Variational problems with constraints.

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Summary. - *The problem considered is that of maximizing $\int_0^T F(x, y)dt$ subject to $\dot{x} = G(x, y)$, $x(0) = c$, and $0 \leq y \leq x$. An essentially new feature is determining in what regions $y = x$, $0 < y < x$, and $y = 0$.*

§ 1. Introduction.

In this paper we wish to treat the general problem of determining the maximum over y of the integral

$$(1.1) \quad \int_0^T F(x, y)dt$$

where $x(t)$ and $y(t)$ are real-valued functions subject to the relations

$$(1.2) \quad \begin{aligned} a. \quad & \dot{x} = G(x, y), \quad x(0) = c \\ b. \quad & 0 \leq y \leq x. \end{aligned}$$

This type of problem arises in the theory of dynamic programming (see [1] where further references are given) in connection with the mathematical treatment of various multistage decision processes.

The novelty and difficulty of the paper arises from the presence of the constraints in (1.2b). We are interested not only in the question of the existence of a solution, which is not difficult to establish by means of classical techniques, but to a much greater degree in the structure of the solution.

The essentially new feature is a study of the regions in which $y = 0$, $0 < y < x$ and $y = x$. Specifically, we find general conditions of fairly simple type under which the solution has the form

$$\begin{aligned} y(t) &= x(t), & 0 \leq t \leq t_1 \\ 0 < y(t) &< x(t), & t_1 < t < t_2 \\ y(t) &= 0. & t_2 \leq t \leq T, \end{aligned}$$

where one or more of these intervals may possibly be degenerate, e. g. $t_2 = T$.

§ 2. Preliminary Reductions.

We make the following assumptions throughout concerning F and G :

- (2.1) a. F and G are in $C^{(2)}$ - i. e., have continuous second partial derivatives for all $x > 0$ and $0 \leq y < x$;
 b. There exist constants p, q, r such that $px \leq G(x, y) \leq qx + r$, for $x > 0$ and $0 \leq y \leq x$;
 c. G_y is of one sign: Either $G_y > 0$ or $G_y < 0$ for all $x > 0$ and $0 \leq y \leq x$.

In (b) we may clearly take $q > 0$. Then

$$0 < ce^{pt} \leq x(t) \leq (c + rq^{-1})e^{qt} - rq^{-1};$$

i. e., $x(t)$ is bounded above and below uniformly on $(0, T)$ by positive constants:

$$(2.2) \quad 0 < m \leq x(t) \leq M.$$

$F(x, y)$ and $G(x, y)$ are bounded from $m \leq x \leq M$, $0 \leq y \leq x$. Thus the integral (1.1) is uniformly bounded, as is $\dot{x} = G(x, y)$, in the set of all $x(t), y(t)$ satisfying (1.2a) and (1.2b).

We wish to consider two further versions of the problem (1.1), each equivalent to the original. Define $\varphi = y/x$ and

$$f(x, \varphi) = F(x, \varphi x), \quad g(x, \varphi) = G(x, \varphi x).$$

Our maximum problem then becomes:

$$(2.3) \quad \max_0^T \int_0^T f(x, \varphi) dt, \quad x(0) = c, \quad \dot{x} = g(x, \varphi), \quad 0 \leq \varphi \leq 1.$$

By (2.1) and (2.2), each choice of $\varphi(t)$, $0 \leq \varphi(t) \leq 1$, uniquely determines $x(t)$. We may therefore write:

$$J[\varphi] = \int_0^T f(x, \varphi) dt$$

and the problem is to maximize $J[\varphi]$.

By (2.1) (c), G has a continuous inverse $\gamma(x, \dot{x})$ with respect to y ; i. e., $\dot{x} = G[x, \gamma(x, \dot{x})]$. In addition, the condition $0 \leq y \leq x$ is equivalent to $G(x, 0) \leq \dot{x} \leq G(x, x)$ or $G(x, x) \leq \dot{x} \leq G(x, 0)$ according as $G_y > 0$ or $G_y < 0$. Let

$$k(\dot{x}, x) = F[x, \gamma(x, \dot{x})].$$

The maximum problem is then in the form:

$$(2.4) \quad \max_0^T \int_0^T k(x, \dot{x}) dt, \quad x(0) = c, \quad u(x) \leq \dot{x} \leq v(x).$$

In each case we use whichever form of the problem, (1.1), (1.2), (2.3), or (2.4), seems most convenient.

§ 3. Existence of a Solution.

By *concavity* of a function $\psi(u)$ we mean $\psi[\sigma u_1 + (1 - \sigma)u_2] \geq \sigma\psi(u_1) + (1 - \sigma)\psi(u_2)$ for all u_1, u_2 , $0 < \sigma < 1$. *Strict concavity* means that strict inequality holds unless $u_1 = u_2$. Reversing the inequality, we get the corresponding notions of convexity and strict convexity.

We use the semi-continuity method of proving existence. The basic lemma we need is:

(3.1) LEMMA. - Let $k(x, \dot{x})$ be continuous in $(x, \dot{x})$ and concave in $\dot{x}$. Let $x_n(t)$ converge to $x(t)$ uniformly on $(0, T)$, and $u(x_n) \leq \dot{x}_n \leq v(x_n)$ for almost all t and $n = 1, 2, \dots$, where $u(x)$ and $v(x)$ are continuous. Then $u(x) \leq \dot{x} \leq v(x)$ for almost all t and

$$\limsup_n \int_0^T k(x_n, \dot{x}_n) dt \leq \int_0^T k(x, \dot{x}) dt.$$

PROOF: The sequence $\{x_n(t)\}$ is uniformly bounded since it converges uniformly. Hence $u(x_n)$ and $v(x_n)$ are uniformly bounded, since $u(x)$ and $v(x)$ are continuous. The first conclusion of the lemma may be rewritten in the form

$$\int_{t_1}^{t_2} u(x) dt \leq x(t_2) - x(t_1) \leq \int_{t_1}^{t_2} v(x) dt$$

for all t_1, t_2 in $(0, T)$, upon which it follows immediately from the corresponding inequality for $x_n(t)$ and uniform convergence. The second, and main, assertion is a corollary of a much more general semicontinuity theorem due to MCSHANE [7, Thm. (4.1)].

With Lemma (3.1) we can readily deduce the first main existence theorem:

(3.2) THEOREM. - Let the conditions (2.1) hold. Let $k(x, \dot{x})$ be concave in $\dot{x}$ for each x . Then the maximum problem (1.1) subject to (1.2) has a solution $x^*(t), y^*(t)$.

PROOF: Let S denote the value of the sup (1.1) subject to (1.2), or equivalently, the sup (2.4). Let $\{x_n(t)\}$ denote a maximizing sequence. By (2.2), $\{x_n(t)\}$ is uniformly bounded; hence, so is $\{\dot{x}_n(t)\}$. By ASCOLI's theorem, a subsequence of $\{x_n(t)\}$ converges uniformly to a limit $x^*(t)$. By (3.1), $x^*(t)$ is admissible for the problem (2.4). Thus $\int_0^T k(x^*, \dot{x}^*) dt \leq S$. However, by (3.1),

$$S = \lim_n \int_0^T k(x_n, \dot{x}_n) dt \leq \int_0^T k(x^*, \dot{x}^*) dt.$$

Hence, $\int_0^T k(x^*, \dot{x}^*) dt = S$, which proves (3.2) upon setting $y^* = \gamma(x^*, \dot{x}^*)$.

(3.3) COROLLARY. - *Let the conditions (2.1) hold. Then the maximum problem (1.1) subject to (1.2) has a solution $x^*(t)$, $y^*(t)$ provided*

$$F_{yy} - F_y G_y^{-1} G_{yy} \leq 0$$

for $x > 0$, $0 \leq y \leq x$.

PROOF: We have

$$k_{\dot{x}}(x, \dot{x}) = G_y^{-2} [F_{yy} - F_y G_y^{-1} G_{yy}] \leq 0,$$

which implies k is concave in $\dot{x}$.

In the existence theory for classical problems of the calculus of variations, some concavity assumption as in (3.2) is invariably necessary. In the present instance, we have the condition $0 \leq y \leq x$, a boundedness condition of a sort not imposed in the classical researches. This permits us to prove existence for certain non-concave problems as well, in particular when $k(x, \dot{x})$ is convex rather than concave in $\dot{x}$. The procedure is to reduce the problem with integrand k to one with integrand $\bar{k}$, the smallest function concave in $\dot{x}$ such that $\bar{k} \geq k$. Since k is convex in $\dot{x}$, we have

$$(3.4) \quad \bar{k}(x, \dot{x}) = \frac{[\dot{x} - u(x)]k[x, v(x)] + [v(x) - \dot{x}]k[x, u(x)]}{v(x) - u(x)}.$$

$\bar{k}$ is linear in $\dot{x}$, $\bar{k} \geq k$, and $\bar{k} = k$ for $\dot{x} = u(x)$ or $v(x)$.

In Lemmas (3.5)-(3.8), the standard assumptions (2.1) are understood implicitly to hold.

(3.5) LEMMA. - *Let $k(x, \dot{x})$ be continuous and convex in $\dot{x}$. Then*

$$\sup_0^T \int_0^T k(x, \dot{x}) dt = \sup_0^T \int_0^T \bar{k}(x, \dot{x}) dt,$$

where in each case the sup is over the set of $x(t)$ satisfying $x(0) = c$, $u(x) \leq \dot{x} \leq v(x)$.

A proof of Lemma (3.5) may be given along well known lines (see BUSEMANN-MAYER [4, page 184, Thm. 1]) and will not be reproduced in detail here. The essence of the proof is, given $\varepsilon > 0$, to exhibit a maximizing sequence $\{x_n(t)\}$ for $\int_0^T \bar{k}(x, \dot{x}) dt$ such that for almost all t either $u(x_n) \leq \dot{x}_n \leq u(x_n) + \varepsilon$ or $v(x_n) - \varepsilon \leq \dot{x}_n \leq v(x_n)$.

From (3.5) we obtain immediately an existence theorem for the convex case:

(3.6) COROLLARY. - *If the right hand sup in (3.5) is attained by $x^*(t)$ such that $\dot{x}^* = u(x^*)$ or $\dot{x}^* = v(x^*)$ almost everywhere, then the left hand sup is likewise attained by $x^*(t)$.*

(3.7) LEMMA. - Suppose the right hand sup in (3.5) is attained by $x^*(t)$. Let

$$A(x) = \frac{v(x)k[x, u(x)] - u(x)k[x, v(x)]}{v(x) - u(x)},$$

and

$$E = [t \mid u(x^*) < x^* < v(x^*)].$$

Then $A'(x^*) = 0$ for almost all $t \in E$.

The proof of (3.7) is an easy consequence of the considerations of Section 4 to follow, and so is deferred until the (see Remark, end Section 4).

(3.8) LEMMA. - If $A(x)$ and E are as in (3.7) and the zeros of $A'(x)$ are at most countable in number, then $\dot{x}^*(t) = 0$ for almost all $t \in E$.

PROOF: Let $x_1, \dots, x_n, \dots$ denote the zeros of $A'(x)$, and

$$E_i = [t \in E \mid x^*(t) = x_i], \quad i = 1, 2, \dots$$

By (3.7) $E = N \cup E_1 \cup E_2 \dots$, where N is of measure 0. Evidently $x^*(t) = 0$ at each point of E_i which is not isolated and at which $\dot{x}^*$ exists, hence, at almost all points of E_i , $i = 1, 2, \dots$. Thus, $\dot{x}^*(t) = 0$ for almost all $t \in E$.

(3.9) THEOREM. - Let the conditions (2.1) hold. Let:

$$(a) \quad F_{yy} - F_y G_y^{-1} G_{yy} \geq 0,$$

$$(b) \quad G \text{ is of one sign (either } G > 0 \text{ or } G < 0),$$

$$(c) \quad \text{The zeros of } A'(x) \text{ are at most countable in number,}$$

for $x > 0$ and $0 \leq y \leq x$. Then the maximum problem (1.1) subject to (1.2) has a solution $x^*(t)$, $y^*(t)$.

PROOF: Theorem (3.2) implies that $\sup_0^T \int_0^x k(x, \dot{x}) dt$ subject to $x(0) = c$, $u(x) \leq \dot{x} \leq v(x)$ is attained by some $x^*(t)$. $G > 0$ implies $u(x) > 0$, since $u(x) = G(x, 0)$ or $u(x) = G(x, x)$; hence, $\dot{x}^* > 0$. By (3.8), $\dot{x}^* = u(x^*)$ or $v(x^*)$ for almost all t . Apply (3.6). The case $G < 0$ is similar.

(3.10) THEOREM - The conclusion of (3.9) holds if (b) and (c) of (3.9) are replaced by:

$$(b') \quad A'(x) \neq 0 \text{ for } x > 0.$$

The proof is as in (3.9), using (3.7) in place of the condition $G > 0$.

In general, $\sup_0^T \int_0^x k(x, \dot{x}) dt$ is not attained even when $\sup_0^T \int_0^x \bar{k}(x, \dot{x}) dt$ is attained, although by (3.5) the two are equal. For example, let

$$k(x, \dot{x}) = \dot{x}^2 \exp [-(x-1)^2],$$

$$u(x) \equiv -1, \quad v(x) \equiv 1, \quad c = 1.$$

Then $\bar{k}(x, \dot{x}) = \exp[-(x-1)^2]$ and $\sup_0^T \int \bar{k}(x, \dot{x}) dt = T$ is attained by $x^*(t) \equiv 1$. But $k(x, \dot{x}) \leq 1$ for $-1 \leq \dot{x} \leq 1$, with equality if and only if $x=1$ and $x=\pm 1$. Thus $\int_0^T k(x, \dot{x}) dt = T$ implies $x(t) \equiv 1$ and $x(t) = \pm 1$ for almost all t , which is impossible. This example is a modification of a classical one due to CARATHÉODORY [5].

§ 4. Necessary Conditions for a Maximum.

Consider first the form (2.3) of the problem. Define

$$(4.1) \quad H(t) = f_\varphi[x(t), \varphi(t)] + g_\varphi[x(t), \varphi(t)] \int_t^T \{ f_x[x(s), \varphi(s)] \\ \exp \int_t^s g_x[x(u), \varphi(u)] du \} ds.$$

We state a first necessary condition for a maximum in terms of H :

(4.2) THEOREM. - Let (2.1) hold. Let $\varphi^*(t)$ be a solution of the maximum problem (2.3). Then, except for a set of t of measure 0,

$$\begin{aligned} H(t) &\geq 0 && \text{whenever } \varphi^*(t) = 1 \\ &= 0 && \text{» } 0 < \varphi^*(t) < 1 \\ &\leq 0 && \text{» } \varphi^*(t) = 0. \end{aligned}$$

PROOF: Let $\eta(t)$ be an arbitrary admissible variation - i. e., any measurable function such that $0 \leq \varphi^*(t) + \eta(t) \leq 1$ for all t . For $0 \leq \varepsilon \leq 1$, let $x^*(t)$ and $x_\varepsilon(t)$ denote the solutions of the differential equations $\dot{x} = g[x, \varphi^*(t)]$ and $\dot{x} = g[x, \varphi^*(t) + \varepsilon\eta(t)]$, respectively, subject to $x^*(0) = x_\varepsilon(0) = c$. Let $\xi_\varepsilon = x_\varepsilon - x^*$. Since $g \in C^{(2)}$ and $m \leq x \leq M$, $0 \leq \varphi \leq 1$, it follows by standard arguments that there exists K such that on $(0, T)$,

$$|\xi_\varepsilon(t)| < K\xi.$$

Then

$$\begin{aligned} \dot{\xi}_\varepsilon &= g(x_\varepsilon, \varphi^* + \varepsilon\eta) - g(x^*, \varphi^*) \\ &= g_x(x^*, \varphi^*)\xi_\varepsilon + g_\varphi(x^*, \varphi^*)\varepsilon\eta + Q(t, \xi_\varepsilon, \varepsilon\eta), \end{aligned}$$

where $Q(t, u, v)$ is a quadratic form in u, v with bounded coefficients. From now on let $g_x(t)$ stand for $g_x[x^*(t), \varphi^*(t)]$, and so on. Let $\xi(t)$ be the solution of

$$\dot{\xi} = g_x(t)\xi + g_\varphi(t)\eta(t), \quad \xi(0) = 0.$$

Then

$$\xi(t) = \int_0^t g_\varphi(s) \eta(s) \frac{P(t)}{P(s)} ds,$$

where $P(t) = \exp \int_0^t g_x(s) ds$.

$$\dot{\xi}_\varepsilon - \varepsilon \dot{\xi} = g_x(t)(\xi_\varepsilon - \varepsilon \xi) + Q,$$

or

$$\xi_\varepsilon(t) - \varepsilon \xi(t) = \int_0^t Q[s, \xi_\varepsilon(s), \varepsilon \eta(s)] \frac{P(t)}{P(s)} ds,$$

which implies $|\xi_\varepsilon(t) - \varepsilon \xi(t)| \leq K_1 \varepsilon^2$ for suitable K_1 .

Making the usual variational argument we now find that, since $\phi^*(t)$ maximizes,

$$\int_0^T [f_x(t) \xi(t) + f_\varphi(t) \eta(t)] dt \leq 0.$$

Using integration by parts

$$\begin{aligned} \int_0^T H(t) \eta(t) dt &= \int_0^T \left[f_\varphi(t) \eta(t) + \frac{g_\varphi(t) \eta(t)}{P(t)} \int_0^t f_x(s) P(s) ds \right] dt \\ &= \int_0^T \left[f_\varphi(t) \eta(t) + f_x(t) P(t) \int_0^t \frac{g_\varphi(s) \eta(s)}{P(s)} ds \right] dt \\ &= \int_0^T [f_\varphi(t) \eta(t) + f_x(t) \xi(t)] dt \leq 0. \end{aligned}$$

Since this is true for an arbitrary admissible variation $\eta(t)$, the conclusion of (4.2) follows.

Any interval (t_1, t_2) on which $H(t) = 0$ identically is termed an *Euler region*. The justification for this term is the following simple corollary of (4.2):

(4.3) THEOREM. - *In an Euler region the following equivalent forms of the classical Euler equation hold almost everywhere:*

- (a) $G_\nu \frac{d}{dt} (F_\nu / G_\nu) = F_x G_\nu - F_\nu G_x$
- (b) $g_\varphi \frac{d}{dt} (f_\varphi / g_\varphi) = f_x g_\varphi - f_\varphi g_x$
- (c) $\frac{d}{dt} k_x = k_x.$

PROOF: Let

$$h(t) = \frac{H(t)P(t)}{g_\varphi(t)} = \frac{f_\varphi(t)P(t)}{g_\varphi(t)} + \int_t^T f_x(s)P(s)ds,$$

where $P(t) = \exp \int_0^t g_x(u)du$ as in (4.2). By direct calculation we find that, whenever the derivatives involved exist,

$$\begin{aligned} (4.4) \quad h'(t) &= P(t) \left[\frac{d}{dt} (f_\varphi/g_\varphi) + \frac{f_\varphi g_x - f_x g_\varphi}{g_\varphi^2} \right] \\ &= P(t) \left[\frac{d}{dt} (F_\nu/G_\nu) + \frac{F_\nu G_x - F_x G_\nu}{G_\nu^2} \right] \\ &= P(t) \left[\frac{d}{dt} k_x - k_x \right]. \end{aligned}$$

In an Euler region, $h(t) = H(t) = 0$, since $g_\varphi \neq 0$ and $P \neq 0$. Hence $h'(t) = 0$, which yields (4.3). Note that in the case $\frac{d}{dt} (f_\varphi/g_\varphi)$ necessarily exists almost everywhere since the derivatives of $P(t)$ and $\int_t^T f_x(s)P(s)ds$ exist almost everywhere.

REMARK: The proof of (3.7) omitted earlier is now immediate. Apply (4.2) to

$$g(x, \varphi) = v(x)\varphi + u(x)(1 - \varphi) \text{ and } \bar{f}(x, \varphi) = \bar{k}[x, \bar{g}(x, \varphi)].$$

$H(t) = 0$, and therefore $h(t) = 0$, for almost all $t \in E$. Since $\bar{f}$ and $\bar{g}$ are linear in φ , $\frac{d}{dt} \left(\frac{\bar{f}_\varphi}{\bar{g}_\varphi} \right)$ exists almost everywhere and by (4.4)

$$h'(t) = P(t) \left[\frac{d}{dt} \bar{k}_x - \bar{k}_x \right] = -P(t)A'[x^*(t)].$$

(Since the right side of this expression is continuous, $h'(t)$ is continuous, a remark we shall use later. $h'(t) = 0$ at any point of density (see SAKS, [9, p. 128]) of E , and so at almost all points of E .)

§ 5. Uniqueness.

A solution $x^*(t)$, $\varphi^*(t)$ of the necessary condition (4.2) in general yields at most a local maximum even when k is strictly concave in x . Nor is the solution of the problem of absolute maximum necessarily unique. We avoid these phenomena under the following much stronger conditions:

(5.1) THEOREM. - Let (2.1) hold. Let f be concave jointly in (x, φ) , and $f_x > 0$; let g be strictly concave in (x, φ) . Then if $\varphi^*(t)$ satisfies the necessary conditions of (4.2), φ^* is the unique ⁽¹⁾ solution of the maximum problem (2.3).

(¹) Except for modifications of φ^* on a null set of t .

PROOF: Let η be an arbitrary non-null admissible variation. Continuing the notation of the proof of (4.2), we have

$$\xi_\varepsilon(t) - \varepsilon \xi(t) = \int_0^t Q[s, \xi_\varepsilon(s), \varepsilon \eta(s)] \frac{P(t)}{P(s)} ds < 0$$

since $Q < 0$ whenever $\xi_\varepsilon(s) \neq 0$ or $\varepsilon \eta(s) \neq 0$ by strict concavity of g . Since f is concave,

$$f(x, \varphi) - f(x^*, \varphi^*) \leq (x - x^*)f_x(x^*, \varphi^*) + (\varphi - \varphi^*)f_\varphi(x^*, \varphi^*).$$

Since $f_x > 0$, $f_x \xi_\varepsilon < \varepsilon f_x \xi$ for $0 < \varepsilon \leq 1$. In particular for $\varepsilon = 1$,

$$\begin{aligned} J[\varphi^* + \eta] - J[\varphi^*] &= \int_0^T \{ f[x_1(t), \varphi^*(t) + \eta(t)] - f[x^*(t), \varphi^*(t)] \} dt \\ &\leq \int_0^T [\xi_1(t)f_x(t) + \eta(t)f_\varphi(t)] dt \\ &< \int_0^T [\xi(t)f_x(t) + \eta(t)f_\varphi(t)] dt \\ &= \int_0^T H(t)\eta(t) dt \leq 0. \end{aligned}$$

Thus, $J[\varphi^* + \eta] < J[\varphi^*]$ ⁽²⁾.

§ 6. Continuity Properties of Solutions.

The existence theory of Section 3 yields a solution $x^*(t)$ with bounded derivative $\dot{x}^*(t)$. It is often of great importance to demonstrate continuity properties of $x^*(t)$, or equivalently of $y^*(t)$ or $\varphi^*(t)$. In this section we obtain some theorems in this direction, following closely the assumptions and methods used to get analogous properties in classical variational problems.

Our first theorem treats the concave case:

(6.1) THEOREM. - Any solution $x^*(t)$, $y^*(t)$ under the hypotheses of Corollary (3.3) is continuous if k_{xx} is strictly concave in x , in particular, if

$$k_{xx} = G_y^{-2} [F_{yy} - F_y G_y^{-1} G_{yy}] > 0$$

for $x > 0$, $0 \leq y \leq x$.

⁽²⁾ Slightly modifying the argument establishes uniqueness when f is strictly concave, $f_x \geq 0$, and g concave.

PROOF: Suppose for definiteness $g_\varphi = xG_\nu > 0$. The case $g_\varphi < 0$ is similar. From the definition (4.1) of $H(t)$, we have for almost all t ,

$$(6.2) \quad H_1(t) = \frac{H(t)}{g_\varphi(t)} = \frac{f_\varphi(t)}{g_\varphi(t)} + I(t) = k_1(t) + I(t)$$

where $I(t)$ is continuous and, as before, $f_\varphi(t) = f_\varphi[x^*(t), \varphi^*(t)]$ and so on. Since $k(x, \dot{x})$ is strictly concave in $\dot{x}$, k_x is strictly decreasing in $\dot{x}$. This leads to the following fundamental observation: if $\{\ell'_n\}$, $\{\ell''_n\}$ are two sequences tending to t such that the quantities in (6.2) exist for all ℓ'_n , ℓ''_n , and if $\lim \dot{x}^*(\ell'_n)$, $\lim \dot{x}^*(\ell''_n)$ exist and $\lim \dot{x}^*(\ell'_n) < \lim \dot{x}^*(\ell''_n)$, then $\lim H_1(\ell'_n) < \lim H_1(\ell''_n)$. Let N denote a set of measure 0 such that for $t \notin N$ the necessary conditions (4.2) hold. Since $g_\varphi > 0$, they also hold with H_1 in place of H .

Since N is of measure 0, every $t \in (0, T)$ is a limit point of $(0, T) - N$. Suppose for some $t \in (0, T)$,

$$(6.3) \quad \liminf_{\ell' \rightarrow t, \ell' \notin N} \dot{x}^*(\ell') < \limsup_{\ell'' \rightarrow t, \ell'' \notin N} \dot{x}^*(\ell'').$$

Let $\{\ell'_n\}$, $\{\ell''_n\}$ be two sequences from $(0, T) - N$ tending to t over which $\dot{x}^*(\ell')$ tends to its lim inf and its lim sup, respectively. Suppose $\lim H_1(\ell'_n) > 0$. Then, by (4.2), $\dot{x}^*(\ell'_n) = v[x^*(\ell'_n)]$ for large n . Thus,

$$\liminf_{\ell' \rightarrow t, \ell' \notin N} \dot{x}^*(\ell') = \lim_n \dot{x}^*(\ell'_n) = \lim_n v[x^*(\ell'_n)] = v[x^*(t)].$$

However, $\dot{x}^*(t) \leq v[x^*(t)]$ for all t ; hence

$$\limsup_{\ell'' \rightarrow t, \ell'' \notin N} \dot{x}^*(\ell'') \leq \lim_{\ell'' \rightarrow t} v[x^*(\ell'')] = v[x^*(t)],$$

contradicting (6.3). Thus, $\lim H_1(\ell'_n) \leq 0$. Similarly, $\lim H_1(\ell''_n) \geq 0$; however, $\lim H_1(\ell''_n) < \lim H_1(\ell'_n)$, which shows that the hypothesis (6.3) is impossible. Thus,

$$(6.4) \quad \lim_{\substack{\ell' \rightarrow t \\ \ell' \notin N}} \dot{x}^*(\ell') = X(t)$$

exists for all $t \in (0, T)$. $X(t)$ is continuous. For, if not we are led at once to a point t for which (6.3) holds. Finally, for almost all t , $\dot{x}^*(t)$ tends to $\dot{x}^*(t)$ over a set of ℓ' having density 1 at t , since $\dot{x}^*(t)$ is approximately continuous almost everywhere (see SAKS [9, p. 132]). Thus, $\dot{x}^*(t) = X(t)$ almost everywhere, which means the derivative of $x^*(t)$ exists and equals the continuous function $X(t)$ for all t . $y^*(t) = \gamma[x^*(t), X(t)]$ is the continuous also. This proves (6.1).

From (6.1) we can immediately deduce stronger differentiability properties of $x^*(t)$:

(6.5) COROLLARY. - Suppose in Theorem (6.1) that $k_{xx} < 0$. Then $\ddot{x}^*(t)$ is continuous in each interval (t_1, t_2) on which $H(t) > 0$, $H(t) = 0$, or $H(t) < 0$.

PROOF: If $H > 0$ on (t_1, t_2) , then $\dot{x}^* = u(x^*)$; hence $\ddot{x} = u'(x^*)\dot{x}^*$ is continuous. The case $H < 0$ on (t_1, t_2) is similar. If (t_1, t_2) is an Euler region the $\frac{d}{dt}k_{\dot{x}} - k_x = 0$, or $\ddot{x}^* = \frac{k_x - k_{xx}\dot{x}^*}{k_{xx}}$, on (t_1, t_2) . Since $k_{xx} \neq 0$, $\ddot{x}^*$ is again continuous.

The question of under what conditions the solution $x^*(t)$, $y^*(t)$ consists of a finite number of intervals on which $H > 0$, $H = 0$, and $H < 0$ respectively, is undecided for the strictly concave case just considered. However, for the convex case (which reduces to the linear case) we resolve this question as follows:

(6.6) THEOREM. - Let the hypotheses of Theorem (3.9) be strengthened to suppose $A'(x)$ has only a finite number n of zeros. Let $x^*(t)$, $y^*(t)$ be a solution to the maximum problem (1.1) subject to (1.2). Then there exists a decomposition of $(0, T)$ into $n + 2$ or fewer disjoint intervals I_j such that for each j , either $y^*(t) = x^*(t)$ for almost all $t \in I_j$ or $y^*(t) = 0$ for almost all $t \in I_j$.

PROOF: By the Remark, end Section 4,

$$h'(t) = P(t) \left[\frac{d}{dt} \bar{k}_{\dot{x}} - \bar{k}_x \right] = -P(t)A'[x^*(t)].$$

Hence, $h'(t)$ is continuous. Since G is strictly of one sign (hypothesis (b) of (3.9)), $x^*(t)$ is strictly either increasing or decreasing. Hence the number of zeros of $h'(t)$ does not exceed n , since $P(t) \neq 0$. The number of zeros of $h(t)$ is then, at most, $n + 1$, which means that $(0, T)$ can be divided into $n + 2$ or fewer intervals on each of which $H > 0$ or $H < 0$ - i. e., $y^* = x^*$ or $y^* = 0$.

REMARK: The conclusion of (6.6) also holds if the hypotheses of (3.10) hold. In this case $h'(t)$ is one sign; consequently, there are at most two intervals I_1, I_2 .

§ 7. Solutions of a Particular Type.

Having determined the existence and general character of solutions, there remains the difficult problem of determining those regions in which $x^*(t)$, $y^*(t)$ is given by the Euler equation and those for which $y^*(t)$ is on the boundary. In this section, we solve an important special case of this problem; namely, we find general conditions on F and G insuring that the

solution is of the form:

$$(7.1) \quad \begin{aligned} y^*(t) &= x^*(t), & 0 \leq t \leq t_1 \\ 0 < y^*(t) < x^*(t), & t_1 < t < t_2 \\ y^*(t) &= 0, & t_2 \leq t \leq T. \end{aligned}$$

We allow the possibility that one or more of these regions is missing. For example, in the convex case we generally have $t_1 = t_2$ (discontinuous solution). In the strictly concave case the Euler region (t_1, t_2) necessarily appears since the solution is continuous, except when $y^*(t) = 0$ identically or $y^*(t) = 1$ identically.

(7.2) LEMMA. - Let $x^*(t)$, $y^*(t)$ satisfy the necessary condition (4.2). Suppose $p(t) > 0$ exists such that $w(t) = p(t)H(t)$ is continuous on $(0, T)$ and $w'(t) \leq 0$ on any interval (t', t'') in which $y^*(t) = x^*(t)$ or in which $y^*(t) = 0$. Then $x^*(t)$, $y^*(t)$ is of the form (7.1).

PROOF: Since $p(t) > 0$, $w(t) > 0$ implies $y^*(t) = x^*(t)$, and $w(t) < 0$ implies $y^*(t) = 0$. Thus it suffices to consider the case when $w(t)$ has a zero t_1 . Suppose $w(t) < 0$ on some interval (t', t'') to the left of t_1 , with $w(t'') = 0$. Then $y^*(t) = 0$ on (t', t'') , whence $w(t)$ is non-increasing on (t', t'') . Hence $\lim_{t \rightarrow t''-} w(t) < 0$, contradicting the continuity of w at t'' . Thus, $w(t) \geq 0$ or $t \leq t_1$. Similarly, $w(t) < 0$ for $t \geq t_1$. Since t_1 is an arbitrary zero, the zeros of $w(t)$ lie in an interval (t_1, t_2) which proves (7.2) since $H(t)$ and $w(t)$ differ by the positive multiple $p(t)$.

(7.3) THEOREM. - Let $F(x, y)$, $G(x, y)$ be in $C^{(2)}$ and satisfy the following conditions for $x > 0$ and $0 \leq y \leq x$:

- (1) $0 \leq G(x, y) \leq qx + r$, q, r constants.
- (2) $F_x \geq 0$, $F_y \leq 0$,
- (3) $G_x \geq 0$, $G_y > 0$,
- (4) $G_y F_{yy} - F_y G_{yy} < 0$,
- (5) $F_{xy} \leq 0$, $G_{xy} \leq 0$.

Then every solution $x^*(t)$, $y^*(t)$ is continuous and is of the form (7.1).

PROOF: Let $x^*(t)$, $y^*(t)$ be a solution. By (6.1) and (6.5), $y^*(t)$ is continuous, and $y^*(t)$ is continuous on any interval in which $y^*(t) = x^*(t)$ or in which $y^*(t) = 0$. Let $h(t)$ be as in the proof of (4.3). By (4.4)

$$(7.4) \quad \begin{aligned} h'(t) &= P(t) \left[\frac{d}{dt} (F_y / G_y) + \frac{F_y G_x - F_x G_y}{G_y} \right] \\ &= \frac{P(t)}{G_y^2} \{ [G_y F_{xy} - F_y G_{xy}] G + [G_y F_{yy} - F_y G_{yy}] y^* + [F_y G_x - F_x G_y] G_y \}, \end{aligned}$$

where F_ν , G_ν , etc. are evaluated at $x^*(t)$, $y^*(t)$. By (1), (2), (3), and (5), $[G_\nu F_{x\nu} - F_\nu G_{x\nu}]G \leq 0$. On an interval where $y^*(t) = 0$, $[G_\nu F_{\nu\nu} - F_\nu G_{\nu\nu}]y^* = 0$; where $y^* = x^*$, we have $y^* = \dot{x}^* = G$, and this term is negative by (1) and (4). By (2) and (3), $[F_\nu G_x - F_x G_\nu]G_\nu \leq 0$. Hence, $h'(t) \leq 0$ on any interval in which $y^*(t) = x^*(t)$ or in which $y^*(t) = 0$. Since $y^*(t)$ is continuous, $h(t)$ is continuous. By Lemma (7.2), $x^*(t)$, $y^*(t)$ is of the form (7.1).

REMARK: The proof still holds without change if we assume that (2), (5), and the part $G_x \geq 0$ of (3) hold only when $y = 0$ or $y = x$.

Under suitably strengthened conditions, none of the intervals $(0, t_1)$, (t_1, t_2) , (t_2, T) degenerate:

(7.5) THEOREM. - *Let the hypotheses of Theorem (7.3) be strengthened as follows:*

(2') $F_x > 0$ and $F_\nu < 0$, for $x > 0$, $0 \leq y \leq x$.

Then $t_2 < T$. There exists T_0 such that if $T > T_0$, then $t_1 > 0$.

PROOF: By the definition (4.1),

$$H(t) = f_\varphi(t) + g_\varphi(t) \int_t^T f_x(s) \frac{P(s)}{P(t)} ds,$$

where the notation $f_\varphi(t)$, $P(t)$, etc. is as in Section 4.

$$H(T) = f_\varphi(T) = F_\nu(T)x(T) < 0,$$

whereas $H(t_2) = 0$. Thus, $t_2 < T$. Let

$$(7.6) \quad h_1(t) = \frac{F_\nu(t)}{G_\nu(t)} P_1(t) + \int_t^T F_x(s) P_1(s) ds,$$

where $P_1(t) = \exp \int_0^t G_x(u) du$. Then, since $g_x = G_x + \varphi G_\nu$,

$$\begin{aligned} P(t) &= \exp \int_0^t g_x(u) du = \exp \int_0^t [G_x(u) + \varphi^*(u) G_\nu(u)] du \\ &= P_1(t) \exp \int_0^t \varphi^*(u) G_\nu(u) du. \end{aligned}$$

On (t_2, T) , $\varphi^*(t) = 0$; hence

$$\begin{aligned} \frac{P(s)}{P(t)} &= \exp \int_t^s g_x(u) du = \exp \int_t^s G_x(u) du \\ &= \frac{P_1(s)}{P_1(t)}, \quad t_2 \leq t \leq s \leq T. \end{aligned}$$

Since $f_x = F_x + \varphi F_y$, we find that

$$h_i(t) = \frac{P_i(t)}{P(t)} h(t)$$

on (t_2, T) . In particular, $h_i(t_2) = h(t_2) = H(t_2) = 0$. Upon differentiating (7.6) and using (4.4), we obtain

$$(7.7) \quad h_i'(t) = \frac{P_i(t)}{P(t)} h'(t)$$

on $(0, T)$. Hence, $h_i(t) = h(t) = H(t) = 0$ in the Euler region (t_1, t_2) .

$$\min_{0 \leq y \leq c} \frac{F_y(c, y)}{G_y(c, y)} = \mu$$

is finite by (7.3) (3). By (2'), $F_x \geq \delta > 0$ for $m \leq x < M$, $0 \leq y \leq x$ ($M, m > 0$ as in (2.2)), since F_x is continuous. Then

$$h_i(0) = \frac{F_y(0)}{G_y(0)} + \int_0^T F_x(s) P_i(s) ds \geq \mu + \int_0^T \delta ds = \mu + \delta T.$$

For $T > T_0 = -\frac{\mu}{\delta}$, $h_i(0) > 0$ which implies $t_1 > 0$.

The solution $x^*(t)$, $y^*(t)$ can be shown to be of form (7.1) under a variety of different conditions, using the same argument as in (7.3). For example:

(7.8) THEOREM. - *The conclusion of Theorem (7.3) holds if (2), (3), (5) are replaced by:*

$$(2a) \quad F_y G_x - F_x G_y \leq 0$$

$$(3a) \quad G_y > 0$$

$$(5a) \quad G_y F_{xy} - F_y G_{xy} \leq 0.$$

(7.9) THEOREM. - *The conclusion of Theorem (7.3) holds if (2) and (5) are replaced by:*

$$(2b) \quad F_x \geq 0, \quad F_x + F_y \geq 0$$

$$(5b) \quad F_{xy} \leq 0, \quad G_{xy} \leq 0, \quad F_{xy} + F_{yy} \leq 0, \quad G_{xy} + G_{yy} \leq 0.$$

The proof of (7.8) is almost identical with that of (7.3). To obtain (7.9), differentiate

$$\frac{H(t)}{x^*(t)} = F_y(t) + G_y(t) \int_t^T [F_x(s) + \varphi^*(s) F_y(s)] \frac{P(s)}{P(t)} ds$$

rather than $h(t)$, and proceed as in (7.3).

Theorem (7.3) treats the concave case. For the convex case, we have:

(7.10). THEOREM - Let $F(x, y)$, $G(x, y)$ be in $C^{(2)}$ and satisfy the following conditions for $x > 0$ and $0 \leq y \leq x$:

- (1) $px \leq G(x, y) \leq qx + r$, p, q, r constants
- (2) $G_y > 0$
- (3) $G_y F_{yy} - F_y G_{yy} \leq 0$
- (4) $A' > 0$, where

$$A(x) = \frac{G(x, x)F(x, 0) - G(x, 0)F(x, x)}{G(x, x) - G(x, 0)}.$$

Then every solution $x^*(t)$, $y^*(t)$ is of the form (7.1). Furthermore, $t_1 = t_2$.

PROOF: We merely note that $h'(t) = -P(t)A[x^*(t)]$ as in the proof of (6.6), and apply Lemma (7.2). By (6.6), $t_1 = t_2$.

REMARK: If $G_y < 0$ rather than $G_y > 0$ in (2), then the conclusion of (7.10) still holds provided (x, x) and $(x, 0)$ are interchanged in the definition of $A(x)$ (see (3.7)).

§ 8. The case $G_y(x, 0^+) = +\infty$.

Until now, we have always assumed $F, G \in C^{(2)}$ for $x > 0$ and $0 \leq y \leq x$. We wish to consider also the case $F, G \in C^{(2)}$ for $y > 0$ and $G_y(x, 0^+) = +\infty$ uniformly in x - i. e., on the interval $0 < m \leq x \leq M$, $G_y(x, y) \geq R(y)$ where $\lim_{y \rightarrow 0} R(y) = +\infty$. This case is in some respects simpler than the previous one, in that often $y^*(t) = 0$ only for $t = T$.

We consider as usual the problem (1.1) subject to (1.2). We look as in Section 7 for conditions under which the solution is of the form (7.1).

(8.1) THEOREM. - Let $F(x, y)$ be in $C^{(2)}$. Let $G(x, y)$ be continuous for $x > 0$ and $0 \leq y \leq x$. Let $G(x, y)$ be in $C^{(2)}$ for $0 < y \leq x$, and $G_y(x, 0^+) = +\infty$ uniformly for $m \leq x \leq M$ ($m, M > 0$ are as in (2.2)). For $0 < y \leq x$, let (1), (3), and (5) of (7.3) hold, and in addition

- (2') $F_x > 0$, $F_y < 0$,
- (4') $F_{yy} \leq 0$, $G_{yy} < 0$.

Then the problem (1.1)-(1.2) has a solution $x^*(t)$, $y^*(t)$ of the form (7.1). $y^*(t)$ is continuous, $y^*(T) = 0$, and $t_2 = T$. There exists T_0 such that if $T > T_0$, then $t_1 > 0$.

PROOF: Given $0 < \varepsilon < 1$, let

$$G(x, y; \varepsilon) = G[x, y + \varepsilon(x - y)],$$

and consider the problem

$$(8.2) \quad \max \int_0^T F(x, y) dt, \quad x(0) = c, \quad \dot{x} = G(x, y; \varepsilon), \quad 0 \leq y \leq x.$$

The hypotheses (3.3) and (7.3) hold for $F(x, y)$, $G(x, y; \varepsilon)$; hence a solution $x_\varepsilon^*(t)$, $y_\varepsilon^*(t)$ of (8.2) with the form (7.1) exists. In the Euler region $(t_{1\varepsilon}, t_{2\varepsilon})$ we have $h'(t) = 0$. The last expression in (7.4) shows that $\dot{y}_\varepsilon^* \leq 0$ in $(t_{1\varepsilon}, t_{2\varepsilon})$, since the coefficient of $\dot{y}^*$ is negative as are the remaining terms. Thus, $y_\varepsilon^*(t)$ is decreasing on $(t_{1\varepsilon}, t_{2\varepsilon})$. On $(0, t_{1\varepsilon})$, $y_\varepsilon^*(t) = x_\varepsilon^*(t) = r_\varepsilon(t)$ the solution of $\dot{x} = G(x, x; \varepsilon)$, $x(0) = c$, and on $(t_{2\varepsilon}, T)$, $y_\varepsilon^*(t) = 0$. Choose a sequence $\{\varepsilon_n\}$ for which $t_{1\varepsilon}$ and $t_{2\varepsilon}$ tend respectively to limits t_1 and t_2 and for which y_ε^* tends to a limit $y^*(t)$ except for a countable set of values of t . The latter is possible by HELLY's theorem. Since $\dot{x}_\varepsilon^*(t)$ is uniformly bounded, we may extract a further subsequence for which $x_\varepsilon^*(t)$ converges uniformly to a limit $x^*(t)$, by ASCOLI's theorem. Then, for this latter subsequence,

$$(8.3) \quad \lim_{\varepsilon \rightarrow 0} \int_0^T F[x_\varepsilon^*(t), y_\varepsilon^*(t)] dt = \int_0^T F[x^*(t), y^*(t)] dt,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_0^t G[x_\varepsilon^*(s), y_\varepsilon^*(s); \varepsilon] ds = \int_0^t G[x^*(s), y^*(s)] ds$$

for $0 \leq t \leq T$. In particular, $\dot{x}^* = G(x^*, y^*)$ almost everywhere. In addition, $0 \leq y^*(t) \leq x^*(t)$. We show that $x^*(t)$, $y^*(t)$ is a solution. Let $x(t)$, $y(t)$ be any pair satisfying (1.2), and $x_\varepsilon(t)$ the unique solution of $\dot{x} = G[x, y(t); \varepsilon]$, $x(0) = c$, $0 < \varepsilon < 1$. Since $G_\nu > 0$,

$$G[x, y(t); \varepsilon] \geq G[x, y(t)]$$

for all $x \geq y(t)$ and all t . Hence, $x_\varepsilon(t) \geq x(t) \geq y(t)$. Since $F_x > 0$ and $x_\varepsilon^*(t)$, $y_\varepsilon^*(t)$ maximizes (8.2),

$$\int_0^T F[x(t), y(t)] dt < \int_0^T F[x_\varepsilon(t), y(t)] dt \leq \int_0^T F[x_\varepsilon^*(t), y_\varepsilon^*(t)] dt.$$

By (8.3),

$$\int_0^T F[x(t), y(t)] dt \leq \int_0^T F[x^*(t), y^*(t)] dt,$$

which shows that $x^*(t)$, $y^*(t)$ is a solution of (1.1).

Consider the functions $h_{1\varepsilon}(t)$ defined as in (7.6). We find that as ε describes the same sub-sequence of $\{\varepsilon_n\}$ as above,

$$\lim_{\varepsilon \rightarrow 0} h_{1\varepsilon}(t) = h_1(t) = \frac{F_\nu(t)}{G_\nu(t)} P_1(t) + \int_t^T F_x(s) P_1(s) ds$$

exists except for a countable set of t . It is understood that $F_y G_y^{-1} = 0$ whenever $y^* = 0$. Furthermore,

$$(8.4) \quad \begin{aligned} h_1(t) &\geq 0 \text{ whenever } y^*(t) = x^*(t) \\ &= 0 \text{ whenever } 0 < y^*(t) < x^*(t) \\ &\leq 0 \text{ whenever } y^*(t) = 0 \end{aligned}$$

for almost all t , since the corresponding relations hold for $h_{i\epsilon}(t)$, $x_{i\epsilon}^*(t)$, $y_{i\epsilon}^*(t)$. The same proof as for (6.1) shows that $y^*(t)$ is continuous. For $t_2 \leq t < T$, $y^*(t) = 0$; consequently, $h_1(t) = \int_t^T F_x(s) P_1(s) ds > 0$, contradicting (8.4). Thus, $t_2 = T$, and $h_1(t) \geq 0$ for $t \in (0, T)$. $F_y < 0$, and hence $F_y \leq \delta < 0$ for $m \leq x \leq M$, $0 \leq y \leq x$. Therefore, $y^*(T) = 0$ since otherwise $0 < G_y(T) < \infty$ and

$$h_1(T) = P_1(T) \frac{F_y(T)}{G_y(T)} < 0.$$

That $t_1 > 0$ for large T is proved as in (7.5).

§ 9. Examples.

To illustrate the theory developed in the preceding sections, we consider a few particular cases, typical of simple control problem arising in the theory of dynamic programming.

EXAMPLE 1: $F(x, y) = x - y$, $G(x, y) = b(y)$, where

- (a) $b(y)$ is continuous, $y \geq 0$;
- (b) For $y > 0$, $b(y) \in C^{(2)}$, $b(y) \geq 0$, $b'(y) > 0$, $b''(y) < 0$; and
- (c) $b'(0^+) = +\infty$.

For instance, $b(y) = \sqrt{y}$ satisfies these conditions. The condition of Theorem (8.1) hold. The Euler equation is $\frac{d}{dt} \frac{1}{b} = -1$ which together with the end condition $y^*(T) = 0$ gives $b[y^*(t)] = \frac{1}{T-t}$, $t_1 \leq t \leq T$. Let $r(t)$ denote the solution of $\dot{x} = b(x)$, $x(0) = c$. Let t_1 be a positive root of $b[r(t)] = \frac{1}{T-t}$, provided $b(c) < 1/T$; let $t_1 = 0$ otherwise. Let $\beta(y)$ denote the inverse of $b'(y)$: $\beta[b'(y)] = y = b'[\beta(y)]$. Then the solution for Example 1 is given by:

$$(9.1) \quad \begin{aligned} y^*(t) &= r(t), \quad 0 \leq t \leq t_1 \\ &= \beta\left(\frac{1}{T-t}\right), \quad t_1 \leq t \leq T, \quad \dot{x}^*(t) = b[y^*(t)], \quad x^*(0) = c. \end{aligned}$$

Thus, the solution consists of an interval $y^* = x^*$ followed by an Euler region, in case $T > 1/b(c)$. Otherwise, only the Euler region appears.

EXAMPLE 2: $F(x, y) = x - y$, $G(x, y) = b(y)$, where

- (a) $b(y) \in C^{(2)}$, $0 \leq y$
 (b) $b(y) \geq 0$, $b'(y) > 0$, $b''(y) < 0$, $0 \leq y$.

The conditions of Theorem (7.5) hold. Let t_1 be defined as before and $t_2 = T - \frac{1}{b'(0)}$. The solution for Example 2 is given by:

$$(9.2) \quad \begin{aligned} y^*(t) &= r(t), \quad 0 \leq t \leq t_1 \\ &= \beta \left(\frac{1}{T-t} \right), \quad t_1 \leq t \leq t_2 \\ &= 0, \quad t_2 \leq t \leq T, \quad \dot{x}^*(t) = b[y^*(t)], \quad x^*(0) = c. \end{aligned}$$

If $T \leq 1/b'(0)$, then $y^*(t) \equiv 0$. If $1/b'(0) \leq T \leq 1/b'(c)$, then both the Euler region and the interval $y^*(t) = 0$ appear. If $T > 1/b'(c)$, then all three regions appear.

EXAMPLE 3: $F(x, y) = x - y$, $G(x, y) = \sqrt{x} + \sqrt{y}$. The conditions of (8.1) hold. Thus, the solution is continuous, is of the form $y^* = x^*$ on $(0, t_1)$, and is given by the Euler equation on (t_1, T) with $y^*(T) = 0$. The Euler equation reduces to $\ddot{x} = 0$; hence, on (t_1, T) ,

$$(9.3) \quad x^*(t) = \alpha t + \beta, \quad y^*(t) = (\alpha - \sqrt{\alpha t + \beta})^2.$$

Since $y^*(T) = 0$, $\beta = \alpha^2 - \alpha T$.

Suppose $t_1 > 0$. Then $x^*(t_1) = y^*(t_1)$, which gives $t_1 = T - 3/4\alpha$ and $y^*(t_1) = \alpha^2/4$. On $(0, t_1)$, $x^*(t) = y^*(t) = r(t)$, where $r(t) = (t + \sqrt{c})^2$ is the solution of $\dot{r} = 2\sqrt{r}$, $r(0) = c$. This gives $\alpha^2/4 = (t_1 + \sqrt{c})^2$, which together with $t_1 = T - 3/4\alpha$ determines

$$\begin{aligned} t_1 &= \frac{2T - 3\sqrt{c}}{5} \\ \alpha &= 4/5(T + \sqrt{c}) \\ \beta &= \alpha^2 - \alpha T. \end{aligned}$$

Note that necessarily $2T - 3\sqrt{c} > 0$. If $2T - 3\sqrt{c} \leq 0$, then $t_1 = 0$ (only the Euler region appears). In this case, α and β are determined from the conditions $x^*(0) = c$, $y^*(T) = 0$ to be

$$\begin{aligned} \alpha &= \frac{T + \sqrt{T^2 + 4c}}{2} \\ \beta &= c. \end{aligned}$$

In the preceding examples, concavity played a vital role (e. g., $b'' < 0$ in Examples 1 and 2). Let us now consider a convex case, which is the continuous analogue of a dynamic programming problem considered in [2].

EXAMPLE 4: $F(x, y) = K(y) + L(x - y)$, $G(x, y) = -ay - b(x - y)$, where for $x \geq 0$ $K(x)$, $L(x) \in C^{(2)}$, $K(0) = L(0) = 0$, $K'' \geq 0$, $L'' \geq 0$, and $b > a > 0$.

As before (see (3.9), (6.6), etc.) we reduce the convex problem to a linear one. Since G is linear, this may be done by linearizing F . Replace $F(x, y)$ by

$$\begin{aligned}\bar{F}(x, y) &= \frac{y}{x} K(x) + \frac{x-y}{x} L(x) \\ &= \varphi K(x) + (1 - \varphi) L(x) \\ &= \bar{f}(x, \varphi).\end{aligned}$$

Let

$$A(x) = \frac{bK(x) - aL(x)}{b - a}.$$

Assume that $A'(x) = 0$ for only a finite number of n of values of $x \in (0, c)$. (Note that we need consider only values of x in this interval since $G < 0$). By Theorem (3.9) a solution $x^*(t)$, $y^*(t)$ exists. By theorem (6.6), $(0, T)$ can be split into $n + 2$ or fewer intervals on each of which $y^*(t) = x^*(t)$ or $y^*(t) = 0$. As in (4.3), let

$$h(t) = \frac{\bar{f}_\varphi(t)}{g_\varphi(t)} P(t) + \int_t^T \bar{f}_x(s) P(s) ds,$$

where $P(t) = \exp \int_0^t g_x(u) du = \exp \int_0^t \frac{x^*(u)}{x^*} du = \frac{x^*(t)}{c}$, $g_x(x, \varphi) = -a\varphi - b(1 - \varphi) = \frac{x}{x}$.

In particular,

$$h(T) = \frac{\bar{f}_\varphi(T)}{g_\varphi(T)} P(T) = \frac{K(T) - L(T)}{(b - a)c},$$

where $K(T) = K[x^*(T)]$, etc. By the Remark, end Section 4,

$$h'(t) = -P(t)A'(t) = -\frac{x^*(t)}{c} A'(t);$$

hence,

$$\begin{aligned}h(t) &= h(T) - \int_t^T h'(s) ds \\ &= \frac{K(T) - L(T)}{(b - a)c} + \int_t^T A'(s) \frac{x^*(s)}{c} ds.\end{aligned}$$

Since $b > a$, $g_\varphi(x, \varphi) = (b - a)x > 0$; hence, $H(t)$ and $h(t) = \frac{H(t)P(t)}{g_\varphi(t)}$ have the same sign. Then $y^*(t) = x^*(t)$ when $h(t) > 0$ and $y^*(t) = 0$ when $h(t) < 0$. If $K(T) - L(T) > 0$, then $y^*(t) = x^*(t)$ on some maximal final interval (t_1, T) . On (t_1, T) , $\dot{x}^* = -ax^*$. Except when $t_1 = 0$, $h(t_1) = 0$; or

$$\begin{aligned} 0 = h(t_1) &= \frac{K(T) - L(T)}{(b - a)c} - \frac{1}{ac} \int_{t_1}^T A'(s) \dot{x}^*(s) ds \\ &= \frac{K(T) - L(T)}{(b - a)c} - \frac{1}{ac} A(t) \Big|_{t_1}^T, \end{aligned}$$

whence $A(t_1) = K(T)$. If $t_1 > 0$, then $y^*(t) = 0$ on some maximal interval (t_2, t_1) . Unless $t_2 = 0$, $h(t_2) = 0$. By the same reasoning as before, $A(t_1) = A(t_2)$. Continuing this process, we conclude that if $K(T) - L(T) > 0$, then $y^*(t)$ jumps from $x^*(t)$ to 0 or vice-versa whenever $A(t) - K(T)$ changes sign. In case $K(T) - L(T) < 0$, this occurs when $A(t) - L(T)$ changes sign.

For large T , $y^*(t)$ has at least j jumps, where j is the number of times $A(x)$ changes sign on $(0, c)$. For, there exist $\varepsilon_0 > 0$, $\delta_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$ and $0 \leq \delta < \delta_0$, then $A(x) - \varepsilon$ changes sign at least j times on (δ, c) . Choose $\delta < \delta_0$ such that $|K(x)| < \varepsilon_0$, $|L(x)| < \varepsilon_0$ for $0 < x \leq \delta$. Let $T_0 = -\frac{\log \delta/c}{a}$. Then $x^*(T) \leq ce^{-aT} \leq \delta$ if $T \geq T_0$. Thus, $|K(T)| = |K[x^*(T)]| < \varepsilon_0$, so that $A(x) - K(T)$ changes sign at least j times on (δ, c) . Since $x^*(t)$ is continuous, $A(t) - K(T) = A[x^*(t)] - K[x^*(T)]$ likewise has at least j sign changes on $(0, T)$ (similarly for $A(t) - L(T)$). Thus, $y^*(t)$ has at least j jumps.

Since $K(x)$, $L(x)$, a , b can be so chosen that j is arbitrarily large, we have an example in which the solution has arbitrarily many jumps between intervals where $y^* = x^*$ and $y^* = 0$, respectively.

§ 10. The n -dimensional case.

Let us consider now the modifications required in the existence theory and necessary conditions to treat the n -dimensional problem

$$(10.1) \quad \max_0^T \int_0^T f(x, \varphi) dt,$$

where $x = [x^1, \dots, x^n]$, $\varphi = [\varphi^1, \dots, \varphi^m]$ are subject to

$$(10.2) \quad \begin{aligned} (a) \quad & \dot{x}^i = g_i(x, \varphi), \quad x^i(0) = c_i \quad i = 1, 2, \dots, n \\ (b) \quad & 0 \leq \varphi^j \leq 1, \quad j = 1, 2, \dots, m \leq n. \end{aligned}$$

We suppose that for all (x, φ) , $0 \leq \varphi \leq 1$,

$$(10.3) \quad \begin{aligned} (a) \quad & f, g_i \in C^{(2)}, \quad i = 1, 2, \dots, n \\ (b) \quad & \text{For } i = 1, 2, \dots, n, \text{ there exist } p_i, q_i, r_i \text{ such that } (p_i, x) \leq \\ & \leq g_i(x, \varphi) \leq (q_i, x) + r_i. \end{aligned}$$

Here (p_i, x) denotes the inner product of the constant vector p_i with x , and similarly for q_i . r_i is a scalar. As in (2.2), (10.3) (b) imposes a uniform bound on $x(t)$ - i. e., $m \leq x^i(t) \leq M$ for all $i = 1, 2, \dots, n$ where m and M are independent of $\varphi(t)$.

Introduce the following notation: g_x, g_φ denote, respectively, the matrices $[g_{ix^j}], [g_{i\varphi^j}]$; $f_x = [f_{x^1}, \dots, f_{x^n}]$; g_{x^j}, g_{φ^j} denote the j^{th} column of g_x, g_φ , respectively.

The generalization of the existence theorem (3.2) is as follows: Suppose there exist $i_1, \dots, i_m$ such that, for each x , the mapping $\varphi \rightarrow [g_{i_1}(x, \varphi), \dots, g_{i_m}(x, \varphi)]$ maps the unit hypercube $0 \leq \varphi^j \leq 1$ one to one with non-vanishing Jacobian onto a convex region R_x . We may then solve for $\varphi = \gamma(x, \dot{x})$ and set as in (2.4),

$$k(x, \dot{x}) = f[x, \gamma(x, \dot{x})].$$

If $k(x, \dot{x})$ is concave in $\dot{x}$, the maximum is attained. Unfortunately, it is difficult to state conditions that R_x be convex and $k(x, \dot{x})$ concave in $\dot{x}$ which have generality and simplicity comparable to the one dimensional case.

For the linear case these difficulties do not occur, and we may state:

(10.4) THEOREM. - Let (10.3) hold. Then sufficient conditions for the maximum of (10.1) subject to (10.2) to be attained are that, for every x ,

- (1) $f(x, \varphi)$ is concave as a function of φ ,
- (2) $g_i(x, \varphi)$ is linear as a function of φ - i. e.,
 $g_i(x, \varphi) = a_i(x) + (b_i(x), \varphi)$, $i = 1, 2, \dots, n$.

PROOF. - Let $\{x_n(t), \varphi_n(t)\}$ denote a maximizing sequence. As in (3.2), we find a subsequence of n such that $x_n(t)$ and $\Phi_n(t) = \int_0^t \varphi_n(s) ds$ converge uniformly to $x^*(t), \Phi^*(t)$ respectively. Letting $\varphi^*(t) = \dot{\Phi}^*(t)$, an argument similar to (3.2) shows that $x^*(t), \varphi^*(t)$ maximizes.

To generalize the necessary conditions (4.2) to the n -dimensional case, let $x^*(t), \varphi^*(t)$ be a solution of (10.1) subject to (10.2). Let $X(t)$ denote a matrix whose columns are n linearly independent solutions of the linear homogeneous equation $\dot{x} = g_x(t)x$, where as usual $g_x(t)$ stands for $g_x[x^*(t), \varphi^*(t)]$ and similarly for $f_x(t), g_\varphi(t)$, etc. Let

$$(10.5) \quad \lambda(t) = X^{-1'}(t) \int_t^T X'(s) f_x(s) ds,$$

where Y' denotes the transpose of a matrix Y . For $j = 1, 2, \dots, m$, let

$$(10.6) \quad \begin{aligned} H_j(t) &= f_{\varphi^j}(t) + (g_{\varphi^j}(t), \lambda(t)) \\ &= f_{\varphi^j}(t) + \sum_{i=1}^n g_{i\varphi^j} \lambda_i(t). \end{aligned}$$

(10.7) THEOREM. - Let (10.3) hold. A necessary condition that $x^*(t)$, $\varphi^*(t)$ be a solution of (10.1) subject to (10.2) is that, for almost all t and all $j = 1, 2, \dots, m$,

$$\begin{aligned} H_j(t) &\geq 0 \text{ whenever } \varphi^j(t) = 1 \\ &= 0 \quad \gg \quad 0 < \varphi^j(t) < 1 \\ &\leq 0 \quad \gg \quad \varphi^j(t) = 0. \end{aligned}$$

PROOF: We make a variation $\eta(t)$ on a particular $\varphi^{j*}(t)$, holding $\varphi^{k*}(t)$ fixed for $k \neq j$. As in the proof of (4.2), we find that

$$(10.8) \quad \int_0^T [(f_x, \xi) + f_{\varphi^j} \eta] dt \leq 0,$$

where

$$(10.9) \quad \dot{\xi} = g_x \xi + g_{\varphi^j} \eta, \quad \xi(0) = 0.$$

The solution of (10.9) may be written

$$\xi(t) = X(t) \int_0^t X^{-1}(s) g_{\varphi^j}(s) \eta(s) ds.$$

Integrating by parts,

$$\begin{aligned} (10.10) \quad \int_0^T (f_x, \xi) dt &= \int_0^T \left(f_x(t), X(t) \int_0^t X^{-1}(s) g_{\varphi^j}(s) \eta(s) ds \right) dt \\ &= \int_0^T \left(X'(t) f_x(t), \int_0^t X^{-1}(s) g_{\varphi^j}(s) \eta(s) ds \right) dt \\ &= \int_0^T \left(\int_t^T X'(s) f_x(s) ds, X^{-1}(t) g_{\varphi^j}(t) \eta(t) \right) dt \\ &= \int_0^T \eta(t) \left(X^{-1}(t) g_{\varphi^j}(t), \int_0^T X'(s) f_x(s) ds \right) dt \\ &= \int_0^T \eta(t) (g_{\varphi^j}(t), \lambda(t)) dt. \end{aligned}$$

By (10.8),

$$\int_0^T \eta(t) [(g_{\varphi^j}(t), \lambda(t)) + f_{\varphi^j}(t)] dt \leq 0.$$

Since $\eta(t)$ is an arbitrary admissible variation of $\varphi^j(t)$, and j is arbitrary, (10.7) follows.

If we multiply (10.5) by X' , differentiate, and recall that $X = g_x X$, we obtain a linear differential equation for λ :

$$(10.11) \quad \dot{\lambda} = -f_x - g'_x \lambda, \quad \lambda(T) = 0.$$

Except for the inequality constraints $0 \leq \varphi^j \leq 1$, our problem would be a special case of the classical problem of LAGRANGE. The $\lambda_i(t)$ correspond to the Lagrangian multipliers ⁽³⁾.

As pointed out by A. S. MENGEL, if one defines

$$F(x, \varphi, \dot{x}, \lambda) = f(x, \varphi) - (\dot{x} - g(x, \varphi), \lambda),$$

the necessary conditions become

$$(10.12) \quad \begin{aligned} F_{x^i} &= \frac{d}{dt} F_{\dot{x}^i}, & i &= 1, 2, \dots, n \\ F_{\varphi^j} &\geq 0 & \text{whenever } \varphi^j &= 1 \\ &= 0 & \text{» } 0 < \varphi^j < 1 \\ &\leq 0 & \text{» } \varphi^j &= 0, \quad j = 1, 2, \dots, m. \end{aligned}$$

This says, in effect, that F satisfies the first necessary condition for a maximum, subject only to the constraints $0 \leq \varphi^j \leq 1$ and the end conditions.

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⁽³⁾ An account of the BOLZA problem, which includes the LAGRANGE problem, is to be found in BLISS [3]. The problem of LAGRANGE with inequalities as side conditions has been examined by PENNISI [8] (see also references cited there). The overlap between the present paper and [8] seems to be small, since the objectives and assumptions made are considerably different.