1.4) *is in* $C(\Omega)$.

From Th. 3 we can deduce the following result on continuous dependence:

TH. 4. – *Let be* $\{f_\alpha\}$ $\{K_\alpha\}$ *two sequences such that*

$$\lim_{\alpha \to +\infty} f_\alpha = f \quad \text{in } \mathfrak{L}^r(\Omega) , \qquad r > N , \qquad f_\alpha(x) \geqslant 0 \quad \text{a.e. in } \Omega$$

$$\lim_{\alpha \to +\infty} K_\alpha(x; \xi) = K(x, \xi) \quad \text{in } \mathfrak{L}^\infty(\Omega \times \boldsymbol{R}_\xi^N) ,$$

$$K_\alpha(x; \xi) \geqslant K_0 > 0 \quad \text{a.e. in } \Omega \times \boldsymbol{R}_+^N$$

and u_α *the solution to the problem*

$$(1.4_n) \qquad \begin{cases} \langle A u_\alpha, v - u_\alpha \rangle \geqslant \langle f, v - u_\alpha \rangle , \\ \forall v \in H_\Gamma^1(\Omega) , \quad v(x) \leqslant M_\alpha u_\alpha(x) \quad \text{a.e. in } \Omega , \\ u_\alpha \in H_\Gamma^1(\Omega) , \quad u_\alpha(x) \leqslant M_\alpha u_\alpha(x) \quad \text{a.e. in } \Omega . \end{cases}$$

Where

$$M_\alpha \varphi(x) = \operatorname*{Inf}_{\substack{x+\xi \in \Omega \\ \xi \geqslant 0}} \left(\varphi(x+\xi) + K_\alpha(x, \xi)\right) .$$

We have

$$\lim_{\alpha \to +\infty} u_\alpha = u \quad \text{in } \mathfrak{L}^\infty(\Omega)$$

where u *is the solution of* (1.4).

Let be now $a_{ii}^p \in \mathcal{L}^\infty(\Omega)$, $i, j = 1, \ldots, N$, such that

$$\beta|\xi|^2 \geqslant \sum_{ij=1}^{N} a_{ij}^p(x)\xi_i\xi_j \geqslant \alpha|\xi|^2 \qquad \text{a.e. in } \Omega .$$

We define $A^p\colon H^1_\Gamma(\Omega) \to (H^1_\Gamma(\Omega))^*$ by

$$\langle A^p u, v\rangle = \int_\Omega \sum_{ij=1}^{n} a_{ij}^p(x)\, u_{x_i}(x)\, v_{x_j}(x)\, dx + \lambda \int_\Omega u(x)\, v(x)\, dx$$

We denote by u_p the solution to the problem

$$(1.4_p) \qquad \begin{cases} \langle A^p u_p, v - u_p\rangle \geqslant \langle f, v - u_p\rangle , \\ \forall v \in H^1_\Gamma(\Omega) \qquad v(x) \leqslant M u_p(x) \qquad \text{a.e. in } \Omega , \\ u_p \in H^1_\Gamma(\Omega) \qquad u_p(x) \leqslant M u_p(x) \qquad \text{a.e. in } \Omega . \end{cases}$$

We suppose that A^p G-converges to A (for the definition of G-convergence [8], [9], [21]).

TH. 5. – *If* $K_0 > 0$, $K(x; \xi) = K(\xi) \in C(\boldsymbol{R}^N_+)$, $M\colon C(\bar{\Omega}) \to C(\bar{\Omega})$; *we have*

$$\lim_{p\to+\infty} u_p = u \qquad \text{in } C(\bar{\Omega})$$

where u *is the solution of* (1.4).

REM. 1. – The result of Th. 5 improves a previous result of A. BENSOUSSAN, J. L. LIONS [3], in which the A suppose

$$\lim_{p\to+\infty} a_{ij}^p = a_{ij} \quad \text{in } \mathcal{L}^\infty(\Omega) \qquad i, j = 1, \ldots, N$$

In the case of homogeneisation with f also non positive and $K(x, \xi) = K_0 > 0$ we have a more precise estimate.

Let be $b_{ij}(x)$ $i, j = 1, \ldots, N$ in $C^1(\boldsymbol{R}^N)$ periodic of period $P = \prod_{i=1}^{N} [0, y_i]$ such that

$$\beta|\xi|^2 \geqslant \sum_{ij=1}^{n} b_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2$$

in $\boldsymbol{R}^N$ $\forall \xi \in \boldsymbol{R}^N$.

We choose

$$a_{ij}^p(x) = b_{ij}(px)$$

and

$$a_{ij}(x) = q_{ij}$$

where q_{ij} are the constant coefficients of the homogeneised operator [5], [9], [13], [21].
In this case we have:

TH. 6. – *Let be f also non positive, $K(x,\xi)=K_0>0$, $\Gamma=\partial\Omega$*

$$g(x)=\operatorname{Inf}\Big(\operatorname*{Inf}_{\substack{x+\xi\in\Omega\\ \xi\geqslant 0}} f(x+\xi),0\Big)$$

We suppose $g(x)\in\mathfrak{L}^r(\Omega)$ and $\underline{u}(x)>-K_0$ in Ω where $\underline{u}(x)$ is the solution to the problem

$$A\underline{u}=g\,.$$

We have

$$\|u_p-u\|_{\mathfrak{L}^\infty}\leqslant Cp^\gamma\,,\qquad p\geqslant\bar{p}_\gamma$$

where

$$\gamma=-\frac{\bar{\alpha}}{2(N-2+3\bar{\alpha})}(1-\delta)\,,\qquad\forall\delta>0$$

($\bar{\alpha}$ is a possible Hölder coefficient in the De Giorgi-Nash theorem relative to α, β, Ω, f) and $\bar{p}_\gamma$ is an integer dependent on δ.

In the § 2 we show the Th. 3 and the Coroll. 1; in the § 3 we give a proof of Th. 4 by the result of Th. 3 and the result of continuous dependence for variational inequalities [17], III, Th. 1.4; in the § 4 we show the Th. 5 by Th. 3 and some preliminary lemmas on G-convergence for variational inequalities; in the § 5 we give a proof of Th. 6 by Th. 3, the estimate on G-convergence for variational inequalities [7], and a result of J. L. JOLY, U. MOSCO, G. TROIANIELLO [10], [15], on the regularity of solution to quasi-variational inequalities.

REM. 2. – A result analogous to Th. 3 for the problem in stochastic control related to (1.4) has been given by J. L. MENALDI [14]; unfortunately the equivalence between the two type of problems and approximations ask a regularity, which generally we are not able to show.

REM. 3. – The Th. 3, 4, 5 can be extended to operators of the type

$$\langle Au,v\rangle=\int_\Omega\Big\{\sum_{ij=1}^N\big(a_{ij}(x)u_{x_i}(x)+b_j(x)u(x)\big)v_{x_j}(x)\sum_{i=1}^N\big(d_i(x)u_{x_i}(x)+c(x)u(x)\big)v(x)\Big\}dx$$

$u,v\in H^1_\Gamma(\Omega)$ where

$$\langle Au,u\rangle\geqslant\alpha\|u\|^2_{H^1_\Gamma}\qquad\forall u\in H^1_\Gamma(\Omega),\ \alpha>0 \tag{1.5}$$

$$a_{ij},\ b_j\ d_i,\ c\in\mathfrak{L}^\infty(\Omega)\,,\qquad c(x)\geqslant 0\qquad\text{a.e. in }\Omega\,.$$

The Th. 3, 4 can be also extended to the case Ω unbounded if (1.5) is valid, and $K(x,\xi)\to+\infty$ for $|\xi|\to+\infty$ uniformly in x, and to the parabolic case if we consider the maximum solutions.

2. – Proof of Th. 3.

Consider the problem

$$(2.1)\qquad \begin{cases} \langle A\overline{w}, v-\overline{w}\rangle \geqslant \langle f, v-\overline{w}\rangle \\ \forall v\in H^1(\Omega) \quad v(x)=\varphi(x) \quad \text{on } \Gamma; \quad v(x)\leqslant\psi(x) \quad \text{a.e. in } \Omega \\ \overline{w}\in H^1(\Omega) \quad \overline{w}(x)=\varphi(x) \quad \text{on } \Gamma; \quad \overline{w}(x)\leqslant\psi(x) \quad \text{a.e. in } \Omega \end{cases}$$

where $\varphi\in H^{\frac{1}{2}}(\partial\Omega)$, $\psi\in\mathfrak{L}^\infty(\Omega)$ such that there is

$$v\in H^1(\Omega)\,,\quad v(x)=\varphi(x)\quad \text{on } \Gamma \quad \text{and} \quad v(x)\leqslant\psi(x)\quad \text{a.e. in } \Omega\,.$$

Let be $\overline{w}$ the solution of the problem (2.1); we indicate $\overline{w}=S(\psi,f,\varphi)$.

The Th. 3 is shown if we show

$$(2.2)\qquad u^n+Q\leqslant\frac{\theta_0^{-n}+C}{\theta_0^{-n}}(u+Q)\quad \text{a.e. in } \Omega$$

where $0<\theta_0<1$ and $C,Q>0$ are suitable constants.

We show (2.2) by induction.

From [4] we have $u(x)\geqslant 0$ a.e. in Ω; let be $P=\operatorname{Sup} u^0(x)<+\infty$; if $CQ\geqslant P$, we have (2.2) for $n=0$.

We suppose now (2.2) for $n-1$ and we show (2.2) for n.

We have

$$(2.3)\qquad u^{n-1}+Q\leqslant\frac{\theta_0^{-n+1}+C}{\theta_0^{-n}}(u+Q)\quad \text{a.e. in } \Omega$$

Let be $w^n=u^n+Q$, $w^{n-1}=u^{n-1}+Q$, $w=u+Q$.

We have

$$(2.4)\qquad w^n=S(Mw^{n-1}, f+\lambda Q, Q)$$

We observe that (2.2) is equivalent to

$$(2.5)\qquad \frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,w^n\leqslant w\,.$$

We have

$$\text{(2.6)}\qquad \frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,w^n = S\left(\frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,Mw^{n-1},\quad \frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,(f+\lambda Q)\,,\quad \frac{\theta_0^{-n}}{\theta_n^{0-}+C}\,Q\right)$$

$$w = S(w,\,(f+\lambda Q),\,Q)\,.$$

From [17] Th. 1.4, we have

$$\text{(2.7)}\qquad \frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,w^n \leqslant S\left(\frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,Mw^{n-1},\quad f+\lambda Q,\ Q\right).$$

As in [12], pg. 165, we have

$$\text{(2.8)}\qquad M(\alpha w^{n-1}) \geqslant \frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,Mw^{n-1}\qquad (\alpha\geqslant 0)$$

if

$$\frac{1-\theta_0^{-n}/(\theta_0^{-n}+C)}{\theta_0^{-n}/(\theta_0^{-n}+C)-\alpha} \geqslant \frac{P+Q}{K_0} = \overline{P}$$

then

$$\text{(2.9)}\qquad \alpha = \max\left(\frac{\overline{P}\theta_0^{-n}-C}{\overline{P}(\theta_0^{-n}+C)},\,0\right).$$

From (2.7), (2.8), (2.9) we have

$$\text{(2.10)}\qquad \frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,w^n \leqslant S(M(\alpha w^{n-1}),\quad f+\lambda Q,\,Q)$$

then from (2.3)

$$\text{((2.11)}\qquad \frac{\theta_0^{-n}}{\theta_0^{-n}+C}\,w^n \leqslant S\left(M\left(\alpha\,\frac{\theta_0^{-n+1}+C}{\theta_0^{-n+1}}\,w\right),\quad f+\lambda Q,\,Q\right).$$

From (2.6) we have (2.5) if

$$\text{(2.12)}\qquad \alpha\,\frac{\theta_0^{-n+1}+C}{\theta_0^{-n+1}} \leqslant 1\,.$$

We have (2.12) if $\alpha=0$; if $\alpha\neq 0$ we can write (2.12) as

$$\frac{\overline{P}\theta_0^{-n}-C}{\overline{P}(\theta_0^{-n}+C)}\,\frac{\theta_0^{-n+1}+C}{\theta_0^{-n+1}} \leqslant 1$$

$$\overline{P}\theta_0^{-2n+1}+\overline{P}C\theta_0^{-n}-C\theta_0^{-n+1}-C^2 \leqslant \overline{P}\theta_0^{-2n+1}+\overline{P}C\theta_0^{-n+1}$$

$$\overline{P}\theta_0^{-n}-(\overline{P}+1)\theta_0^{-n+1}-C^2 \leqslant 0$$

then we can choose

$$\theta_0 = \frac{\bar{P}}{\bar{P}+1} = \frac{P+Q}{P+Q+K_0}$$

and we have (2.12) and (2.5).

The result is shown. ■

We give now the proof of Coroll. 1.

Lemma 1. – *We consider $u = S(\psi, f, 0)$, where $f \in \mathcal{L}^\infty(\Omega)$ and $\psi \in C(\bar{\Omega})$, $\psi \geq 0$ on Γ; we have $u \in C(\bar{\Omega})$.*

From [15] we have the result if $A\psi \in \mathcal{L}^p(\Omega)$, $p > N$.

We can suppose $\varphi = 0$ in Γ we have a sequence $\{\psi_n\}$ such that

$$\|A\psi_n\|_{\mathcal{L}^\infty} \leq C \qquad \lim_{n \to +\infty} \psi_n = \psi \quad \text{in } C(\bar{\Omega}),$$

then from [17] Th. 1.4; we have

$$\lim_{n \to +\infty} S(\psi_n, f, 0) = S(\psi, f, 0) \quad \text{in } \mathcal{L}^\infty(\Omega)$$

then $u \in C(\bar{\Omega})$. ■

From [18] we have $u^0 \in C^\alpha(\bar{\Omega})$ with $0 < \alpha < 1$; then from the lemma 1 and [14], [12] we have $u^n \in C(\Omega)$.

From Th. 3 we have

$$\lim_{n \to +\infty} u^n = u \quad \text{in } \mathcal{L}^\infty(\Omega)$$

then $u \in C(\bar{\Omega})$.

Rem. 4. – From the proof of Th. 3 we have that we can choose in the result

$$\theta_0 = \frac{\bar{P}}{\bar{P}+K_0} + \delta \quad \forall \delta > 0 .$$

In our proof we can not choose $\delta = 0$, why for $\delta \to 0$ we have $Q \to 0$ and then $C \to +\infty$.

3. – Proof of Th. 4.

From Th. 3 the Th. 4 is shown if we show that

$$\lim_{\alpha \to +\infty} u^n_\alpha = u^n \quad \text{in } \mathcal{L}^\infty(\Omega), \qquad \forall n. \tag{3.1}$$

We show (3.1) by induction.

For $n = 0$ we have

$$Au_\alpha^0 = f_\alpha\,, \qquad Au^0 = f$$

then from [18], [19] we have

$$\lim_{\alpha\to+\infty} u_\alpha^0 = u^0 \quad \text{in } \mathcal{L}^\infty(\Omega)\,.$$

We suppose now the result to be valid for $n-1$ and we show the result for n.

The function u_α^n is the solution of the problem

$$(3.2)\qquad \begin{cases} \langle Au_\alpha^n, v - u_\alpha^n\rangle \leqslant \langle f_\alpha, v - u_\alpha^n\rangle\,, \\ \forall v \in H_\Gamma^1(\Omega) \quad v(x) \leqslant M_\alpha^{n-1}(x) \quad \text{a.e. in } \Omega\,, \\ u_\alpha \in H_\Gamma^1(\Omega) \quad u_\alpha(x) \leqslant M_\alpha^{n-1}(x) \quad \text{a.e. in } \Omega\,. \end{cases}$$

We indicate $w_\alpha = u_\alpha^n - A^{-1}f_\alpha$; then w_α is the solution to the problem

$$(3.3)\qquad \begin{cases} \langle Aw_\alpha, v - w_\alpha\rangle \geqslant 0 \\ \forall v \in H_\Gamma^1(\Omega) \quad v(x) \leqslant \psi_\alpha(x) \quad \text{a.e. in } \Omega \\ w_\alpha \in H_\Gamma^1(\Omega) \quad w_\alpha(x) \leqslant \psi_\alpha(x) \quad \text{a.e. in } \Omega\,, \end{cases}$$

where

$$\psi_\alpha(x) = Mu_\alpha^{n-1}(x) - A^{-1}f_\alpha(x)\,.$$

Being

$$\lim_{\alpha\to+\infty} f_\alpha = f \quad \text{in } \mathcal{L}^r(\Omega)$$

we have [18], [19],

$$(3.4)\qquad \lim_{\alpha\to+\infty} A^{-1}f_\alpha = A^{-1}f \quad \text{in } C(\bar{\Omega})\,.$$

Being the result valid for $n-1$, we have

$$(3.5)\qquad \lim_{\alpha\to+\infty} Mu_\alpha^{n-1} = Mu^{n-1} \quad \text{in } \mathcal{L}^\infty(\Omega)$$

then

$$(3.6)\qquad \lim_{\alpha\to+\infty} (Mu_\alpha^{n-1} - A^{-1}f_\alpha) = (Mu^{n-1} - A^{-1}f) \quad \text{in } \mathcal{L}^\infty(\Omega)$$

then from [17] Th. 1.4

$$(3.7)\qquad \lim_{\alpha\to+\infty} w_\alpha = w \quad \text{in } \mathcal{L}^\infty(\Omega)$$

where w is the solution to the problem

$$(3.8)\qquad \begin{cases} \langle Aw, v-w\rangle \geqslant 0 \\ \forall v\in H^1_\Gamma(\Omega) \qquad v(x)\leqslant \psi(x) \\ w\in H^1_\Gamma(\Omega) \qquad w(x)\leqslant \psi(x) \end{cases}$$

where

$$\psi(x) = Mu^{n-1}(x) - A^{-1}f(x)\,.$$

We have

$$(3.9)\qquad w(x) = u^n(x) - A^{-1}f(x)$$

then from (3.4), (3.7) we have

$$(3.10)\qquad \lim_{\alpha\to+\infty} u^n_\alpha = u^n \quad \text{in } \mathcal{L}^\infty(\Omega) \quad \blacksquare$$

REM. 5. – The result of Th. 3 seem to be the first result on continuous dependence for solutions of quasi-variational inequalities without hypothesis on monotonicity for the sequences $\{f_\alpha\}$ $\{K_\alpha\}$.

4. – Proof of Th. 5.

We show at first the following lemma on G-convergence for variational inequalities:

LEMMA 2. – *We consider the following problems*

$$(4.1_p)\qquad \begin{cases} \langle A^p w_p, v-w_p\rangle \geqslant \langle f, v-w_p\rangle \\ \forall v\in H^1_\Gamma(\Omega) \qquad v(x)\leqslant \psi_p(x) \qquad \text{a.e. in } \Omega \\ w_p\in H^1_\Gamma(\Omega) \qquad w_p(x)\leqslant \psi_p(x) \qquad \text{a.e. in } \Omega \end{cases}$$

$$(4.1)\qquad \begin{cases} \langle Aw, v-w\rangle \geqslant \langle f, v-w\rangle \\ \forall v\in H^1_\Gamma(\Omega) \qquad v(x)\leqslant \psi_p(x) \qquad \text{a.e. in } \Omega \\ w_p\in H^1_\Gamma(\Omega) \qquad w_p(x)\leqslant \psi_p(x) \qquad \text{a.e. in } \Omega \end{cases}$$

where ψ_p, $\psi\in C(\bar\Omega)$

$$\lim_{p\to+\infty}\psi_p = \psi \quad \textit{in } C(\bar\Omega)$$

then

$$\lim_{p\to+\infty} w_p = w \quad \textit{in } C(\bar\Omega)$$

where $w_p(w)$ *is the solution to* (4.1_p), $((4.1))$.

We suppose at first $\|A^p\psi_p\|_{\mathcal{L}^\infty}\leqslant$ Cst., $\|A\psi\|_{\mathcal{L}^\infty}\leqslant$ Cst.

From [16] we have

$$\lim_{p\to+\infty} w_p = w \quad \text{in } \mathcal{L}^2(\Omega)$$

and fron [15] Th.

$$\|A^p w_p\|_{\mathcal{L}^\infty} \leqslant \text{Cst.}$$

then from [18] we have $\|w_p\|_{C^\alpha} \leqslant \text{Cst.}$, $0<\alpha<1$, and

$$\lim_{p\to+\infty} w_p = w \quad \text{in } C(\bar{\Omega}) .$$

We suppose now $\psi_p = \psi \in W^{1,p}(\Omega)$, $p>N$, we can also suppose $\psi|_\Gamma = 0$.

We prolongate ψ to a function in $W^{1,p}(\boldsymbol{R}^N)$ then from the lemma of [9] we have a sequence $\{\psi_{pn}\}$ $\{\psi_n\}$ such that ψ_{pn}, $\psi_n|_\Gamma = 0$ and

$$(4.2) \qquad \begin{cases} \|A^p\psi_p\|_{\mathcal{L}^\infty}\,, \quad \|A\psi_n\|_{\mathcal{L}^\infty} \leqslant Cn^{\frac12} \\ \|\psi_{pn}-\psi\|_{\mathcal{L}^\infty}\,, \quad \|\psi_n-\psi\|_{\mathcal{L}^\infty} \leqslant Cn^{-\frac12} \\ \lim\limits_{p\to+\infty} \psi_{pn} = \psi_n \quad \text{in } C(\bar{\Omega}) \qquad \forall n . \end{cases}$$

We consider now the problems

$$(4.3_p) \qquad \begin{cases} \langle Aw_{pn}, v-w_{pn}\rangle \geqslant \langle f, v-w_{pn}\rangle \\ \forall v\in H^1_\Gamma(\Omega) \qquad v(x)\leqslant \psi_{pn}(x) \qquad \text{a.e. in } \Omega \\ w_{pn}\in H^1_\Gamma(\Omega) \qquad w_{pn}(x)\leqslant\psi_{pn}(x) \qquad \text{a.e. in } \Omega , \end{cases}$$

$$(4.3) \qquad \begin{cases} \langle Aw_n, v-w_n\rangle \geqslant \langle f, v-w_n\rangle \\ \forall v\in H^1_\Gamma(\Omega) \qquad v(x) \leqslant \psi_n(x) \qquad \text{a.e. in } \Omega \\ w_n\in H^1_\Gamma(\Omega) \qquad w_n(x)\leqslant\psi_n(x) \qquad \text{a.e. in } \Omega . \end{cases}$$

From the first part of the proof we have

$$(4.4) \qquad \lim_{p\to+\infty} w_{pn} = w_n \quad \text{in } C(\bar{\Omega}) \qquad \forall n .$$

From (4.2) and Lemma 1.4 of [9] we have

$$(4.5) \qquad \begin{cases} \|w_{pn}-w_p\|_{\mathcal{L}^\infty} \leqslant Cn^{\frac12} \\ \|w_n-w\|_{\mathcal{L}^\infty} \leqslant Cn^{\frac12} \end{cases}$$

then

$$\lim_{p\to+\infty} w_p = w \quad \text{in } C(\bar\Omega)\,. \tag{4.6}$$

We suppose now $\psi_p,\ \psi \in W^{1,p}(\Omega),\ \psi_p \neq \psi$.

We indicate by $\bar w_p$ the solution to the problem

$$\begin{cases} \langle A^p \bar w_p, v - \bar w_p\rangle \geqslant \langle f, v - \bar w_p\rangle\,, \\ \forall v \in H^1_\Gamma(\Omega) \qquad v(x) \leqslant \psi(x) \qquad \text{a.e. in } \Omega\,, \\ \bar w_p \in H^1_\Gamma(\Omega) \qquad \bar w_p(x) \leqslant \psi(x) \qquad \text{a.e. in } \Omega\,. \end{cases} \tag{4.7$_p$}$$

From (4.6) we have

$$\lim_{p\to+\infty} \bar w_p = w \quad \text{in } C(\bar\Omega)$$

and from Th. 1.4 [17]

$$\|w_p - \bar w_p\|_{L^\infty} \leqslant \|\psi_p - \psi\|_{L^\infty}$$

then

$$\lim_{p\to+\infty} w_p = w \quad \text{in } C(\bar\Omega)\,. \tag{4.8}$$

We consider now the general case $\psi_p,\ \psi \in C(\bar\Omega)$.
Being

$$\lim_{p\to+\infty} \psi_p = \psi \quad \text{in } C(\bar\Omega) \tag{4.9}$$

we have two sequences $\{\psi_{pk}\}$ $\{\psi_k\}$ such that

$$\lim_{k\to+\infty} \psi_{pk} = \psi_p \quad \text{in } C(\bar\Omega) \text{ uniformly in } p\,. \tag{4.10}$$

$$\lim_{k\to+\infty} \psi_k = \psi \quad \text{in } C(\bar\Omega) \tag{4.11}$$

$$\lim_{p\to+\infty} \psi_{pk} = \psi_k \quad \text{in } C(\bar\Omega)\,. \tag{4.12}$$

We indicate now by w_{pk} (w_k) the solution to the problem

$$\begin{cases} \langle A^p w_{pk}, v - w_{pk}\rangle \geqslant \langle f, v - w_{pk}\rangle \\ \forall v \in H^1_\Gamma(\Omega) \qquad v(x) \leqslant \psi_{pk}(x) \qquad \text{a.e. in } \Omega \\ w_{pk} \in H^1_\Gamma(\Omega) \qquad w_{pk}(x) \leqslant \psi_{pk}(x) \qquad \text{a.e. in } \Omega\,, \end{cases} \tag{4.13$_p$}$$

$$\begin{cases} \langle A w_u, v - w_k\rangle \geqslant \langle f, v - w_k\rangle \\ \forall v \in H^1_\Gamma(\Omega) \qquad v(x) \leqslant \psi_k(x) \qquad \text{a.e. in } \Omega \\ w_k \in H^1_\Gamma(\Omega) \qquad w_k(x) \leqslant \psi_k(x) \qquad \text{a.e. in } \Omega)\,. \end{cases} \tag{4.13}$$

From (4.10) and Th. 1.4 [17] we have

$$\lim_{k\to+\infty} w_{pk} = w_p \quad \text{in } C(\bar{\Omega}) \text{ uniformly in } p \tag{4.14}$$

From (4.11), (4.12) and Th. 1.4 [17] we have also

$$\begin{aligned} &\lim_{k\to+\infty} w_k = w \quad &&\text{in } C(\bar{\Omega}) \\ &\lim_{k\to+\infty} w_{pk} = w_k \quad &&\text{in } C(\bar{\Omega}) . \end{aligned} \tag{4.15}$$

From (4.14) and (4.15) we have

$$\lim_{p\to+\infty} w_p = w \quad \text{in } C(\bar{\Omega}) \quad \blacksquare$$

We give now the proof of Th. 5.

From [9], [21] we have

$$\lim_{p\to+\infty} u_p^0 = u^0 \quad \text{in } \mathfrak{L}^r(\Omega) \tag{4.16}$$

where

$$A^p u_p^0 = f . \tag{4.17}$$

From [18] we have $\|u_p^0\|_{C^\alpha} \leqslant \text{Cst.}$, $0 < \alpha < 1$, then

$$\lim_{p\to+\infty} u_p^0 = u^0 \quad \text{in } C(\bar{\Omega}) . \tag{4.18}$$

We show now that

$$\lim_{p\to+\infty} u_p^n = u^n \quad \text{in } C(\bar{\Omega}) \tag{4.19}$$

where u_p^n is the solution to the problem.

$$\begin{cases} \langle A^p u_p^n, v - u_p^n\rangle \geqslant \langle f, v - u_p^n\rangle , \\ \forall v \in H^1_\Gamma(\Omega) \quad v(x) \leqslant M u_p^{n-1}(x) \quad \text{a.e. in } \Omega , \\ u_p^n \in H^1_\Gamma(\Omega) \quad u_p^n(x) \leqslant M u_p^{n-1}(x) \quad \text{a.e. in } \Omega . \end{cases} \tag{4.20$_n$}$$

From (4.18) we have the result for $n = 0$.

We suppose now the result for $n-1$ and we show the result for n.

Being

$$\lim_{p\to+\infty} u_p^{n-1} = u^{n-1} \quad \text{in } C(\bar{\Omega}) \tag{4.21}$$

we have

$$\lim_{p\to+\infty} Mu_p^{n-1} = Mu^{n-1} \quad \text{in } C(\bar{\Omega}) \tag{4.22}$$

then from Th. 1.4 [17]

$$\lim_{p\to+\infty} u_p^n = u^n \quad \text{in } C(\bar{\Omega}) . \tag{4.23}$$

From the Th. 3 we have also

$$\begin{cases} 0\leqslant u_p^n - u_p \leqslant K\theta_0^n \\ 0\leqslant u^n - u \leqslant K\theta_0^n \end{cases} \tag{4.24}$$

where the constant K don't depend on p, n.

From (4.23), (4.24) we have

$$\lim_{p\to+\infty} u_p = u \quad \text{in } C(\bar{\Omega}) . \quad \blacksquare$$

5. – Proof of Th. 6.

Let be $B: H_0^1(\Omega)\to H^{-1}(\Omega)$ an elliptic operator with constant coefficients; we have [10], [15]:

Lemma 3. – *Let be* $f\in \mathcal{L}^r(\Omega)$, $r > n/2$

$$g(x) = \operatorname{Inf}\Big(\operatorname*{Inf}_{\xi\geqslant 0} \tilde{f}(x+\xi), 0\Big) \in \mathcal{L}^r(\Omega)$$

where $\tilde{f}$ *is the prolongate of* f *to* $\mathbf{R}^N$ *by* 0.

Let be u *the solution to the problem*

$$B\underline{u} = g .$$

We suppose $\underline{u}(x) \geqslant -K_0$ *a.e. in* Ω.

We consider the problem.

$$\begin{cases} \langle Bu, v-u\rangle \geqslant \langle f, v-u\rangle , \\ \forall v \in H_0^1(\Omega) \quad v(x)\leqslant Mu(x) \quad \text{a.e. in } \Omega , \\ u\in H_0^1(\Omega) \quad u(x) \leqslant Mu(x) \quad \text{a.e. in } \Omega . \end{cases} \tag{5.1}$$

The problem (5.1) *has a unique solution* $u\in W^{2,p}(\Omega)$ *and if we construct* u^n *as in*

Th. 1 *we have*:

$$\|Bu^n\|_{\mathfrak{L}^r} \leqslant \operatorname{Max}\left(\|f\|_{\mathfrak{L}^r}, \|g\|_{\mathfrak{L}^r}\right).$$

Lemma 4. – *The result of Th.* 3 *is again valid for the problem*

$$\text{(5.2)} \qquad \begin{cases} \langle Au, v-u\rangle \geqslant \langle f, v-u\rangle \\ \forall v \in H_0^1(\Omega) \qquad v(x) \leqslant Mu(x) \qquad \text{a.e. in } \Omega \\ u \in H_0^1(\Omega) \qquad u(x) \leqslant Mu(x) \qquad \text{a.e. in } \Omega \end{cases}$$

(*where A is as in Th.* 3 *and f as in the Lemma* 3) *if* $u(x) > -K_0$ *in* Ω.

We observe that $\underline{u}(x)$ is continuous being $g \in \mathfrak{L}^p(\Omega)$, then we have

$$\text{(5.3)} \qquad \min_{x\in\overline{\Omega}} u(x) > -K_0 .$$

Let be $w = u - \underline{u}$, w is the solution to the problem

$$\text{(5.4)} \qquad \begin{cases} \langle Aw, v-w\rangle \geqslant \langle f-g, v-w\rangle \\ \forall v \in H_0^1(\Omega) \qquad v(x) \leqslant M'w(x) \qquad \text{a.e. in } \Omega \\ w \in H_0^1(\Omega) \qquad w(x) \leqslant M'w(x) \qquad \text{a.e. in } \Omega \end{cases}$$

where

$$\text{(5.5)} \qquad M'\varphi(x) = K_0 - \overline{u}(x) + \operatorname*{Inf}_{\substack{x+\xi\in\Omega\\ \xi\geqslant 0}} \left(\varphi(x+\xi) + \overline{u}(x+\xi)\right).$$

Let be now

$$K(x;\xi) = K_0 - \underline{u}(x) + \underline{u}(x+\xi).$$

From (5.3) we have

$$K(x;\xi) \geqslant \delta > 0 .$$

From (5.5) we have

$$\text{(5.6)} \qquad M'\varphi(x) = \operatorname*{Inf}_{\substack{x+\xi\in\Omega\\ \xi\geqslant 0}} \left(\varphi(x+\xi) + K(x;\xi)\right).$$

Finally we observe that $f-g \geqslant 0$.

Then the Th. 3 is valid for the problem (5.4) and we have

$$\text{(5.7)} \qquad 0 \leqslant w^n - w \leqslant K\theta_0^n$$

where w^0 is the solution to the problem

(5.8) $$Aw^0 = f - g$$

and w^n is the solution to the problem

(5.9) $$\begin{cases} \langle Aw^n, v - w^n \rangle \geq \langle f - g, v - w^n \rangle , \\ \forall v \in H_0^1(\Omega) \quad v(x) \leq M' w^{n-1}(x) \quad \text{a.e. in } \Omega , \\ w^n \in H_0^1(\Omega) \quad w^n(x) \leq M' w^{n-1}(x) \quad \text{a.e. in } \Omega . \end{cases}$$

From (5.5), (5.8), (5.9) we have

$$w^n = u^n - u , \qquad \forall n$$

then from (5.7)

$$0 \leq u^n - u \leq K\theta_0^n . \qquad \blacksquare$$

LEMMA 5. – *Let be $u \in H_0^{1,\infty}(\Omega)$ we have $Mu \in H^{1,\infty}(\Omega)$ with*

$$Mu|_\Gamma \leq 0 , \qquad \|Mu\|_{1,\infty} \leq \|u\|_{1,\infty} .$$

Let be $u \in H_0^{1,\infty}(\Omega)$, then $Mu \in C(\bar{\Omega})$ [14] (12).

Let be x, $x + he_i \in \Omega$, where e_i is the unit vector in the direction of the x_i, $\xi \geq 0$.

We can suppose $h \geq 0$, the proof for $h \leq 0$ is analogous.

If $x + \xi$ and $x + he_i + \xi$ are in $\bar{\Omega}$ we have:

(5.10) $$|u(x + he_i + \xi) - u(x + \xi)| \leq C|h|$$

where $C = \|u\|_{1,\infty}$.

If $x + he_i + \xi \notin \Omega$, let be $\bar{h} = \theta h$, $0 < \theta \leq 1$, the supremum of the η such that the segment $[x + \xi, x + \eta e_i + \xi[$ is in Ω.

We $x + \bar{h}e_i + \xi \in \partial\Omega$ then $u(x + \bar{h}e_i + \xi) = 0$. We have

$$|u(x + \xi)| = |u(x + \xi) - u(x + \bar{h}e_i + \xi)| \leq C|\bar{h}| \leq C|h| .$$

Being $x + h \in \Omega$ there is $\xi' \geq 0$ such that $x + h + \xi' \in \partial\Omega$ then

(5.11) $$|u(x + he_i + \xi') - u(x + \xi)| \leq C|h| .$$

We can show analogously (5.11) in the case $x + \xi \notin \Omega$, $x + he_i + \xi \in \Omega$.

If $x + \xi$, $x + he_i + \xi \in \partial\Omega$ we have

(5.12) $$|u(x + he_i + \xi) - u(x + \xi)| = 0 .$$

Then for x, $x+he_i \in \Omega$, $\xi \geqslant 0$, $x+\xi \in \Omega$ there is a $\xi' \geqslant 0$ such that

$$|u(x+he_i+\xi')-u(x+\xi)| \leqslant C|h| . \tag{5.13}$$

The result of (5.13) is valid for $h \geqslant 0$ or $h \leqslant 0$, then

$$|Mu(x+he_i)-Mu(x)| \leqslant C|h|$$

then

$$\|Mu\|_{1,\infty} \leqslant \|u\|_{1,\infty} .$$

Being now $u|_\Gamma = 0$ we have for all $x \in \Gamma$

$$Mu(x) \leqslant 0 . \quad ■$$

From [7], [8] we have

LEMMA 5. – *Let be $\psi \in W^{1,r}(\Omega)$, $r > N$; we consider the problems*

$$\begin{cases} \langle A^p u_p, v-u_p\rangle \geqslant \langle f, v-u_p\rangle \\ \forall v \in H_0^1(\Omega) \qquad v(x) \leqslant \psi(x) \qquad \text{a.e. in } \Omega \\ u_p \in H_0^1(\Omega) \qquad u_p(x) \leqslant \psi(x) \qquad \text{a.e. in } \Omega \end{cases} \tag{5.14_p}$$

$$\begin{cases} \langle Au, v-u\rangle \geqslant \langle f, v-u\rangle \\ \forall v \in H_0^1(\Omega) \qquad v(x) \leqslant \psi(x) \qquad \text{a.e. in } \Omega \\ u \in H_0^1(\Omega) \qquad u(x) \leqslant \psi(x) \qquad \text{a.e. in } \Omega \end{cases} \tag{5.14}$$

where A and A^p is as in Th. 5.

Let be $u(u_p)$ the solution to the problem (5.14), ((5.14$_p$)) we have

$$\|u_p - u\|_{L^\infty} \leqslant C p^{\bar{\alpha}/2(N-2+3\bar{\alpha})}$$

where C is a constant dependent as α, β and $\|\psi\|_{1,p}$ and $\bar{\alpha}$ is an Hölder exponent for A^p, A and $W^{-1,p}$ in the De Giorgi-Nash theorem.

Let be now u_p^0 the solution to the problem

$$A^p u_p^0 = f \tag{5.15}$$

and u_p^n, $n \geqslant 1$, the solution to the problem

$$\begin{cases} \langle A^p u_p^n, v-u_p^n\rangle \geqslant \langle f, v-u_p^n\rangle , \\ \forall v \in H_0^1(\Omega) \qquad v(x) \leqslant Mu_p^{n-1}(x) \qquad \text{a.e. in } \Omega , \\ u_p^n \in H_0^1(\Omega) \qquad u_p^n(x) \leqslant Mu_p^{n-1}(x) \qquad \text{a.e. in } \Omega . \end{cases} \tag{5.16}$$

We show that

$$\|u_p^n - u^n\|_{\mathfrak{L}^\infty} \leqslant C(np^{-\bar{\alpha}/2(N-2+3\bar{\alpha})} + p^{-\frac{1}{4}}) . \tag{5.17}$$

From [7] we have (5.17) for $n = 0$.

We suppose now (5.17) valid for $n-1$.

We have

$$\|Mu_p^{n-1} - Mu^{n-1}\|_{\mathfrak{L}^\infty} \leqslant C\big((n-1)p^{-\bar{\alpha}/2(N-2+3\bar{\alpha})} + p^{-\frac{1}{4}}\big) . \tag{5.18}$$

Let be now $\bar{u}_p^0$ the solution to the problem

$$\begin{cases} \langle A^p \bar{u}_p^n, v - \bar{u}_p^n \rangle \geqslant \langle f, v - \bar{u}_p^n \rangle , \\ \forall v \in H_0^1(\Omega) \qquad v(x) \leqslant Mu^{n-1}(x) , \\ \bar{u}_p^n \in H_0^1(\Omega) \qquad \bar{u}_p^n(x) \leqslant Mu^{n-1}(x) . \end{cases} \tag{5.19}$$

From the lemma 3, 5, 6 we have

$$\|\bar{u}_p^n - u^n\|_{\mathfrak{L}^\infty} \leqslant C' p^{-\bar{\alpha}/2(N-2+3\bar{\alpha})} \tag{5.20}$$

where C' depends only on f, g and we can suppose $C > C'$.

From Th. 1.4 [17] we have also

$$\|u_p^n - \bar{u}_p^n\|_{\mathfrak{L}^\infty} \leqslant C\big((n-1)p^{-\bar{\alpha}/2(N-2+3\bar{\alpha})} + p^{-\frac{1}{4}}\big) \tag{5.21}$$

then

$$\|u_p^n - u^n\|_{\mathfrak{L}^\infty} \leqslant C(np^{-\bar{\alpha}/2(N-2+3\bar{\alpha})} + p^{-\frac{1}{4}}) . \tag{5.22}$$

From the lemma 4 we have also

$$\left.\begin{aligned} \|u_p^n - u_p\|_{\mathfrak{L}^\infty} &\leqslant \frac{K}{n^s} \\ \|u^n - u\|_{\mathfrak{L}^\infty} &\leqslant \frac{K}{n^s} \end{aligned}\right. \qquad \forall s > 0, \; n \geqslant \bar{n}_s \tag{5.23}$$

where K is a constant independent on p.

From (5.22), (5.23) we have

$$\|u_p - u\|_{\mathfrak{L}^\infty} \leqslant C(np^{-\bar{\alpha}/2(N-2+3\alpha)} + p^{-\frac{1}{4}}) + K/n^s .$$

Choosing $n^{s+1/s}$ as the first integer $\leqslant p^{-\bar{\alpha}/4(N-2+3\bar{\alpha})}$ we have

$$\|u_p - u\|_{\mathfrak{L}^\infty} \leqslant C p^{-\bar{\alpha}/4(N-2+3\bar{\alpha})\; s/(s+1)} . \qquad \blacksquare$$

REFERENCES

[1] A. BENSOUSSAN - J. L. LIONS, *Nouvelle formulation des problèmes de contrôle impulsionnel*, Comptes Rendus Acad. Sc. Paris, A **276** (1973), pp. 1189-1192.
[2] A. BENSOUSSAN - J. L. LIONS, *Nouvelles méthodes en contrôle impulsionnel*, Appl. Math. Opt., **1**, (4) (1975), pp. 289-312.
[3] A. BENSOUSSAN - J. L. LIONS, *Inéquations quasi-variationelles dépendantes d'une paramètre*, Ann. Sc. Norm. Sup. Pisa.
[4] A. BENSOUSSAN - M. GOURSAT - J. L. LIONS, *Contrôle impulsionnel et I.Q.V. stationnaires*, Comptes Rendus Acad. Sc. Paris, A **276** (1973), pp. 1279-1281.
[5] A. BENSOUSSAN - J. L. LIONS - G. PAPANICOLAU, *Sur quelques phenomènes asymptotiques stationnaires*, Comptes Rendus Acad. Sc. Paris, A **281** (1975), pp. 89-94.
[6] M. BIROLI, *Sur la G-convergence pour des inéquations quasi-variationnelles*, Boll. U.M.I., (5), **14**-A (1977), pp. 540-551.
[7] M. BIROLI, *An estimate on convergence for an homogeneisation problem for variational inequalities*, in print Rend. Sem. Mat. Padova.
[8] M. BIROLI, *G-convergence for elliptic equations, variational inequalities and quasi-variational inequalities*, in print Rend. Sem. Mat. Fis. Milano.
[9] E. DE GIORGI - S. SPAGNOLO, *Sulla convergenza degli integrali dell'energia per operatori ellittici del* 2- *ordine*, Boll. U.M.I., (4) **8** (1973), pp. 391-411.
[10] J. L. JOLY - U. MOSCO - G. TROIANIELLO, *Un résultat de régularité pour un'inéquation quasi-variationelle intervenant dans un problème de contrôle impulsionnel*, Comptes Rendus Acad. Sc. Paris, A **279** (1974), pp. 937-941.
[11] TH. LAESTCH, *An uniqueness theorem for elliptic quasi-variational inequalities*, J. Func. An., **18** (1975), pp. 286-287.
[12] J. L. LIONS, *Sur quelques questions d'analyse, mécanique et contrôle optimal*, Les presses de l'Université de Montréal, 1976.
[13] J. L. LIONS, *Asymptotic behavior of variational inequalities with highly oscillating coefficients*, in *Applications of Methods of Functional Analysis to Problems in Mechanics*, Lecture Notes 503, Springer, 1976.
[14] J. L. MENALDI, *Sur le problème de contrôle impulsionnel et l'inéquation quasi-variationelle degénerée associée*, Comptes Rendus Acad. Sc. Paris.
[15] U. MOSCO, *Implicit variational problems and quasi-variational inequalities*, in *Nonlinear Operator and Calculul of Variations - Bruxelles 1975*, Lecture Notes 543, Springer, 1976.
[16] F. MURAT, *Sur l'homogeneisation d'inéquations elliptiques du 2ᵉ ordre ...*, Lab. An. Num. Paris VI, no. 76013.
[17] J. P. PUEL, *Thèse*, Lab. An. Num. Paris VI, 1975.
[18] G. STAMPACCHIA, *Problemi al contorno con soluzioni hölderiane*, Ann. Mat. Pura Appl., **51** (1960), pp. 1-32.
[19] G. STAMPACCHIA, *Equations elliptiques du second ordre à coefficients discontinues*, Les presse de l'Université de Montréal, 1965.
[20] L. TARTAR, *Inéquations quasi-variationelles abstraites*, Comptes Rendus Acad. Sc. Paris, A **278** (1974), pp. 1193-1196.
[21] L. TARTAR, *Cours Péccot*, Collège de France, 1977.
[22] G. TROIANIELLO, In print.