On the Optimal Controls of a Class of Systems Governed by Second Order Parabolic Partial Delay-Differential Equations with First Boundary Conditions (*).

K. L. Teo (Australia) - N. U. Ahmed (Canada)

Summary. – *In this paper we consider the question of existence of optimal controls for a class of systems governed by second order parabolic partial delay-differential equations with first boundary conditions and with controls appearing in the coefficients. In Theorems* 2.2 *and* 2.3 *we present existence and uniqueness of solutions of the first boundary problems. In Theorems* 3.1 *and* 3.2 *we prove that whenever the coefficients of the system converge in the* w^*-*topology* (L^1 *topology on* L^∞) *the corresponding solutions converge weakly in an appropriate Sobolev space. Using these basic results we present two theorems* (*Theorems* 4.1 *and* 4.2) *on the existence of optimal controls.*

1. – Introduction.

It appears from the recent literature on the existence of optimal controls for systems governed by partial differential equations, that there are not many results available, in case the system has control dependent coefficients with delayed arguments (2, p. 262; 4). In this paper we consider this question for a class of distributed parameter systems governed by second order partial delay-differential equations of parabolic type with homogeneous boundary conditions and with controls appearing in the coefficients. The coefficients associated with the second order terms are assumed to be continuous and those appearing in the first and zero-order terms are assumed bounded measurable and contain controls and delays in their arguments.

In Theorems 2.2 and 2.3 we present the existence and uniqueness of solutions to the boundary value problems. In Theorems 3.1 and 3.2 we show that whenever the coefficients of the problems converge in the w^*-sense in L^∞ space equipped with w^*-topology (L^1 topology on L^∞) the corresponding solutions converge weakly in the Sololev space $W_\lambda^{2,1}$ ($n+2<\lambda<\infty$) giving a form of continuous dependence of solutions on parameters.

Using these results, we prove two existence theorems (Theorems 4.1 and 4.2) for optimal controls from the class of bounded measurable functions for a control problem to be stated later.

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Let Ω be a domain in n-dimensional Euclidean space R^n, bounded by a smooth surface $\partial\Omega$ satisfying the following properties: each point of $\partial\Omega$ is locally representable by functions with Hölder continuous second order partial derivatives.

We denote the coordinates of a point x in R^n by $x_1, \dots, x_n$, time by t.

Let T be a fixed time instant, $T < \infty$, and $h_\varkappa$, $(\varkappa = 0, 1, \dots, \nu)$, be certain given numbers such that

$$0 = h_0 < h_1 < \dots < h_\nu < T\,.$$

We denote the intervals

$$I_0 = [-h_\nu, 0]\,, \qquad I_1 = (0, T)\,, \qquad I_2 = [-h_\nu, T]\,.$$

Now let us consider the following second order linear delayed partial differential equations of parabolic type:

$$(1)\quad \begin{cases} \varphi_t(u)(x,t) = \sum\limits_{i,j=1}^{n} a_{ij}(x,t)\cdot\varphi_{x_i x_j}(u)(x,t) + \sum\limits_{\varkappa=0}^{\nu}\sum\limits_{i=1}^{n} b_{i,\varkappa}\big(x, t-h_\varkappa, u(x, t-h_\varkappa)\big)\cdot \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot\varphi_{x_i}(u)(x, t-h_\varkappa) \\ \qquad + \sum\limits_{\varkappa=0}^{\nu} c_\varkappa\big(x, t-h_\varkappa, u(x, t-h_\varkappa)\big)\cdot\varphi(u)(x, t-h_\varkappa) \\ \qquad + \sum\limits_{\varkappa=0}^{\nu} f_\varkappa\big(x, t-h_\varkappa, u(x, t-h_\varkappa)\big)\,, \qquad \text{for } (x,t)\in\Omega\times I_1 \\ \varphi(u)(x,t) = \varphi_0(x,t)\,, \qquad \text{for } (x,t)\in\Omega\times I_0 \\ \varphi(u)(x,t) = 0\,, \qquad \text{for } (x,t)\in\partial\Omega\times I_2 \\ u\in D\,, \end{cases}$$

where

$$\psi_t \triangleq \frac{\partial\psi}{\partial t}\,, \qquad \psi_{x_i} \triangleq \frac{\partial\psi}{\partial x_i}\,, \qquad \psi_{x_i x_j} \triangleq \frac{\partial^2\psi}{\partial x_i\,\partial x_j}\,, \qquad (i, j = 1, \dots, n)\,,$$

and D is the set of admissible controls to be defined later.

Let $\hat{u}$ be a bounded measurable function from $\bar{\Omega}\times[-h_\nu, 0)$ into R^r and let U be a non-empty compact convex subset of R^r. Now let us define the set of admissible controls on $\bar{\Omega}\times[-h_\nu, T]$ by

$$\begin{aligned} D \triangleq \{u\colon u \text{ measurable on } \bar{\Omega}\times[-h_\nu, T],\ u(x,t) = \hat{u}(x,t) \\ \text{almost everywhere on } \bar{\Omega}\times[-h_\nu, 0) \text{ and } u(x,t)\in U \\ \text{almost everywhere on } \bar{\Omega}\times[0, T]\}. \end{aligned}$$

With this preparation, we state the optimal control problem considered in the present paper, P, as: Given the dynamic system (1), find a control $u_0 \in D$ that minimizes the cost functional

$$J(u) = \iint_{\Omega\times(0,T)} \left\{\sum_{\varkappa=0}^{\nu} F^{\varkappa}(x, t, u(x, t-h_\varkappa), \varphi(u)(x, t-h_\varkappa), \varphi_x(u)(x, t-h_\varkappa))\right\} dx\, dt\,,$$

where

$$\sum_{\varkappa=0}^{\nu} F^{\varkappa}(x, t, u(x, t-h_\varkappa), \varphi(u)(x, t-h_\varkappa), \varphi_x(u)(x, t-h_\varkappa)) =$$
$$= \sum_{\varkappa=0}^{\nu}\left\{\sum_{l=1}^{r} \eta_l^{\varkappa}(x, t)\cdot u_l(x, t-h_\varkappa) + V_0^{\varkappa}(x, t)\cdot\varphi(u)(x, t-h_\varkappa) + \sum_{i=1}^{n} V_i^{\varkappa}(x, t)\cdot\varphi_{x_i}(u)(x, t-h_\varkappa)\right\}$$

$$\eta_l^{\varkappa} \in L^1(\Omega\times(0, T), R^1)\,, \quad (l = 1, \ldots, r) \quad \text{and} \quad V_i^{\varkappa} \in L^1(\Omega\times(0, T), R^1)\,,$$
$$(\varkappa = 0, 1, \ldots, \nu; \quad \text{and} \quad i = 1, \ldots, n)\,.$$

2. – Existence and uniqueness of solutions.

In this section, we shall recall certain results on the existence and uniqueness of solutions of the system (1) and certain other systems related to it.

With reference to system (1), it is assumed that, for each $u \in D$, the coefficients and data are defined and measurable on their appropriate domains. Before stating more specific assumptions, we shall introduce some useful notations.

$|0|$ denotes the Lebesgue measure of the measurable set 0 of any finite dimensional Euclidean space. Let E be any connected subset of an s-dimensional Euclidean space R^s and denote by $C^l(E)$ the class of all l times continuously differentiable functions on E, where $1 \leqslant l < \infty$ is an integer. Further, let $C_0^l(E)$ be the class of functions from $C^l(E)$ with compact support on E.

For any $Z \in R^n$, let $|Z| \triangleq \left(\sum_{i=1}^{n} Z^2\right)^{\frac{1}{2}}$. Let $z_x \triangleq [z_{x_1}, \ldots, z_{x_n}]$ denote the gradient of the scalar valued function z on R^n.

Let E be as before and denote by $L^\delta(E)$ the Banach space consisting of all measurable functions on E that are δth-power ($\delta \geqslant 1$) integrable on E. The norm in it is defined by the equalities

$$\|z\|_{\delta,E} = \left\{\int_E |z(y)|^\delta dy\right\}^{1/\delta} \quad \text{for } 1 \leqslant \delta < \infty$$

and

$$\|z\|_{\infty,E} = \operatorname*{ess\,sup}_E |z(y)) \quad \text{for } \delta = \infty\,.$$

Measurability and integrability are to be understood in the sense of Lebesgue. The elements of $L^\delta(E)$ are the equivalence classes of the functions on E (functions belonging to the same equivalence class are equal almost everywhere).

Let λ be a real number such that $1 \leqslant \lambda < \infty$ and denote by $W_\lambda^{2,1}(E)$ the Banach space of functions from $L^\lambda(E)$ having generalized derivatives of the form $(\partial^r/\partial t^r)(\partial^s/\partial x^s)$ with any r and s satisfying the inequality $2r + s \leqslant 2$. The norm in it is defined by the equality

$$\|z\|_{\lambda,E}^{(2)} = \|z\|_{\lambda,E} + \left\|\frac{\partial z}{\partial t}\right\|_{\lambda,E} + \sum_{i=1}^{n}\left\|\frac{\partial z}{\partial x_i}\right\|_{\lambda,E} + \sum_{i,j=1}^{n}\left\|\frac{\partial^2 z}{\partial x_i\,\partial x_j}\right\|_{\lambda,E}.$$

Note that if z is only a function of x defined in Ω then we denote $W_\lambda^2(\Omega)$ the Banach space of functions from $L^\lambda(\Omega)$ having generalized derivatives of the form $\partial^s/\partial x^s$ with $s = 0, 1$, and 2. The norm in it is defined by the equality

$$\|z\|_{\lambda,\Omega}^{(2)} = \|z\|_{\lambda,\Omega} + \sum_{i=1}^{n}\left\|\frac{\partial z}{\partial x_i}\right\|_{\lambda,\Omega} + \sum_{i,j=1}^{n}\left\|\frac{\partial^2 z}{\partial x_i\,\partial x_j}\right\|_{\lambda,\Omega}.$$

For any nonintegral positive number λ, $H^{\lambda,\lambda/2}(\bar{E})$ denotes the Banach space of functions z that are continuous on $\bar{E}$ and have derivatives of the form

$$D_t^\alpha \cdot D_x^\beta \cdot z \triangleq \frac{\partial^\alpha}{\partial t^\alpha}\cdot\left(\frac{\partial^{\beta_1+\dots+\beta_n}}{\partial x_1^{\beta_1}\dots\partial x_1^{\beta_n}}\cdot z\right), \qquad \sum_{i=1}^{n}\beta_i = \beta,$$

β_i nonnegative integer, $2\alpha + \beta < \lambda$, and have a finite norm

$$|z|_{\bar{E}}^{\lambda} \triangleq \|z\|_{\bar{E}}^{(\lambda)} + \sum_{j=0}^{[\lambda]}\|z\|_{\bar{E}}^{(j)}.$$

Note that $[\lambda]$ denotes the largest integral part of λ and

$$\|z\|_{\bar{E}}^{(0)} \triangleq |z|_{\bar{E}}^{(0)} = \max_{\bar{E}}|z(x,t)|,$$

$$\|z\|_{\bar{E}}^{(j)} \triangleq \sum_{(2\alpha+\beta=j)}|D_t^\alpha\cdot D_x^\beta\cdot z|_{\bar{E}}^{(0)},$$

$$\|z\|_{\bar{E}}^{(\lambda)} \triangleq \|z\|_{x,\bar{E}}^{(\lambda)} + \|z\|_{t,\bar{E}}^{(\lambda/2)},$$

$$\|z\|_{x,\bar{E}}^{(\lambda)} \triangleq \sum_{(2\alpha+\beta=[\lambda])}\|D_t^\alpha\cdot D_x^\beta\cdot z\|_{x,\bar{E}}^{(\lambda-[\lambda])},$$

$$\|z\|_{t,\bar{E}}^{(\lambda/2)} \triangleq \sum_{0<\lambda-2\alpha-\beta<2}\|D_t^\alpha\cdot D_x^\beta\cdot z\|_{t,\bar{E}}^{(\lambda-2\alpha-\beta)/2},$$

$$\|z\|_{x,\bar{E}}^{(\gamma)} \triangleq \max_{(x,t),(x',t)\in\bar{E}}\frac{|z(x,t)-z(x',t)|}{|x-x'|^\gamma}, \qquad 0<\gamma<1,$$

$$\|z\|_{t,\bar{E}}^{(\gamma)} \triangleq \max_{(x,t),(x,t')\in\bar{E}}\frac{|z(x,t)-z(x,t')|}{|t-t'|^\gamma}, \qquad 0<\gamma<1.$$

Throughout this paper, the coefficients and data of the system (1) are assumed to satisfy the following assumptions *which will be referred to collectively as* (A).

(i) $a_{ij}(\cdot,\cdot)$, $(i,j=1,\ldots,n)$, are continuously on $\bar{Q}$, where $Q \triangleq \Omega\times I_1$ and $\bar{Q}$ is the closure of Q;

(ii) there exist numbers $\alpha_u>\alpha_l>0$ such that

$$\alpha_u|Z|^2 \geqslant \sum_{i,j=1}^{n} a_{ij}(x,t)\cdot Z_i\cdot Z_j \geqslant \alpha_l\cdot|Z|^2$$

for all $Z\in R^n$ uniformly on $\bar{Q}$ (uniformly parabolic), where $|Z|^2 \triangleq \sum_{i=1}^{n}|Z_i|^2$;

(iii) $\max\limits_{1\leqslant i,j\leqslant n} \dfrac{|a_{ij}(x,t)-a_{ij}(x',t')|}{|t-t'|+|x-x'|} \leqslant M\,,$

where $t,t'\in[0,T]$; $x,x'\in\bar{\Omega}$; and M is a constant;

(iv) $b_{i,\varkappa}(\cdot,\cdot-h_\varkappa,\cdot)$, $c_\varkappa(\cdot,\cdot-h_\varkappa,\cdot)$, $(i=1,\ldots,n;\ k=0,1,\ldots,\nu)$, are bounded measurable on $\bar{Q}\times U$ and continuous on U for almost all $(x,t)\in\bar{Q}$;

(v) $\varphi_0\in C^2(\bar{\Omega}\times[-h_\nu,0])$ and $\varphi_0(x,t)=0$ for all $x\notin\bar{\Omega}_0$ and $t\in[-h_\nu,0]$, where Ω_0 is a compact subset of Ω.

Consider a linear second order partial differential equation of parabolic type described by

$$(2)\qquad \begin{cases} L\cdot\Phi(x,t)=f(x,t)\,, & (x,t)\in Q\\ \Phi(x,0)=\Phi_0(x)\,, & x\in\Omega\\ \Phi(x,t)=0\,, & (x,t)\in\partial\Omega\times[0,T]\end{cases}$$

where the operator L is defined by

$$L\cdot\psi(x,t)=\psi_t(x,t)-\left\{\sum_{i,j=1}^{n}a_{ij}(x,t)\cdot\psi_{x_ix_j}(x,t)+\sum_{i=1}^{n}b_i(x,t)\cdot\psi_{x_i}(x,t)+c(x,t)\cdot\psi(x,t)\right\}.$$

It is assumed that a_{ij}, $(i,j=1,\ldots,n)$, satisfy the conditions (i), (ii) of (A); $\Phi_0\in W^2_\lambda(\Omega)$, $(\lambda>n+2)$, and such that $\Phi_0(x)=0$ on $\partial\Omega$; and b_i, $(i=1,\ldots,n)$, c are bounded measurable on $\bar{Q}$. *These conditions will be referred to collectively as* (A').

For brevity, the statement: « C depends on the structure of the differential equation of the system (2) » will be used to mean that C is determined by the quantities α_l,α_u, M, and the bounds of the functions b_i, $(i=1,\ldots,n)$, and c where α_l,α_u, and M are as defined in (A).

Let E be a connected subset of $\bar{Q}$ and the statement: « C depends on the structure of the system (1) in E » will be used to mean that C is determined by the quantities α_l,α_u, M, and the bounds for the functions $b_{i,0}$, $(i=1,\ldots,n)$, and c_0 evaluated in E. On the other hand, the statement: « C depends on the structure of the differential equation of the system (1) » will be used to mean that C is determined by the quantities α_l, α_u, M, and the bounds for the functions $b_{i,\varkappa}$, $c_\varkappa$, $(i=1, \ldots, n;\ \varkappa=0, 1, \ldots, \nu)$.

In the sequel, we need

DEFINITION 2.1. – A function $\Phi \in W_\lambda^{2,1}(Q)$ with $n+2<\lambda<\infty$ is said to be a solution of the system (2) if it satisfies the first equation of (2) almost everywhere and the rest of the two equations everywhere. We remark that a similar definition applies to system (1).

In this paper, it is understood that ψ_t and $\psi_{x_i x_j}$ denote, respectively, the generalized derivative of ψ with respect to t and the generalized derivative of ψ_{x_i} with respect to x_j.

Note that the existence and uniqueness of solutions in $W_\lambda^{2,1}(Q)$, $(n+2<\lambda<\infty)$ of the system (2) are known (1, Theorem 9.1, pp. 341-342). Further, it follows easily from (1, the estimate (9.3) of Theorem 9.1, p. 342 and the first estimate on the page 343) that the solution of the system (2) satisfies two well-known a priori estimates given below in Theorem 2.2.

THEOREM 2.2. – Consider the system (2). If the assumption (A′) is satisfied, then there exists a unique solution $\Phi \in W_\lambda^{2,1}(Q)$. Further, Φ satisfies

$$\|\Phi\|_{\lambda,Q}^{(2)} \leqslant M_1 \left[\left\{ \iint_Q |f(x,t)|^\lambda \, dx\, dt \right\}^{1/\lambda} + \|\Phi_0\|_{\lambda,\Omega}^{(2)} \right]; \tag{3}$$

and

$$|\Phi|_Q^{1+\mu} \leqslant M_2 \left[\left\{ \iint_Q |f(x,t)|^\lambda \, dx\, dt \right\}^{1/\lambda} + \|\Phi_0\|_{\lambda,\Omega}^{(2)} \right], \tag{4}$$

for all $\lambda > n+2$ and $\mu = 1-(n+2)/\lambda$, where, if Q and $\partial\Omega$ are regarded as given, the constant M_1 depends only on the structure of the differential equation of the system (2) whereas the constant M_2 depends on M_1 and λ.

PROOF. – The proof of the theorem follows from [1, the estimate 9.3 of Theorem 9.1, p. 342 and the first estimate on page 343].

With the help of Theorem 2.2, the existence and uniqueness of solution of the system (1) together with two a priori estimates are proved in (5, Theorem 2.3). For convenience in further references, these results are quoted without proof in the following theorem.

THEOREM 2.3. – Under the assumption (A), the system (1) admits, for each $u \in D$, a unique solution $\varphi(u)$ satisfying the following estimates:

$$\|\varphi(u)\|_{\lambda,Q}^{(2)} \leqslant N_1; \tag{5}$$

and

$$|\varphi(u)|_Q^{1+\mu} \leqslant N_2 \tag{6}$$

for all $\lambda > n+2$ and $\mu = 1-(n+2)/\lambda$, where the constants N_1 and N_2 depend only on n, λ, Q, $\partial\Omega$, the structure of the differential equation of the system (1), the bounds for the functions $f_\varkappa$, $(\varkappa = 0, 1, \dots, \nu)$,

$$|\varphi_0|^{(0)}_{\bar\Omega\times[-h_\nu,0]}, \qquad \left|\frac{\partial\varphi_0}{\partial x_i}\right|^{(0)}_{\bar\Omega\times[-h_\nu,0]}, \qquad (i=1,\dots,n) \text{ and } \|\varphi_0(\cdot, 0)\|^{(2)}_{\lambda,\Omega}.$$

3. - Certain preparatory results.

Consider the following sequence of first boundary value problems:

$$\begin{cases} L^\sigma\cdot\Phi^\sigma(x,t) = f^\sigma(x,t)\,, & (x,t)\in Q \\ \Phi^\sigma(x,0) = \Phi_0^\sigma(x)\,, & x\in\Omega \\ \Phi^\sigma(x,t) = 0\,, & (x,t)\in\partial\Omega\times[0,T]\,, \end{cases} \tag{7}$$

where the operator L^σ is defined by

$$L^\sigma\psi(x,t) = \frac{\partial\psi(x,t)}{\partial t} - \left\{\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial^2\psi(x,t)}{\partial x_i\partial x_j} + \sum_{i=1}^n b_i^\sigma(x,t)\frac{\psi(x,t)}{\partial x_i} + c^\sigma(x,t)\,\psi(x,t)\right\}.$$

For each positive integer σ, let Φ^σ denote the corresponding solution of the system (7).

In the sequel, we need

THEOREM 3.1. - Consider the system (7). Suppose that their coefficients and data satisfy the assumption (A′) uniformly with respect to σ and that b_i^σ, $(i=1,\dots,n)$, c^σ and f^σ converge, respectively, to b_i^*, $(i=1,\dots,n)$, c^* and f^* in the weak $*$ topology of $L^\infty(Q)$. Further, it is assumed that

$$\Phi_0^\sigma \to \Phi_0^*$$

in the norm topology of $L^2(\Omega)$, where $\Phi_0^*\in W_\lambda^2(\Omega)$, $\Phi_0^*(x)=0$ on $\partial\Omega$, and $n+2<\lambda<\infty$. Then there exists a subsequence of $\{\Phi^\sigma\}_{\sigma=1}^\infty$ (which is denoted by the original sequence) so that

$$\Phi^\sigma,\ \Phi^\sigma_{x_i} \to \Phi^*,\ \Phi^*_{x_i} \text{ uniformly on } \bar Q\,, \qquad (i=1,\dots,n)\,,$$

and

$$\Phi^\sigma_t,\ \Phi^\sigma_{x_ix_j} \to \Phi^*_t,\ \Phi^*_{x_ix_j} \text{ weakly in } L^\lambda(Q)\,, \qquad (i,j=1,\dots,n)\,,$$

where $\Phi^* \in W_\lambda^{2,1}(Q)$ is the solution of the system

$$
(8) \qquad \begin{cases} \Phi_t^*(x,t) - \Big\{ \sum_{i,j=1}^n a_{ij}(x,t)\,\Phi^*_{x_ix_j}(x,t) + \sum_{i=1}^n b_i^*(x,t)\,\Phi^*_{x_i}(x,t) + \\ \qquad + c^*(x,t)\,\Phi^*(x,t)\Big\} = f^*(x,t)\,, & (x,t)\in Q \\ \Phi^*(x,0) = \Phi_0^*(x)\,, & x\in\Omega \\ \Phi^*(x,t) = 0\,, & (x,t)\in\partial\Omega\times[0,T]\,. \end{cases}
$$

Proof. – By hypotheses, the coefficients and data of the systems (7) satisfy the assumption (A′) uniformly with respect to σ. Thus, it follows from Theorem 2.2 that, for each positive integer σ, the system (7), admits a unique solution Φ^σ satisfying the following estimates.

$$
(9) \qquad \|\Phi^\sigma\|^{(2)}_{\lambda,Q} \leqslant M_1\{\|f^\sigma\|_{\lambda,Q} + \|\Phi_0^\sigma\|^{(2)}_{\lambda,\Omega}\}
$$

and

$$
(10) \qquad |\Phi^\sigma|_{\bar{Q}}^{1+\mu} \leqslant M_2\{\|f^\sigma\|_{\lambda,Q} + \|\Phi_0^\sigma\|^{(2)}_{\lambda,\Omega}\}\,,
$$

where $n+2<\lambda<\infty$, $\mu = 1-(n+2)/\lambda$, and M_1, M_2 are positive constants independent of σ.

Clearly, the estimate (10) implies that for $0<\mu<1$ and $\zeta^\sigma = \Phi^\sigma$ or $\Phi^\sigma_{x_i}$, $(i=1,\dots,n)$,

$$
(11) \qquad |\zeta^\sigma(x,t)| + \frac{|\zeta^\sigma(x,t)-\zeta^\sigma(x',t')|}{[|x-x'|^2+|t-t'|]^{\mu/2}} \leqslant M_2' \qquad \text{for all } (x,t),\ x',t')\in\bar{Q},
$$

where the positive constant M_2' is independent of σ.

Thus, it follows from Ascoli-Arzela Theorem that we can extract a system of subsequences $\{\Phi^{\sigma_\varkappa}, \Phi^{\sigma_\varkappa}_{x_i};\ i=1,\dots,n\} \subset \{\Phi^\sigma, \Phi^\sigma_{x_i};\ i=1,\dots,n\}$ so that

$$
\Phi^{\sigma_\varkappa} \to \hat{\Phi} \qquad \text{uniformly on } \bar{Q}\,,
$$

and

$$
\Phi^{\sigma_\varkappa}_{x_i} \to \hat{\Phi}^i\,, \qquad (i=1,\dots,n), \text{ uniformly on } \bar{Q}\,,
$$

as $\varkappa \to \infty$.

In particular, they converge to their corresponding limiting functions weakly in $L^\lambda(Q)$, $n+2<\lambda<\infty$. Further, it is clear that the limiting functions also satisfy the estimate (11). Later, we will show that $\hat{\Phi}^i = \hat{\Phi}_{x_i}$, $(i=1,\dots,n)$, and $\hat{\Phi} \equiv \Phi^*$ on $\bar{Q}$, where Φ^* is the solution of the system (8).

Using the properties of the functions Φ_0^σ and f, it is easily seen that the estimate (9) implies that

$$\left\{\iint_Q |\Phi_t^\sigma(x,t)|^\lambda\, dx\, dt\right\}^{1/\lambda} + \sum_{i,j=1}^{n} \left\{\iint_Q |\Phi_{x_i x_j}^\sigma(x,t)|^\lambda\, dx\, dt\right\}^{1/\lambda} \leqslant M_1', \tag{12}$$

where the positive constant M_1' is independent of σ.

Obviously, $\{\Phi_t^{\sigma_\varkappa}, \Phi_{x_i x_j}^{\sigma_\varkappa}; i, j = 1, \ldots, n\} \subset \{\Phi_t^\sigma, \Phi_{x_i x_j}^\sigma; i, j = 1, \ldots, n\}$ also satisfy the estimate (12). Thus, it is a system of bounded sequences in $L^\lambda(Q)$, which is a reflexive Banach space as $n+2<\lambda<\infty$, and consequently there exists a system of subsequences $\{\Phi_t^{\sigma_{\varkappa_\iota}}, \Phi_{x_i x_j}^{\sigma_{\varkappa_\iota}}; i, j = 1, \ldots, n\}$ and a system of functions $\{\psi, \psi^{ij}; i, j = 1, \ldots, n\} \in L^\lambda(Q)$ so that

$$\Phi_t^{\sigma_{\varkappa_\iota}} \xrightarrow{W} \psi \qquad (W \equiv \text{weakly in } L^\lambda(Q))$$
$$\Phi_{x_i x_j}^{\sigma_{\varkappa_\iota}} \xrightarrow{W} \psi^{ij}$$

as $\iota \to \infty$.

Recall that the limiting functions $\hat{\Phi}, \hat{\Phi}^i$, $(i = 1, \ldots, n)$, also satisfy the estimate (11). Further, for any $z \in C_0^1(Q)$,

$$\iint_Q \Phi_{x_i}^{\sigma_{\varkappa_\iota}}(x,t)\, z(x,t)\, dx\, dt = -\iint_Q \Phi^{\sigma_{\varkappa_\iota}}(x,t)\, z_{x_i}(x,t)\, dx\, dt, \qquad i = 1, \ldots, n,$$
$$\iint_Q \Phi_t^{\sigma_{\varkappa_\iota}}(x,t)\, z(x,t)\, dx\, dt = -\iint_Q \Phi^{\sigma_{\varkappa_\iota}}(x,t)\, z_t(x,t)\, dx\, dt,$$
$$\iint_Q \Phi_{x_i x_j}^{\sigma_{\varkappa_\iota}}(x,t)\, z(x,t)\, dx\, dt = -\iint_Q \Phi_{x_i}^{\sigma_{\varkappa_\iota}}(x,t)\, z_{x_j}(x,t)\, dx\, dt, \qquad i, j = 1, \ldots, n.$$

Since $\{\Phi^{\sigma_{\varkappa_\iota}}, \Phi_{x_i}^{\sigma_{\varkappa_\iota}}; i = 1, \ldots, n\}$ is a subsequence of $\{\Phi^{\sigma_\varkappa}, \Phi_{x_i}^{\sigma_\varkappa}; i = 1, \ldots, n\}$ and since $\Phi^{\sigma_\varkappa}$, $\Phi_{x_i}^{\sigma_\varkappa}$, $(i = 1, \ldots, n)$, converge, respectively, to $\hat{\Phi}, \hat{\Phi}^i$, $(i = 1, \ldots, n)$, uniformly on $\bar{Q}$, it is clear that $\Phi^{\sigma_{\varkappa_\iota}}, \Phi_{x_i}^{\sigma_{\varkappa_\iota}}$, $(i = 1, \ldots, n)$, also converge, respectively, to $\hat{\Phi}, \hat{\Phi}^i$, $(i = 1, \ldots, n)$, uniformly on $\bar{Q}$.

Thus, it is easily verified that $\hat{\Phi}^i = \hat{\Phi}_{x_i}$, $(i = 1, \ldots, n)$, almost everywhere on Q and that ψ and ψ^{ij} are, respectively, the generalized derivative of $\hat{\Phi}$ with respect to t and the generalized derivative of $\hat{\Phi}^i$ with respect to x_j. We will write $\hat{\Phi}_{x_i}$, $\hat{\Phi}_t$ and $\hat{\Phi}_{x_i x_j}$ instead of $\hat{\Phi}^i$, ψ and ψ^{ij} respectively.

In summary, we have already shown that

$$\Phi^{\sigma_{\varkappa_\iota}} \longrightarrow \hat{\Phi}, \qquad \text{uniformly on } \bar{Q}$$
$$\Phi_{x_i}^{\sigma_{\varkappa_\iota}} \longrightarrow \hat{\Phi}_{x_i}, \qquad (i = 1, \ldots, n), \text{ uniformly on } \bar{Q}$$
$$\Phi_t^{\sigma_{\varkappa_\iota}} \xrightarrow{W} \hat{\Phi}_t, \qquad (W \equiv \text{weakly in } L^\lambda(Q))$$
$$\Phi_{x_i x_j}^{\sigma_{\varkappa_\iota}} \xrightarrow{W} \hat{\Phi}_{x_i x_j}, \qquad (i, j = 1, \ldots, n)$$

where $\hat{\Phi}$ satisfies the stimates (11) and (12).

On the other hand, we note that the coefficients and data of the system (8) satisfy the hypothesis (A′). Thus, it follows from Theorem 2.2 that $\Phi^* \in W_\lambda^{2,1}(Q)$ is the *only* solution. Therefore, it remains to show that $\hat{\Phi} \in W_\lambda^{2,1}(Q)$ is also a solution of the system (8). For then we can conclude immediately that $\hat{\Phi} \equiv \Phi^*$ and the proof is thus complete. In order to do so, let us first show that $\hat{\Phi}$ satisfies the differential equation of the system (7). For this, we write

$$\begin{aligned}(13)\qquad &\hat{\Phi}_t - \Big\{\sum_{i,j=1}^n a_{ij}\hat{\Phi}_{x_i x_j} + \sum_{i=1}^n b_i^* \hat{\Phi}_{x_i} + c^*\hat{\Phi}\Big\} - f^* = \\ &= \hat{\Phi}_t - \Big\{\sum_{i,j=1}^n a_{ij}\hat{\Phi}_{x_i x_j} + \sum_{i=1}^n b_i^* \hat{\Phi}_{x_i} + c^*\hat{\Phi}\Big\} - f^* - \\ &+ \Phi_t^{\sigma_\varkappa} + \Big\{\sum_{i,j=1}^n a_{ij}\Phi_{x_i x_j}^{\sigma_{\varkappa_l}} + \sum_{i=1}^n b_i^{\sigma_{\varkappa_l}}\Phi^{\sigma_{\varkappa_l}} + c^{\sigma_{\varkappa_l}}\Phi^{\sigma_{\varkappa_l}}\Big\} + f^{\sigma_{\varkappa_l}} = (\hat{\Phi}_t - \Phi_t^{\sigma_{\varkappa_l}}) - \\ &- \Big\{\sum_{i,j=1}^n a_{ij}(\hat{\Phi}_{x_i x_j} - \Phi_{x_i x_j}^{\sigma_{\varkappa_l}}) + \sum_{i=1}^n (b_i^*\hat{\Phi}_{x_i} - b_i^{\sigma_{\varkappa_l}}\Phi_{x_i}^{\sigma_{\varkappa_l}}) + (c^*\hat{\Phi} - c^{\sigma_\varkappa}\Phi^{\sigma_{\varkappa_l}})\Big\} - (f^* - f^{\sigma_{\varkappa_l}})\end{aligned}$$

almost everywhere on Q.

Recall that both $\Phi^{\sigma_{\varkappa_l}}$ and $\hat{\Phi}$ satisfy the estimates (11) and (12). Thus, in particular, $\Phi^{\sigma_{\varkappa_l}}$, $\Phi_i^{\sigma_{\varkappa_l}} \Phi_{x_i}^{\sigma_{\varkappa_l}}$, $\Phi_{x_i x_j}^{\sigma_{\varkappa_l}}$, $(i, j = 1, \ldots, n)$, $\hat{\Phi}$, $\hat{\Phi}_t$, $\hat{\Phi}_{x_i}$, $\Phi_{x_i x_j}$, $(i, j = 1, \ldots, n)$, are in $L^\lambda(Q)$. Further, by the assumptions on the coefficients and data of the systems (7) and (8), we note that a_{ij}, b_i^*, $b_i^{\sigma_{\varkappa_l}}$, $(i, j = 1, \ldots, n)$, c^*, $c^{\sigma_{\varkappa_l}}$, f^* and $f^{\sigma_{\varkappa_l}}$ are bounded measurable on $\bar{Q}$. Moreover, $|Q| < \infty$. Thus, it can be easily verified that

$$\Big[\hat{\Phi}_t - \Big\{\sum_{i,j=1}^n a_{ij}\hat{\Phi}_{x_i x_j} + \sum_{i=1}^n b_i^*\hat{\Phi}_{x_i} + c^*\hat{\Phi}\Big\} - f^*\Big]$$

and

$$\Big[(\hat{\Phi}_t - \Phi_t^{\sigma_{\varkappa_l}}) - \Big\{\sum_{i,j=1}^n a_{ij}(\hat{\Phi}_{x_i x_j} - \Phi_{x_i x_j}^{\sigma_{\varkappa_l}}) + \sum_{i=1}^n (b_i^*\hat{\Phi}_{x_i} - b_i^{\sigma_{\varkappa_l}}\Phi^{\sigma_{\varkappa_l}}) + (c^*\hat{\Phi} - c^{\sigma_{\varkappa_l}}\Phi^{\sigma_{\varkappa_l}})\Big\} - (f^* - f_f^{\sigma_{\varkappa_l}})\Big]$$

are in $L^\lambda(Q)$, $(n + 2 < \lambda < \infty)$.

Then, if λ' is such that $1/\lambda + 1/\lambda' = 1$, it follows from the expression (13) that for all $z \in L^{\lambda'}(Q)$,

$$\begin{aligned}(14)\qquad &\iint_Q \Big[\hat{\Phi}_t - \Big\{\sum_{i,j=1}^n a_{ij}\hat{\Phi}_{x_i x_j} + \sum_{i=1}^n b_i^*\hat{\Phi}_{x_i} + c^*\hat{\Phi}\Big\} - f^*\Big] z\,dx\,dt = \\ &= \iint_Q \Big[(\hat{\Phi}_t - \Phi_t^{\sigma_{\varkappa_l}})z - \Big\{\sum_{i,j=1}^n a_{ij}(\hat{\Phi}_{x_i x_j} - \Phi_{x_i x_j}^{\sigma_{\varkappa_l}})z + \sum_{i=1}^n (b_i^*\hat{\Phi}_{x_i} - b_i^{\sigma_{\varkappa_l}}\Phi_{x_i}^{\sigma_{\varkappa_l}})z + \\ &+ (c^*\hat{\Phi} - c^{\sigma_{\varkappa_l}}\Phi^{\sigma_{\varkappa_l}})z\Big\} - (f^* - f^{\sigma_{\varkappa_l}})z\Big]\,dx\,dt\,.\end{aligned}$$

However, as it is shown above, we have

$$\Phi^{\sigma_{\varkappa_\iota}},\ \Phi^{\sigma_{\varkappa_\iota}}_{x_i} \to \hat{\Phi},\ \hat{\Phi}_{x_i}, \qquad i = 1, \ldots, n ,$$

uniformly on $\bar{Q}$ and, in particular, weakly in $L^\lambda(Q)$; and

$$\Phi^{\sigma_{\varkappa_\iota}}_t,\ \Phi^{\sigma_{\varkappa_\iota}}_{x_i x_j} \to \hat{\Phi}_t,\ \hat{\Phi}_{x_i x_j}, \qquad i, j = 1, \ldots, n ,$$

weakly in $L^\lambda(Q)$. Further, by hypotheses, $b_i^{\sigma_{\varkappa_\iota}}$, $(i = 1, \ldots, n)$, $c^{\sigma_{\varkappa_\iota}}$, and $f^{\sigma_{\varkappa_\iota}}$ converge respectively, to b_i^*, $(i = 1, \ldots, n)$, c^* and f^* in the weak $*$ topology of $L^\infty(Q)$.

Using these facts and the assumptions of the coefficients and forcing terms of the systems (7) and (8), we can easily show that the expression on the right hand side of (14) converges to zero as $\iota \to \infty$.

Thus, we have

$$\int_Q \Big[\hat{\Phi}_t - \Big\{\sum_{i,j=1}^n a_{ij}\hat{\Phi}_{x_i x_j} + \sum_{i=1}^n b_i^*\hat{\Phi}_{x_i} + c^*\hat{\Phi}\Big\} - f^*\Big] z\, dx\, dt = 0$$

for all $z \in L^{\lambda'}(Q)$ and, in particular, for all $z \in C(\bar{Q})$.

Since, as shown above,

$$\Big[\hat{\Phi}_t - \Big\{\sum_{i,j=1}^n a_{ij}\hat{\Phi}_{x_i x_j} + \sum_{i=1}^n b_i^*\hat{\Phi}_{x_i} + c^*\hat{\Phi}\Big\} - f^*\Big]$$

is an $L^\lambda(Q)$ function, it is easily shown that $\hat{\Phi}$ satisfies the differential equation of the system (8) almost everywhere on Q.

Next, we shall show that $\hat{\Phi}$ satisfies the initial and the boundary conditions of the system (8). For this, we first recall that $\Phi^{\sigma_{\varkappa_\iota}} \to \hat{\Phi}$ uniformly on $\bar{Q}$ as $\iota \to \infty$ and $\Phi^{\sigma_\varkappa}(x, t) = 0$ on $\partial\Omega \times [0, T]$. Thus, it is clear that $\hat{\Phi}(x, t) = 0$ on $\partial\Omega \times [0, T]$, This implies that $\hat{\Phi}$ satisfies the boundary condition of the system (7). To complete the proof, it remains to show that $\hat{\Phi}(x, 0) = \Phi_0^*(x)$ on Ω. For this, let us consider the following system

$$(16)\qquad \begin{cases} \Psi_t(x, t) - \Big\{\displaystyle\sum_{i,j=1}^n a_{ij}(x, t)\Psi_{x_i x_j}(x, t) + \sum_{i=1}^n b_i^*(x, t)\Psi_{x_i}(x, t) + \\ \qquad + c^*(x, t)\Psi(x, t)\Big\} = 0\ , \qquad (x, t) \in Q\ , \\ \Psi(x, 0) = \Phi_0^*(x) - \hat{\Phi}(x, 0)\ , \qquad x \in \Omega\ , \\ \Psi(x, t) = 0\ , \qquad (x, t) \in \partial\Omega \times [0, T]\ . \end{cases}$$

Denoting $\sum_{j=1}^{n}(\partial a_{ij}(x,t))/\partial x_j - b_i^*(x,t)$ by $a_i^*(x,t)$, we can write the system (16) as

$$(16')\quad \begin{cases} \Psi_t - \left\{\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} a_{ij}(x,t)\Psi_{x_i}(x,t)\right) - \sum_{i=1}^{n} a_i^*(x,t)\Psi_{x_i}(x,t) + c^*(x,t)\Psi(x,t)\right\} = 0\,, & (x,t)\in Q \\ \Psi(x,0) = \Phi_0^*(x) - \hat{\Phi}(x,0)\,, & x\in\Omega \\ \Psi(x,t) = 0\,, & (x,t)\in\partial\Omega\times[0,T]. \end{cases}$$

Note that $\Psi \triangleq \Phi^* - \hat{\Phi}$ is the unique solution of the system (16) and hence the system (16′). Thus, it is also a global weak solution of the system (16′) in the sense of Aronson ([3], p. 622). Therefore, by letting $\zeta = 1, s = 0$ and $\mu = \infty$ in ([3], Lemma 1, p. 623), we deduce that

$$(17)\quad \left\{\operatorname*{ess\cdot sup}_{t\in[0,T]}\left(\int_\Omega |\Psi(x,t)|^2 dx\right)^{\frac12}\right\}^2 + \left\{\iint_Q \sum_{i=1}^{n}\left|\frac{\partial\Psi(x,t)}{\partial x_i}\right|^2 dx\,dt\right\} \leqslant e^{\beta T} M_3\left\{\int_\Omega (\Phi(x) - \hat{\Phi}(x,0))^2\right\} dx\,,$$

where the constant β depends at most on n and the structure of the system (16′) while the constant M_3 depends at most on n, T and the structutfe of the system (16′).

However, we note that

$$(18)\quad \iint_Q |\Psi(x,t)|^2 dx\,dt = \int_0^T \left\{\left(\int_\Omega |\Psi(x,t)|^2 dx\right)^{\frac12}\right\}^2 dt \leqslant \int_0^T \operatorname*{ess\cdot sup}_{t\in[0,T]}\left(\int_\Omega |\Psi(x,t)|^2 dx\right)^{\frac12}\Bigg\}^2 dt = \left\{\operatorname*{ess\cdot sup}_{t\in[0,T]}\left(\int_\Omega |\Psi(x,t)|^2 dx\right)^{\frac12}\right\}^2 T\,.$$

Thus, it follows from the estimates (17) and (18) that

$$(19)\quad \iint_Q |\Psi(x,t)|^2 dx\,dt \leqslant M_4\left\{\int_\Omega (\Phi_0^*(x) - \hat{\Phi}(x,0))^2 dx\right\}.$$

Further, the estimate (19) can be reduced to

$$(20)\quad \iint_0 |\Psi(x,t)|^2 dx\,dt \leqslant 2M_4\left\{\int_\Omega (\Phi_0^{\sigma_{\varkappa_l}}(x) - \Phi_0^*(x))^2 dx + \int_\Omega (\Phi_0^{\sigma_{\varkappa_l}}(x) - \hat{\Phi}(x,0))^2 dx\right\}.$$

By the hypothesis, $\Phi_0^{\sigma_{\varkappa_l}} \to \Phi_0^*$ in $L^2(\Omega)$. On the other hand, as shown before, $\Phi^{\sigma_{\varkappa_l}}$ converges to $\hat{\Phi}$ uniformly on $\bar{Q}$. Further, $|Q| < \infty$. Thus, it follows easily from the Lebesgue Bounded Convergence Theorem that

$$\lim_{\iota\to\infty}\int_{\Omega}(\Phi_0^{\sigma_{\varkappa_\iota}}(x)-\Psi(x,0))^2\,dx=0\,.$$

Using these facts and noting that Ψ is differentiable on $\bar{Q}$, we obtain easily shat $\Psi(x,t)\equiv 0$ on $\bar{Q}$. This implies that $\Phi^*(x,t)\equiv\hat{\Phi}(x,t)$ on $\bar{Q}$, and in particular, $\Phi_0^*(x)=\hat{\Phi}(x,0)$ on Ω. Thus, the proof is complete.

THEOREM 3.2. – Let $\{u_l\}_{l=1}^{\infty}\subset D$ and let, for each positive integer l, $\varphi(u_l)$ be the corresponding solution of the system (1). Suppose that the assumption (A) is satisfied. Then, there exists a subsequence $\{u_{l_\iota}\}\subset\{u_l\}$ so that $b_{i,\varkappa}(\cdot,\cdot-h_\varkappa,u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $c_\varkappa(\cdot,\cdot-h_\varkappa,u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $f_\varkappa(\cdot,\cdot-h_\varkappa,u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $(i=1,\dots,n;\varkappa=0,1,\dots,\nu)$, converge, respectively, to $b_{i,\varkappa}^*(\cdot,\cdot-h_\varkappa)$, $c_\varkappa^*(\cdot,\cdot-h_\varkappa)$, $f_\varkappa^*(\cdot,\cdot-h_\varkappa)$, $(i=1,\dots,n;\varkappa=0,1,\dots,\nu)$ in the weak $*$ topology of $L^\infty(Q)$. Further,

$$\left.\begin{array}{ll}\varphi(u_{l_\iota})\to\varphi^* & \\ \varphi_{x_i}(u_{l_\iota})\to\varphi_{x_i}^*\,, & i=1,\dots,n\end{array}\right\}\quad\text{uniformly on }\bar{Q}$$

$$\left.\begin{array}{ll}\varphi_t(u_{l_\iota})\to\varphi_t^* & \\ \varphi_{x_ix_j}(u_{l_\iota})\to\varphi_{x_ix_j}^*, & i,j=1,\dots,n\end{array}\right\}\quad\text{weakly in }L^\lambda(Q)$$

as $\iota\to\infty$, where $\lambda>n+2$ and $\varphi^*\in W_\lambda^{2,1}(Q)$ is the unique solution of the system

$$(21)\qquad\begin{cases}\varphi_t^*(x,t)=\sum_{i,j=1}^{n}a_{ij}(x,t)\varphi_{x_ix_j}^*(x,t)+\sum_{\varkappa=0}^{\nu}\sum_{i=1}^{n}b_{i,\varkappa}^*(x,t-h_\varkappa)\varphi_{x_i}(x,t-h_\varkappa)+\\ \qquad+\sum_{\varkappa=0}^{\nu}c_\varkappa^*(x,t-h_\varkappa)\varphi^*(x,t-h_\varkappa)+\sum_{\varkappa=0}^{\nu}f_\varkappa^*(x,t-h_\varkappa)\,, & (x,t)\in\Omega\times I_1\,,\\ \varphi^*(x,t)=\varphi_0(x,t)\,, & (x,t)\in\Omega\times I_0\,,\\ \varphi^*(x,t)=0\,, & (x,t)\in\partial\Omega\times I_2\,.\end{cases}$$

REMARK 3.3. – By the definition of the class of admissible controls, D, we can easily show that, for almost all $(x,t)\in\Omega\times[-h_\varkappa,0)$,

$$b_{i,\varkappa}^*(x,t)=b_{i,\varkappa}(x,t,\hat{u}(x,t))\,,\qquad c_\varkappa^*(x,t)=c_\varkappa(x,t,\hat{u}(x,t))\,,$$

and

$$f_\varkappa^*(x,t)=f_\varkappa(x,t,\hat{u}(x,t))\,,$$

where $\varkappa=1,\dots,\nu$; $i=1,\dots,n$; and $\hat{u}$ is as defined in the definition of the class of admissible controls, D.

Proof of Theorem 3.2. – By the assumption (A), we note that the system of sequences $\{b_{i,\varkappa}(\cdot,\cdot-h_\varkappa, u_l(\cdot,\cdot-h_\varkappa)), c_\varkappa(\cdot,\cdot-h_\varkappa, u_l(\cdot,\cdot-h_\varkappa)), f_\varkappa(\cdot,\cdot-h_\varkappa, u_l(\cdot,\cdot-h_\varkappa))$ $i=1,\dots,n;\ \varkappa=0,1,\dots,\nu\}_{l=1}^{\infty}$ are bounded uniformly on $\bar{Q}$. Thus, it is compact in the weak $*$ topology of $L^\infty(Q)$. Therefore, there exists a system of subsequences $\{b_{i,\varkappa}(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa)),\ c_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa)),\ f_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa)):$ $i=1,\dots,n;\ \varkappa=0,1,\dots,\nu\}$ and a system of functions $\{b^*_{i,\varkappa}(\cdot,\cdot-h_\varkappa), c^*_\varkappa(\cdot,\cdot-h_\varkappa),$ $f^*_\varkappa(\cdot,\cdot-h_\varkappa): i=1,\dots,n;\ \varkappa=0,1,\dots,\nu\}$ so that

$$b_{i,\varkappa}(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))\,, \quad c_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))\,, \quad f_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))\,,$$
$$(i=1,\dots,;\ \varkappa=0,1,\dots,\nu)\,,$$

converge, respectively, to $b^*_{i,\varkappa}(\cdot,\cdot-h_\varkappa)$, $c^*_\varkappa(\cdot,\cdot-h_\varkappa)$, $f^*_\varkappa(\cdot,\cdot-h_\varkappa)$, $(i=1,\dots,n;\ \varkappa=0,$ $1,\dots,\nu)$, in the weak $*$ topology of $L^\infty(Q)$, where the limiting functions are also bounded by the same bound for the original system of sequences of functions. In particular, $b_{i,\varkappa}(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $c_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $f_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $(i=1,\dots,n;\ \varkappa=0,1,\dots,\nu)$, converge, respectively, to

$$b^*_{i,\varkappa}(\cdot,\cdot-h_\varkappa)\,,\quad c^*_\varkappa(\cdot,\cdot-h_\varkappa)\,,\quad f^*_\varkappa(\cdot,\cdot-h_\varkappa)\,,\qquad (i=1,\dots,n;\ \varkappa=0,1,\dots,\nu)\,,$$

in the weak $*$ topology of $L^\infty(\hat{Q})$, where $\hat{Q}$ is any subset of Q. By Theorem 2.3, we note that the system (21) admits a unique solution satisfying the estimates (5) and (6). Let the solution be denoted by φ^*.

For convenience, let us write down the sequence of the following systems

$$(22)\quad \begin{cases} \dfrac{\partial\varphi(u_{l_\iota})(x,t)}{\partial t} - \displaystyle\sum_{i,j=1}^{n} a_{ij}(x,t)\,\frac{\partial^2\varphi(u_{l_\iota})(x,t)}{\partial x_i\,\partial x_j} - \sum_{\varkappa=0}^{\nu}\sum_{i=1}^{n} b_{i,\varkappa}(x,t-h_\varkappa, u_{l_\iota}(x,t-h_\varkappa)) \\ \cdot\dfrac{\partial\varphi(u_{l_\iota})(x,t-h_\varkappa)}{\partial x_i} - \displaystyle\sum_{\varkappa=0}^{\nu} c_\varkappa(x,t-h_\varkappa, u_{l_\iota}(x,t-h_\varkappa))\,\varphi(u_{l_\iota})(x,t-h_\varkappa) \\ = \displaystyle\sum_{\varkappa=0}^{\nu} f_\varkappa(x,t-h_\varkappa, u_{l_\iota}(x,t-h_\varkappa))\,, & (x,t)\in\Omega\times(0,T) \\ \varphi(u_{l_\iota})(x,t) = \varphi_0(x,t)\,, & (x,t)\in\Omega\times[-h_\nu,0] \\ \varphi(u_{l_\iota})(x,t) = 0\,, & (x,t)\in\partial\Omega\times[-h_\nu,T]. \end{cases}$$

We first consider the systems (22) on $\bar{\Omega}\times[0,h_1]$. Note that $\varphi(u_{l_\iota})(x,t)=\varphi_0(x,t)$ in $\Omega\times[-h_\nu,0]$ and $\varphi(u_{l_\iota})(x,t)=0$ in $\partial\Omega\times[-h_\nu,T]$, where ι is any positive integer and φ_0 is a given function satisfying the property (v) of (A). Thus, $\varphi(u_{l_\iota})(\cdot,\cdot-h_\nu)$, $\varphi_{x_i}(u_{l_\iota})(\cdot,\cdot-h_\varkappa)$, $(i=1,\dots,n;\ \varkappa=1,\dots,\nu)$, are known continuous functions in $\bar{\Omega}\times[0,h_1]$ and equal to zero for all $x\notin\bar{\Omega}_0$ and $t\in[0,h_1]$, where $\bar{\Omega}_0$ is a compact subset of Ω as defined in property (v) of (A). This implies that the sequence of the systems (22) in $\bar{\Omega}\times[0,h_1]$ is only a sequence of first boundary value problems without time delayed arguments, where its forcing term is given by

$$\sum_{\varkappa=1}^{\nu}\sum_{i=1}^{n} b_{i,\varkappa}(\cdot,\cdot-h_\varkappa,\hat{u}(\cdot,\cdot-h_\varkappa))\frac{\partial\varphi_0(\cdot,\cdot-h_\varkappa)}{\partial x_i} + \sum_{\varkappa=0}^{\nu} c_\varkappa(\cdot,\cdot-h_\varkappa,\hat{u}(\cdot,\cdot-h_\varkappa))\,\varphi_0(\cdot,\cdot-h_\varkappa) + \sum_{\varkappa=1}^{\nu} f_\varkappa(\cdot,\cdot-h_\varkappa,\hat{u}(\cdot,\cdot-h_\varkappa)) + f_0(\cdot,\cdot,u_{l_\iota}(\cdot,\cdot)) \triangleq \tilde{f}_{l_\iota}(\cdot,\cdot)\,.$$

Let

$$\tilde{f}^*(x,t) \triangleq \sum_{\varkappa=1}^{\nu}\sum_{i=1}^{n} b_{i,\varkappa}(x,t-h_\varkappa,\hat{u}(x,t-h_\varkappa))\frac{\partial\varphi_0(x,t-h_\varkappa)}{\partial x_i} + \sum_{\varkappa=1}^{\nu} c_\varkappa(x,t-h_\varkappa,\hat{u}(x,t-h_\varkappa))\,\varphi_0(x,t-h_\varkappa) + \sum_{\varkappa=1}^{\nu} f_\varkappa(x,t-h_\varkappa,\hat{u}(x,t-h_\varkappa)) + f_0^*(x,t)\,.$$

In view of the assumption (A), we can easily verify that $\tilde{f}^*$ and $\tilde{f}_{l_\iota}$ are bounded measurable on $\bar{Q}$. Furthermore, $\{\tilde{f}_{l_\iota}\}_{l_\iota=1}^{\infty}$ is bounded uniformly with respect to ι.

Since $b_{i,\varkappa}(\cdot,\cdot-h_\varkappa,\hat{u}(\cdot,\cdot-h_\varkappa))$, and $c_\varkappa(\cdot,\cdot-h_\varkappa,\hat{u}(\cdot,\cdot-h_\varkappa))$, and $f_\varkappa(\cdot,\cdot-h_\varkappa,\hat{u}(\cdot,\cdot-h_\varkappa))$ $(i=1,\dots,n;\ \varkappa=1,\dots,\nu)$, are independent of ι and $f_0(\cdot,\cdot,u_{l_\iota}(\cdot,\cdot))$ converges to $f_0^*(\cdot,\cdot)$ in the weak $*$ topology of $L^\infty(\Omega\times(0,h_1))$, it is clear that $\tilde{f}_{l_\iota}(\cdot,\cdot)$ also converge to $\tilde{f}^*(\cdot,\cdot)$ in the weak $*$ topology of $L^\infty(\Omega\times(0,h_1))$. Recall that $a_{ij}(\cdot,\cdot)$, $(i,j=1,\dots,n)$, are independent of ι and that $b_{i,0}(\cdot,\cdot,u_{l_\iota}(\cdot,\cdot))$, $(i=1,\dots,n)$, and $c_0(\cdot,\cdot,u_{l_\iota}(\cdot,\cdot))$ converge respectively, to $b_{i,0}^*(\cdot,\cdot)$, $(i=1,\dots,n)$, and $c_0^*(\cdot,\cdot)$ in the weak $*$ topology of $L^\infty(\Omega\times(0,h_1))$. Further, the system corresponding to these limiting functions is the system (21) in $\bar{\Omega}\times[0,h_1]$. Thus, it follows from Theorem 2.2 that the system (21) in $\bar{\Omega}\times[0,h_1]$ admits a unique solution $\tilde{\varphi}_1$ satisfying the estimates (3) and (4) with Q and $\bar{Q}$ replaced, respectively, by $\Omega\times(0,h_1)$ and $\bar{\Omega}\times[0,h_1]$. This, in turn, implies that $\tilde{\varphi}_1(x,t)\equiv\varphi^*(x,t)$ in $\bar{\Omega}\times[0,h_1]$. Now, by Theorem 3.1, we have

$$\left.\begin{array}{ll}\varphi(u_{l_\iota})\to\varphi^* & \\ \varphi_{x_i}(u_{l_\iota})\to\varphi^*_{x_i}\,, & (i=1,\dots,n)\end{array}\right\}\ \text{uniformly on } \bar{\Omega}\times[0,h_1]\,,$$

$$\left.\begin{array}{ll}\varphi_{x_ix_j}(u_{l_\iota})\to\varphi^*_{x_ix_j}\,, & (i,j=1,\dots,n)\\ \varphi_t(u_{l_\iota})\to\varphi^*_t & \end{array}\right\}\ \text{weakly in } L^\lambda(\Omega\times(0,h_1))\,,$$

as $\iota\to\infty$, where $n+2<\lambda<\infty$.

Further, we note that

$$\|\varphi^*\|^{(2)}_{\lambda,\Omega\times(0,h_1)}\leqslant M_1\{\|\tilde{f}^*\|_{\lambda,\Omega\times(0,h_1)}+\|\Phi_0\|^{(2)}_{\lambda,\Omega}\}$$

and

$$\|\Phi(u_{l_\iota})\|^{(2)}_{\lambda,\Omega\times(0,h_1)}\leqslant M_1\{\|\tilde{f}_{l_\iota}\|_{\lambda,\Omega\times(0,h_1)}+\|\Phi_0\|^{(2)}_{\lambda,\Omega}\}\,.$$

Since $\{\tilde{f}_{l_\iota}\}$ is bounded uniformly with respect to ι, f^* is a bounded measurable function on $\bar{\Omega}\times[0, h_1]$, and Φ_0 satisfies its corresponding condition of the assumption (A′), it follows readily that

$$\|\varphi^*\|^{(2)}_{\lambda,\Omega\times(0,h_1)}\leqslant M_4 \quad \text{and} \quad \|\varphi(u_{l_\iota})\|^{(2)}_{\lambda,\Omega\times(0,h_1)}\leqslant M_4$$

where the positive constant M_4 is independent of ι.

In particular, we have

$$\text{(23)}\qquad \Big\{\iint\limits_{\Omega\times(0,h_1)}|\hat{\varphi}(x,t)|^\lambda dx\,dt\Big\}^{1/\lambda}+\sum_{i=1}^{n}\Big\{\iint\limits_{\Omega\times(0,h_1)}\Big|\frac{\partial\hat{\varphi}(x,t)}{\partial x_i}\Big|^\lambda dx\,dt\Big\}^{1/\lambda} + \sum_{i,j=1}^{n}\Big\{\iint\limits_{\Omega\times(0,h_1)}\Big|\frac{\partial^2\hat{\varphi}(x,t)}{\partial x_i\partial x_j}\Big|^\lambda dx\,dt\Big\}^{1/\lambda}\leqslant M_4,$$

for all $\hat{\varphi}\in\{\varphi^*, \varphi(u_{l_\iota}), \iota=1,2,\ldots\}$, where $n+2<\lambda<\infty$ and the constant M_4 are independent of ι.

By the Hölder inequality for sum, the estimate (23) can be easily reduced to

$$\text{(24)}\qquad \int_0^{h_1}\Big\{\int_\Omega|\hat{\varphi}(x,t)|^\lambda dx+\sum_{i=1}^{n}\int_\Omega\Big|\frac{\partial\hat{\varphi}(x,t)}{\partial x_i}\Big|^\lambda dx + \sum_{i,j=1}^{n}\int_\Omega\Big|\frac{\partial^2\hat{\varphi}(x,t)}{\partial x_i\partial x_j}\Big|^\lambda dx\Big\}dt\leqslant(n^2+n+1)^{1/\lambda'}M_4^\lambda$$

where λ' is such that $1/\lambda+1/\lambda'=1$.

Now, we claim that there exists an $h_1'\in(h_1/2, h_1)$ so that

$$\text{(25)}\qquad \int_\Omega|\hat{\varphi}(x,h_1')|^\lambda dx+\sum_{i=1}^{n}\int_\Omega\Big|\frac{\partial\hat{\varphi}(x,h)}{\partial x_i}\Big|^\lambda dx + \sum_{i,j=1}^{n}\int_\Omega\Big|\frac{\partial^2\hat{\varphi}(x,h_1')}{\partial x_i\partial x_j}\Big|^\lambda dx\leqslant(n^2+n+1)^{1/\lambda'}\cdot M_4^\lambda\cdot\frac{2}{h_1}$$

for all $\hat{\varphi}\in\{\varphi^*, \varphi(u_{l_\iota}), \iota=1,2,\ldots\}$.

Indeed, suppose it were false. Then, there exists an $\hat{\varphi}\in\{\varphi^*, \varphi(u_{l_\iota}), \iota=1,2,\ldots\}$ so that

$$\int_0^{h_1}\Big\{\int_\Omega|\hat{\varphi}(x,t)|^\lambda dx+\sum_{i=1}^{n}\int_\Omega\Big|\frac{\partial\hat{\varphi}(x,t)}{\partial x_i}\Big|^\lambda dx+\sum_{i,j=1}^{n}\int_\Omega\Big|\frac{\partial^2\hat{\varphi}(x,t)}{\partial x_i\partial x_j}\Big|^\lambda dx\Big\}dt$$
$$\geqslant\int_{h_1/2}^{h_1}\Big\{\int_\Omega|\hat{\varphi}(x,t)|^\lambda dx+\sum_{i=1}^{n}\int_\Omega\Big|\frac{\partial\hat{\varphi}(x,t)}{\partial x_i}\Big|^\lambda dx+\sum_{i,j=1}^{n}\int_\Omega\Big|\frac{\partial\hat{\varphi}(x,t)}{\partial x_i\partial x_j}\Big|^\lambda dx\Big\}dt$$

$$> \int_{h_1/2}^{h_1} \left\{ (n^2 + n + 1)^{1/\lambda'} \cdot M_4^\lambda \cdot \frac{2}{h_1} \right\} dt$$

$$= (n^2 + n + 1)^{1/\lambda'} \cdot M_4^\lambda .$$

A contradiction to the estimate (24), and the estimate (25) is proved.

Next, by using the estimate (25), we obtain readily that

$$(26) \qquad \|\varphi^*(\cdot, h_1')\|_{\lambda,\Omega}^{(2)} \leqslant (n^2 + n + 1)^{1/\lambda' \cdot 1/\lambda} M_4 \left(\frac{2}{h_1}\right)^{1/\lambda} (n^2 + n + 1)$$

$$= (n^2 + n + 1)^{(\lambda^2 + \lambda - 1)/\lambda^2} \cdot M_4 \cdot \left(\frac{2}{h_1}\right)^{1/\lambda} \triangleq G_1 < \infty$$

and

$$(27) \qquad \|\varphi(u_{l_\iota})(\cdot, h_1')\|_{\lambda,\Omega}^{(2)} \leqslant G_1 < \infty$$

for all ι.

Now, let us consider the systems (21) and (22) on $\bar{\Omega} \times [h_1', h_1' + h_1]$. Note that the system (21) on $\bar{\Omega} \times [h_1', h_1' + h_1]$ is only a non-homogeneous first boundary value problem without time delayed argument, while the systems (22) on $\bar{\Omega} \times [h_1', h_1' + h_1]$ are a sequence of non-homogeneous first boundary value problems without time delayed argument. The forcing terms for the system (21) on $\bar{\Omega} \times [h_1', h_1' + h_1]$ and the systems (22) on $\bar{\Omega} \times [h_1', h_1' + h_1]$ are given, respectively, by

$$\sum_{\varkappa=1}^{\nu} \sum_{i=1}^{n} b_{i,\varkappa}^*(\cdot, \cdot - h_\varkappa) \varphi_{x_i}(\cdot, \cdot - h_\varkappa) + \sum_{\varkappa=1}^{\nu} c_\varkappa(\cdot, \cdot - h_\varkappa) \varphi^*(\cdot, \cdot - h_\varkappa) +$$
$$+ \sum_{\varkappa=1}^{\nu} f_\varkappa(\cdot, \cdot - h_\varkappa) + f_0^*(\cdot, \cdot) \triangleq \tilde{f}^2(\cdot, \cdot)$$

and

$$\sum_{\varkappa=1}^{\nu} \sum_{i=1}^{n} b_{i,\varkappa}(\cdot, \cdot - h_\varkappa, u_{l_\iota}(\cdot, \cdot - h_\varkappa)) \varphi_x(u_{l_{l_\iota}})(\cdot, \cdot - h_\varkappa) +$$
$$+ \sum_{\varkappa=1}^{\nu} c_\varkappa(\cdot, \cdot - h_\varkappa, u_{l_\iota}(\cdot, \cdot - h_\varkappa)) \varphi(u_{l_\iota})(\cdot, \cdot - h_\varkappa) +$$
$$+ \sum_{\varkappa=1}^{\nu} f_\varkappa(\cdot, \cdot - h_\varkappa, u_{l_\iota}(\cdot, \cdot - h_\varkappa)) + f_0(\cdot, \cdot, u_{l_\iota}(\cdot, \cdot)) \triangleq \tilde{f}_{l_\iota}^2(\cdot, \cdot) .$$

Note that $b_{i,\varkappa}(\cdot, \cdot - h_\varkappa, u_{l_\iota}(\cdot, \cdot - h_\varkappa))$, $(i = 1, \ldots, n)$, $c_\varkappa(\cdot, \cdot - h_\varkappa, u_{l_\iota}(\cdot, \cdot - h_\varkappa))$, $f_\varkappa(\cdot, \cdot - h_\varkappa, u_{l_\iota}(\cdot -, \cdot h_\varkappa))$, and $f_0(\cdot, \cdot u_{l_\iota}(\cdot, \cdot))$ converge, respectively, to $b_{x_i}^*(\cdot, \cdot - h_\varkappa)$, $(i = 1, \ldots, n)$, $c_\varkappa^*(\cdot, \cdot - h_\varkappa)$

$f_\varkappa^*(\cdot,\cdot-h_\varkappa)$, and $f_0^*(\cdot,\cdot)$ in the weak $*$ topology of $L^\infty(\Omega\times(h_1', h_1'+h_1))$ as $\iota\to\infty$. Further, as shown above, $\varphi_{x_i}(u_{l_\iota})$, $(i=1,\dots,n)$, and $\varphi(u_{l_\iota})$ converge uniformly on $\bar\Omega\times[0,h_1]$ to $\varphi_{x_i}^*$, $(i=1,\dots,n)$, and φ^* respectively. Thus, it follows easily that $\tilde f_{l_\iota}^2(\cdot,\cdot)$ converge to $\tilde f^2(\cdot,\cdot)$ in the weak $*$ topology of $L^\infty(\Omega\times(h_1',h_1'+h_1))$ as $\iota\to\infty$.

By hypotheses, $a_{ij}(\cdot,\cdot)$, $(i,j=1,\dots,n)$, are independent of ι. Further, $b_{i,0}(\cdot,\cdot, u_{l_\iota}(\cdot,\cdot))$, $(i=1,\dots,n)$, and $c_0(\cdot,\cdot,u_{l_\iota}(\cdot,\cdot))$ converge, respectively, to $b_{i,0}^*(\cdot,\cdot)$, $(i==1,\dots,n)$, and $c_0^*(\cdot,\cdot)$ in the weak $*$ topology of $L^\infty(\Omega\times(h_1',h_1'+h_1))$ as $\iota\to\infty$. Further, we note that, corresponding to each integer ι the unique solution $\varphi(u_{l_\iota})$ of the system (22) is also the unique solution of the system (22) on $\bar\Omega\times[h_1',h_1'+h_1]$. Similarly, the unique solution φ^* of the system (21) is also the unique solution of the system (21) on $\bar\Omega\times[h_1',h_1'+h_1]$. Moreover, $\varphi(u_{l_\iota})$ and φ^* satisfy their corresponding versions of the estimates (3) and (4) with Q and $\bar Q$ replaced, respectively, by $\Omega\times(h_1',h_1'+h_1)$ and $\bar\Omega\times[h_1',h_1'+h_1]$.

Thus, it follows from Theorem 3.1 that

$$\left.\begin{array}{ll}\varphi(u_{l_\iota})\to\varphi^* & \\ \varphi_{x_i}(u_{l_\iota})\to\varphi_{x_i}^*, & (i=1,\dots,n)\end{array}\right\}\ \text{uniformly on } \bar\Omega\times[h_1',h_1'+h_1]$$

$$\left.\begin{array}{ll}\varphi_{x_ix_j}(u_{l_\iota})\xrightarrow{W}\varphi_{x_ix_j}^*, & (i,j=1,\dots,n)\\ \varphi_t(u_{l_\iota})\xrightarrow{W}\varphi_t^* & \end{array}\right\}\ \begin{array}{l}W\equiv\text{weakly in}\\ L^\lambda(\Omega\times(h_1',h_1'+h_1),\ n+2<\lambda<\infty)\,,\end{array}$$

as $\iota\to\infty$.

By the same token, we consider the system (21) and the systems (22) in each of the domains $\bar\Omega\times[h_{k+1}',h_{k+1}'+h_1]$, $(k=1,\dots,\varrho)$, and $\bar\Omega\times[h_{\varrho+2}',T]$ successively, where h_{k+1}', $(k=1,\dots,\varrho+1)$, are chosen so that

$$h_{k+1}'\in\left(h_k'+\frac{h_1}{2},\,h_k'+h_1\right)\quad\text{and}\quad\|\varphi(u_{l_\iota})(\cdot,h_{k+1}')\|_{\lambda,\Omega}^{(2)}\,,\qquad(k=1,\dots,\varrho+1),$$

are bounded uniformly with respect to ι, while $\|\varphi^*(\cdot,h_{k+1}')\|_{\lambda,\Omega}^{(2)}$, $(k=1,\dots,\varrho+1)$, are also bounded. Their bounds can be obtained by using the argument similar to that for the estimate (27). Note that the integer ϱ is chosen so that $h_{\varrho+2}'+h_1\geqslant T$, Since T is finite, it is obvious that ϱ is also finite. Thus, it follows from the same argument that, for each $k\in\{1,\dots,\varrho\}$,

$$\left.\begin{array}{ll}\varphi(u_{l_\iota})\to\varphi^* & \\ \varphi_{x_i}(u_{l_\iota})\to\varphi_{x_i}^*, & (i=1,\dots,n)\end{array}\right\}\ \text{uniformly on } \bar\Omega\times[h_{k+1}',h_{k+1}'+h_1]\,,$$

$$\left.\begin{array}{ll}\varphi_{x_ix_j}(u_{l_\iota})\xrightarrow{W}\varphi_{x_ix_j}^*, & (i,j=1,\dots,n)\\ \varphi_t(u_{l_\iota})\xrightarrow{W}\varphi_t^* & \end{array}\right\}\ \begin{array}{l}W\equiv\text{weakly in}\\ L^\lambda(\Omega\times(h_{k+1}',h_{k+1}'+h_1)),\\ (n+2<\lambda<\infty)\,.\end{array}$$

as $\iota\to\infty$.

Finally, we again use the same approach to consider the system (21) and the system (22) in $\bar{\Omega}\times[h'_{\varrho+2}, T]$. Thus, by the same argument, we show that

$$\left.\begin{array}{ll}\varphi(u_{l_\iota}) \to \varphi^* & \\ \varphi_{x_i}(u_{l_\iota}) \to \varphi^*_{x_i}, & (i=1,\dots,n)\end{array}\right\}\text{ uniformly on } \bar{\Omega}\times[h'_{\varrho+2}, T]$$

$$\left.\begin{array}{ll}\varphi_{x_ix_j}(u_{l_\iota}) \xrightarrow{W} \varphi^*_{x_ix_j}, & (i,j=1,\dots,n)\\ \varphi_t(u_{l_\iota}) \xrightarrow{W} \varphi^*_t & \end{array}\right\}\begin{array}{l}W \equiv \text{weakly in}\\ \quad L^\lambda(\Omega\times(h'_{\varrho+2}, T)),\ (n+2<\lambda<\infty)\,.\end{array}$$

as $\iota\to\infty$.

Let $\zeta_\iota \triangleq \varphi(u_{l_\iota})$ or $\varphi_{x_i}(u_{l_\iota})$, $(i=1,\dots,n)$ and $\zeta^*\triangleq\varphi^*$ or $\varphi^*_{x_i}$, $(i=1,\dots,n)$. Then

$$|\zeta_\iota(x,t)-\zeta^*(x,t)| = \Big|\big(\zeta_\iota(x,t)-\zeta^*(x,t)\big)\chi_{[0,h'_1)}(t)+\sum_{\varkappa=1}^{\varrho+1}\big(\varrho_\iota(x,t)-\zeta^*(x,t)\big)\chi_{[h'_\varkappa,h'_{\varkappa+1})}(t)+$$
$$+\big(\varrho_\iota(x,t)-\zeta^*(x,t)\big)\chi_{[h'_{\varrho+2},T)}(t)\Big| \leqslant \big|\big(\zeta_\iota(x,t)-\zeta^*(x,t)\big)\chi_{[0,h'_1)}(t)\big|$$
$$+\sum_{\varkappa=1}^{\varrho+1}\big|\big(\zeta_\iota(x,t)-\zeta^*(x,t)\big)\chi_{[h'_\varkappa,h'_{\varkappa+1})}(t)\big|+\big|\big(\zeta_\iota(x,t)-\zeta^*(x,t)\big)\chi_{[h'_{\varrho+2},T)}(t)\big|\,,$$

where $\chi_E(\cdot)$ is the characteristic function of the measurable set E. Thus, it is clear that

$$|\zeta_\iota(x,t)-\zeta^*(x,t)|\leqslant \max_{\bar{\Omega}\times[0,h_1]}|\zeta_\iota(x,t)-\zeta^*(x,t)|+\sum_{\varkappa=1}^{\varrho+1}\max_{\bar{\Omega}\times[h'_\varkappa,h'_{\varkappa+1}]}|\zeta_\iota(x,t)-\zeta^*(x,t)|$$
$$+\max_{\bar{\Omega}\times[h'_{\varrho+2},T]}|\zeta_\iota(x,t)-\zeta^*(x,t)|\to 0$$

as $\iota\to\infty$.

This shows that $\varphi(u_{l_\iota})$ and $\varphi_{x_i}(u_{l_\iota})$, $(i=1,\dots,n)$, converge, respectively, to φ^* and $\varphi^*_{x_i}$, $(i=1,\dots,n)$, uniformly on $\bar{Q}$.

On the other hand, since $\varphi_{x_ix_j}(u_{l_\iota})$, $\varphi^*_{x_ix_j}$, $(i,j=1,\dots,n)$, $\varphi_t(u_{l_\iota})$ and φ^*_t are elements of $L^\lambda(Q)$, $(n+2<\lambda<\infty)$, it follows that if λ' is such that $1/\lambda+1/\lambda'=1$, then, for any $g\in L^{\lambda'}(Q)$,

$$[\varphi_{x_ix_j}(u_{l_\iota})(x,t)-\varphi^*_{x_ix_j}(x,t)]g(x,t)$$

and

$$[\varphi_t(u_{l_\iota})(x,t)-\varphi^*_t(x,t)]g(x,t)$$

are defined and belong to $L^1(Q)$. Thus,

$$\iint_Q\sum_{i,j=1}^n[\varphi_{x_ix_j}(u_{l_\iota})(x,t)-\varphi^*_{x_ix_j}(x,t)]g(x,t)\,dx\,dt$$
$$=\sum_{i,j=1}^n\int_0^{h'_1}\!\!\int[\varphi_{x_ix_j}(u_{l_\iota})(x,t)-\varphi^*_{x_ix_j}(x,t)]g(x,t)\,dx\,dt$$

$$+\sum_{\varkappa=1}^{\varrho+1}\sum_{i,j=1}^{n}\int_{h'_\varkappa}^{h'_{\varkappa+1}}\int_{\Omega}[\varphi_{x_ix_j}(u_{l_\iota})(x,t)-\varphi^*_{x_ix_j}(x,t)]\,g(x,t)\,dx\,dt$$

$$+\sum_{i,j=1}^{n}\int_{h'_{\varrho+2}}^{T}\int_{\Omega}[\varphi_{x_ix_j}(u_{l_\iota})(x,t)-\varphi^*_{x_ix_j}(x,t)]\,g(x,t)\,dx\,dt\to 0$$

as $\iota\to\infty$, and

$$\iint_Q[\varphi_t(u_{l_\iota})(x,t)-\varphi^*_t(x,t)]g(x,t)\,dx\,dt=\int_0^{h'_1}\int_\Omega[\varphi_t(u_{l_\iota})(x,t)-\varphi^*_t(x,t)]g(x,t)\,dx\,dt+$$

$$+\sum_{\varkappa=1}^{\varrho+1}\int_{h'_\varkappa}^{h'_{\varkappa+1}}\int_\Omega[\varphi_t(u_{l_\iota})(x,t)-\varphi^*_t(x,t)]g(x,t)\,dx\,dt+\int_{h'_{\varrho+2}}^{T}\int_\Omega[\varphi_t(u_{l_\iota})(x,t)-\varphi^*_t(x,t)]g(x,t)\,dx\,dt\to 0$$

as $l_\iota\to\infty$.

These two conclusions imply that $\varphi_{x_ix_j}(u_{l_\iota})$, $(i,j=1,\dots,n)$, and $\varphi_t(u_{l_\iota})$ converge, respectively, to $\varphi^*_{x_ix_j}$, $(i,j=1,\dots,n)$, and φ^*_t in the weak topology of $L^\lambda(Q)$ as $\iota\to\infty$, where $n+2<\lambda<\infty$. This completes the proof.

4. – Existence of optimal controls of the problem P.

In this section, we shall apply the results presented in sections 2 and 3 to prove the existence of an optimal control for the problem P. However, it may be noted that the results to be presented in Theorems 4.1 and 4.2 below only cover the case in which the first and zero order coefficients and the forcing term of the system of the problem P are linear in control variables.

THEOREM 4.1. – Consider the problem P and suppose the assumption (A) is satisfied. If $b_{i,\varkappa}, c_\varkappa, f_\varkappa$, $(i=1,\dots,n;\ \varkappa=0,1,\dots,\nu)$, are linear in u, then the problem P has a solution.

PROOF. – Let $\{u_l\}\subset D$ be a minimizing sequence for J. Since U is compact and convex, there exists a subsequence $\{u_{l_\iota}\}\subset\{u_l\}$ so that u_{l_ι} tends to a limit u^* in the weak $*$ topology of $L^\infty(Q)$, where $u^*(x,t)\in U$ almost everywhere on Q.

By hypotheses, the first and zero order coefficients and the forcing term are linear in u. Thus, $b_{i,\varkappa}(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $c_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, and $f_\varkappa(\cdot,\cdot-h_\varkappa, u_{l_\iota}(\cdot,\cdot-h_\varkappa))$, $(i=1,\dots,n;\ \varkappa=0,1,\dots,\nu)$, converge, respectively, to $b_{i,\varkappa}(\cdot,\cdot-h_\varkappa, u^*(\cdot,\cdot-h_\varkappa))$, $c_\varkappa(\cdot,\cdot-h_\varkappa, u^*(\cdot,\cdot-h_\varkappa))$, $f_\varkappa(\cdot,\cdot-h_\varkappa, u^*(\cdot,\cdot-h_\varkappa))$, $(i=1,\dots,n;\ \varkappa=0,1,\dots,\nu)$ in the weak $*$ topology of $L^\infty(Q)$. Further, the second order coefficients of the systems (1) are independent of ι. Therefore, it follows from Theorem 3.2 that $\varphi(u_{l_\iota})$ and $\varphi_{x_i}(u_{l_\iota})$, $(i=1,\dots,n)$, converge, respectively, to φ^* and $\varphi^*_{x_i}$, $(i=1,\dots,n)$, uniformly on $\bar{Q}$, where φ^* is the unique solution of the system (41) with its first and zero order coefficients and forcing term replaced, respectively, by $b_{i,\varkappa}(\cdot,\cdot - h_\varkappa, u^*(\cdot,\cdot - h_\varkappa))$, $c_\varkappa(\cdot,\cdot - h_\varkappa, u^*(\cdot,\cdot - h_\varkappa))$, and $f_\varkappa(\cdot,\cdot - h_\varkappa, u^*(\cdot,\cdot - h_\varkappa))$, $(i = 1, \ldots, n;$ $\varkappa = 0, 1, \ldots, \nu)$. By the definition of the cost integrand given in section 2 and the facts just established above, we deduce that

$$J(u_{l_\iota}) = \iint_Q \left\{\sum_{\varkappa=0}^{\nu} F^k(x, t, u_{l_\iota}(x, t-h_\varkappa), \varphi(u_{l_\iota})(x, t-h_\varkappa), \varphi_x(u_{l_\iota})(x, t-h_\varkappa))\right\} dx\, dt \to$$

$$J(u^*) = \iint_Q \left\{\sum_{\varkappa=0}^{\nu} F^\varkappa(x, t, u^*(x, t-h_\varkappa), \varphi^*(x, t-h_\varkappa), \varphi_x^*(x, t-h_\varkappa))\right\} dx\, dt$$

as $\iota \to \infty$.

Since $\{u_{l_\iota}\}$ is a minimizing sequence, we conclude that u^* is an optimal control. This completes the proof.

THEOREM 4.2. - Consider the problem P′, which consists of the system (1) and the cost function $\tilde{J}$ given by

$$\tilde{J}(u) = \iint_Q |\varphi - \varphi^d|^\lambda dx\, dt + \iint_Q \left\{\sum_{i=1}^{n} (\varphi_{x_i} - \varphi_{x_i}^d)^2\right\}^{\lambda/2} dx\, dt + \iint_Q (N(x, t)\, u(x, t), u(x, t))\, dx\, dt\,,$$

where φ^d is a fixed (desired) element from $W_\lambda^{1,0}(Q)$ and N is a $(r \times r)$ matrix valued function with $N^* = N$ and there exists an $\alpha \geqslant 0$ so that $(N(x, t)\xi, \xi) \geqslant \alpha|\xi|^2$ uniformly on Q for all $\xi \in R^r$. Suppose all the other assumptions of Theorem 4.1 regarding the system (1) hold. Then there exists an optimal control u that imposes a minimum to the cost function $\tilde{J}$.

PROOF. - Since D is a w^*-compact subset of L_∞, it suffices to show that $\tilde{J}$ is w^*-lower semicontinuous. Let $u_l \xrightarrow{w^*} u$. Then we know from Theorem 3.2 that $\varphi(u_l)$ and $\varphi_{x_i}(u_l)$, $(i = 1, 2, \ldots, n)$, converge uniformly on $\bar{Q}$ to $\varphi(u)$ and $\varphi_{x_i}(u)$, $(i = 1, 2, \ldots, n)$, respectively and that they belong to $L^\lambda(Q)$. Let $\tilde{J}_1$ denote the first two components of $\tilde{J}$. Since $\varphi(u_l)$ and $\varphi_{x_i}(u_l)$, $(i = 1, \ldots, n)$, converge uniformly on $\bar{Q}$, it follows from the Fatou Lemma that

$$\iint_Q \underline{\lim}_{l} |\varphi(u_l) - \varphi^d|^\lambda dx\, dt \leqslant \underline{\lim}_{l} \iint_Q |\varphi(u_l) - \varphi^d|^\lambda dx\, dt$$

and

$$\iint_Q \underline{\lim}_{l} \left(\sum_{i=1}^{n} (\varphi_{x_i}(u_l) - \varphi_{x_i}^d)^2\right)^{\lambda/2} dx\, dt \leqslant \underline{\lim}_{l} \iint_Q \left(\sum_{i=1}^{n} (\varphi_{x_i}(u_l) - \varphi_{x_i}^d)^2\right)^{\lambda/2} dx\, dt$$

Thus

$$\tilde{J}_1(u) \leqslant \underline{\lim}_{l} \tilde{J}_1(u_l)\,.$$

Further since $\tilde{J}_2(u) = \int_Q (Nu, u)\, dx\, dt$ is quadratic and N is positive semidefinite, it is clear that

$$\tilde{J}_2(u) \leqslant \underset{l}{\underline{\lim}}\, \tilde{J}_2(u_l)\,.$$

Therefore we conclude that

$$\tilde{J}(u) \leqslant \underset{l}{\underline{\lim}}\, \tilde{J}(u_l)$$

completing the proof.

REMARK. – In Theorem 4.2, if α is strictly positive, then the optimal control is also unique since ψ is strictly convex.

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