On characterization of probability distributions by means of independent statistics.

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Summary. - *The probability density functions $f_k(x_k) = A_k |x_k|^{p_k-1} e^{-a_k \psi_k(x_k)}$ of independent random variables $x_0, x_1, \dots, x_n$, are characterized by independence of two functions of them.*

TAMHANKAR [3] gave a characterization of the normal distribution symmetric about the origin by means of independent statistics. KOTLARSKI [2] characterized the gamma distribution in a similar way. In this paper the same method is used to characterize other distributions, where the preceding two cases are particular ones.

Let $x_0, x_1, \dots, x_n$ ($n \geq 1$) be independent real random variables (r.v.-s) having probability density functions (p.d.f.-s) f_k satisfying the following conditions:

- (A₁) $f_k(x) > 0$ for $x \in \mathcal{A}_k$, where $\mathcal{A}_k = (-\infty, 0)$ or $(0, +\infty)$
or $(-\infty, 0) \cup (0, +\infty)$; $f_k(x) = 0$ for $x \notin \mathcal{A}_k$;
if $\mathcal{A}_k = (-\infty, 0) \cup (0, +\infty)$ then f_k is even;
 f_k is continuous on $\mathcal{A}_k$;

- (A₂) there exist real numbers p_k such that the following limits exist

$$(1) \quad \lim_{x \rightarrow 0} \frac{f_k(x)}{|x|^{p_k-1}} = A_k > 0.$$

Denote the following transformation

$$(2) \quad \begin{cases} x_0 = y \cdot h_0(y_1, \dots, y_n) \\ x_1 = y \cdot h_1(y_1, \dots, y_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n = y \cdot h_n(y_1, \dots, y_n) \end{cases}$$

(*) Entrata in Redazione il 12 aprile 1969.

for $x_k \in \mathcal{A}_k$, $k = 0, 1, \dots, n$, $y \in \mathcal{B}_0 = (0, \infty)$, $(y_1, \dots, y_n) \in \mathcal{B}$, where $\mathcal{B}$ is a set in the n -dimensional space. Assume that the functions h_k on $\mathcal{B}$ are taken in such a way, that there is possible the change of variables in $(n+1)$ -dimensional integrals, particularly that the JACOBIAN of (2) exists and does not vanish for $y \in \mathcal{B}_0$, $(y_1, \dots, y_n) \in \mathcal{B}$.

Assume further that there exist functions $\Psi(y)$, $y \in \mathcal{B}_0$ transforming $\mathcal{B}_0$ onto $\mathcal{C} = (0, \infty)$ and $\Psi_k(x)$, $x \in \mathcal{A}_k$, $k = 0, 1, \dots, n$ transforming $\mathcal{A}_k$ onto $\mathcal{C}$, satisfying the following conditions:

- (B₁) Ψ is strictly increasing on $\mathcal{B}_0$,
 if $(-\infty, 0) \subset \mathcal{A}_k$ then Ψ_k is decreasing on it,
 if $(0, \infty) \subset \mathcal{A}_k$ then Ψ_k is increasing on it,
 if $\mathcal{A}_k = (-\infty, 0) \cup (0, \infty)$ then Ψ_k is even;

(B₂) there exist limits

$$(3) \quad \lim_{y \rightarrow 0} \Psi(y) = 0, \quad \lim_{x \rightarrow 0} \Psi_k(x) = 0$$

$$(B_3) \quad \Psi(y) = \Psi_0(x_0) + \Psi_1(x_1) + \dots + \Psi_n(x_n),$$

$$x_k \in \mathcal{A}_k, \quad k = 0, 1, \dots, n, \quad y \in \mathcal{B}_0;$$

(B₄) the functions

$$(4) \quad f_k(x) = \begin{cases} A_k |x|^{p_k-1} e^{-a\Psi_k(x)} & x \in \mathcal{A}_k \\ 0 & x \notin \mathcal{A}_k \end{cases}$$

where p_k and A_k are given in (A₂), are p.d.f.-s.

THEOREM. - The necessary and sufficient condition for the r.v.-s x_k ($k = 0, 1, \dots, n$) to be distributed according to the p.d.f.-s (4) is that

(C) y and $(y_1, \dots, y_n)$ are independent.

PROOF. - The $(n+1)$ -dimensional r.v. $(y, y_1, \dots, y_n)$ has its p.d.f. $g(y, y_1, \dots, y_n)$ given on $\mathcal{B}^* = \mathcal{B}_0 \times \mathcal{B}$ by the formula

$$(5) \quad g(y, y_1, \dots, y_n) = f_0(x_0) \cdot f_1(x_1) \dots f_n(x_n) \cdot |J|$$

where $(y, y_1, \dots, y_n)$ on $\mathfrak{B}^*$ and $(x_0, x_1, \dots, x_n)$ on $\mathfrak{A}_0 \times \mathfrak{A}_1 \times \dots \times \mathfrak{A}_n$ are connected by (2), and where the JACOBIAN of (2) is

$$(6) \quad J = \begin{vmatrix} h_0, y \cdot \frac{\partial h_0}{\partial y_1}, \dots, y \cdot \frac{\partial h_0}{\partial y_n} \\ h_1, y \cdot \frac{\partial h_1}{\partial y_1}, \dots, y \cdot \frac{\partial h_1}{\partial y_n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ h_n, y \cdot \frac{\partial h_n}{\partial y_1}, \dots, y \cdot \frac{\partial h_n}{\partial y_n} \end{vmatrix} = y^n \cdot H(y_1, \dots, y_n)$$

$y \in \mathfrak{B}_0,$

$(y_1, \dots, y_n) \in \mathfrak{B},$

where the function H depends only on $(y_1, \dots, y_n)$ but not on y . From the other hand, because of the condition (C) the p.d.f. of $(y, y_1, \dots, y_n)$ may be written in form

$$(7) \quad g(y, y_1, \dots, y_n) = G_0(y) \cdot G(y_1, \dots, y_n) \quad y \in \mathfrak{B}_0$$

$(y_1, \dots, y_n) \in \mathfrak{B}$

where G_0 and G are the p.d.f.s of y and $(y_1, \dots, y_n)$ respectively.

Because the left hand sides of (5) and (7) are equal, then the right hand sides are equal as well, and we may write the following functional equation

$$(8) \quad f_0(x_0) f_1(x_1) \cdot \dots \cdot f_n(x_n) = \gamma_0(y) \cdot \gamma(y_1, \dots, y_n)$$

$$x_k \in \mathfrak{A}_k, y \in \mathfrak{B}_0, (y_1, \dots, y_n) \in \mathfrak{B}$$

where f_k are unknown functions satisfying the conditions (A_1) , (A_2) , and where γ_0 and γ are given by

$$(9) \quad \begin{cases} \gamma_0(y) = \frac{G_0(y)}{y^n}, y \in \mathfrak{B}_0, \\ \gamma(y_1, \dots, y_n) = \frac{G(y_1, \dots, y_n)}{H(y_1, \dots, y_n)}, (y_1, \dots, y_n) \in \mathfrak{B}. \end{cases}$$

In order to solve the equation (8) let us divide both its sides by $\prod_{k=0}^n |x_k|^{p_k-1}$; using (2) we get

$$(10) \quad \varphi_0(x_0) \cdot \varphi_1(x_1) \cdot \dots \cdot \varphi_n(x_n) = \gamma_0^*(y) \cdot \gamma^*(y_1, \dots, y_n)$$

$$x_k \in \mathcal{A}_k, y \in \mathcal{B}_0, (y_1, \dots, y_n) \in \mathcal{B}$$

where

$$(11) \quad \left\{ \begin{array}{l} \varphi_k(x) = \frac{f_k(x)}{|x|^{p_k-1}}, \quad x \in \mathcal{A}_k \\ \gamma_0^*(y) = \frac{\gamma_0(y)}{y_{\sum_{k=0}^n p_k - n - 1}}, \quad y \in \mathcal{B}_0 \\ \gamma^*(y_1, \dots, y_n) = \frac{\gamma(y_1, \dots, y_n)}{\prod_{k=0}^n |h_k(y_1, \dots, y_n)|^{p_k-1}} \quad (y_1, \dots, y_n) \in \mathcal{B} \end{array} \right.$$

Putting $y \rightarrow 0$ and using (B₁), (B₂) and (B₃) we get also $x_k \rightarrow 0$. Putting this into (10) we get

$$(12) \quad \varphi_0(0) \cdot \varphi_1(0) \cdot \dots \cdot \varphi_n(0) = \gamma_0^*(0) \cdot \gamma_1(y_1, \dots, y_n) \quad (y_1, \dots, y_n) \in \mathcal{B}$$

where according to (1) and (11)

$$(13) \quad \left\{ \begin{array}{l} \varphi_k(0) = \lim_{x \rightarrow 0} \varphi_k(x) = \lim_{x \rightarrow 0} \frac{f_k(x)}{|x|^{p_k-1}} = A_k, \quad x \in \mathcal{A}_k \\ \gamma_0^*(0) = \lim_{y \rightarrow 0} \gamma_0^*(y) \quad y \in \mathcal{B}_0. \end{array} \right.$$

Dividing sides by sides (10) and (12) we get

$$(14) \quad \varphi_0^*(x_0) \cdot \varphi_1^*(x_1) \cdot \dots \cdot \varphi_n^*(x_n) = \gamma_0^{**}(y) \quad x_k \in \mathcal{A}_k, \quad y \in \mathcal{B}_0$$

where

$$(15) \quad \left\{ \begin{array}{l} \varphi_k^*(x_k) = \frac{\varphi_k(x_k)}{\varphi_k(0)}, \quad x_k \in \mathcal{A}_k \\ \gamma_0^{**}(y) = \frac{\gamma_0^*(y)}{\gamma_0^*(0)} \quad y \in \mathcal{B}_0. \end{array} \right.$$

Let us put

$$(16) \quad \begin{cases} u_k = \Psi_k(x_k), & \varphi_k^*(x_k) = \lambda_k(\Psi_k(x_k)) & x_k \in \mathfrak{A}_k, u_k \in \mathfrak{C} \\ v = \Psi(y), & \gamma_0^{**}(y) = \lambda(\Psi(y)) & y \in \mathfrak{B}_0, v \in \mathfrak{C} \end{cases}$$

then the equation (14) becomes

$$(17) \quad \lambda_0(u_0) \cdot \lambda_1(u_1) \cdot \dots \cdot \lambda_n(u_n) = \lambda(v), \quad u_k \in \mathfrak{C}, v \in \mathfrak{C}.$$

Putting (16) into (B₃) we get it as

$$(18) \quad v = u_0 + u_1 + \dots + u_n, \quad u_k \in \mathfrak{C}, v \in \mathfrak{C}$$

that is why the equation (17) meets the form

$$(19) \quad \prod_{k=0}^n \lambda_k(u_k) = \lambda\left(\sum_{k=0}^n u_k\right) \quad u_k \in \mathfrak{C}$$

where λ_k are continuous and $\lambda_k(0) = 1$. This is the known PEXIDER equation (see [1], p. 141). Putting $u_j \rightarrow 0$ for $j \neq k$ we get

$$(20) \quad \lambda_k(u_k) = \lambda(u_k) \quad u_k \in \mathfrak{C}.$$

Putting (20) into (19) we get the equation

$$(21) \quad \prod_{k=0}^n \lambda(u_k) = \lambda\left(\sum_{k=0}^n u_k\right) \quad u_k \in \mathfrak{C}$$

for the only function λ on $\mathfrak{C}$, continuous, and for which $\lambda(0) = 1$. This is the known CAUCHY equation (see [1], p. 31).

Its solution is

$$(22) \quad \lambda(u) = e^{-au} \quad u \in \mathfrak{C}$$

where a is a real constant. Putting (22) into (20) we get

$$(23) \quad \lambda_k(u) = e^{-au} \quad u \in \mathfrak{C}.$$

Putting (23) into (16) we get

$$(24) \quad \varphi_k^*(x) = e^{-a \Psi_k(x)} \quad x \in \mathfrak{A}_k.$$

Putting (24) into (15) we get

$$(25) \quad \varphi_k(x) = A_k \cdot e^{-a \Psi_k(x)} \quad x \in \mathfrak{A}_k.$$

Putting (25) into (11) we get

$$(26) \quad f_k(x) = A_k |x|^{p_k-1} e^{-a \Psi_k(x)} \quad x \in \mathcal{A}_k.$$

This ends the proof of the sufficient condition of the theorem.

The proof of the necessary condition of the theorem is obvious by substitution (4) and (6) into (5), using (2) and noting that the right side of (5) becomes the right side of (7).

EXAMPLE 1. - Put

$$(27) \quad \left\{ \begin{array}{ll} x_0 = y \sin y_1 & y \in \mathcal{B}_0, \\ x_1 = y \cos y_1 \sin y_2 & (y_1, \dots, y_n) \in \mathcal{B} \text{ where} \\ x_2 = y \cos y_1 \cos y_2 \sin y_3 & \mathcal{B} = \{(y_1, \dots, y_n); \frac{-\pi}{2} < y_k < \frac{\pi}{2}, \\ \dots & k = 1, 2, \dots, n-1, 0 < y_n < 2\pi\} \\ \dots & \\ x_{n-1} = y \cos y_1 \cos y_2 \dots \cos y_{n-1} \sin y_n \\ x_n = y \cos y_1 \cos y_2 \dots \cos y_{n-1} \cos y_n \end{array} \right.$$

We see that the condition (B₃) becomes $x_0^2 + x_1^2 + \dots + x_n^2 = y^2$. That is why $\Psi(y) = y^2$, $\Psi_k(x_k) = x_k^2$.

The independency of y and $(y_1, \dots, y_n)$ is the necessary and sufficient condition for x_k to be normally distributed with zero mean and common standard deviation (see TAMHANKAR [3]).

EXAMPLE 2. - Put

$$\mathcal{A}_k = (-\infty, 0) \cup (0, \infty), \quad p_k = 1, \quad \Psi(y) = \log(1 + y^2),$$

$$\Psi_k(x_k) = \log(1 + x_k^2) \quad (k = 0, 1, \dots, n).$$

Then we get the characterization of the generalized CAUCHY distribution given by the p.d.f.

$$(28) \quad f(x) = \frac{A}{(1 + x^2)^\alpha}, \quad x \in \mathcal{A}_k$$

where α is a positive constant.

EXAMPLE 3. - Put

$$\mathcal{A}_k = (0, \infty), \quad k = 0, 1, \dots, n,$$

$$(29) \quad \left\{ \begin{array}{ll} x_0 = y(1 - y_1 - \dots - y_n) & y \in \mathcal{B}_0 \\ x_1 = yy_1 & (y_1, \dots, y_n) \in \mathcal{B} \text{ where} \\ x_2 = yy_2 & \mathcal{B} = \{(y_1, \dots, y_n); y_k > 0, k = 1, \dots, n, \\ \cdot \cdot \cdot \cdot & y_1 + \dots + y_n < 1\}. \\ \cdot \cdot \cdot \cdot & \\ x_n = yy_n & \end{array} \right.$$

We see that the condition (B₃) becomes $y = x_0 + x_1 + \dots + x_n$. That is why $\Psi(y) = y$, $\Psi_k(x_k) = x_k$. The independency of y and $(y_1, \dots, y_n)$ is the necessary and sufficient condition for x_k to have the gamma distribution given by the density

$$(30) \quad f_k(x_k) = \begin{cases} \frac{a^{p_k}}{\Gamma(p_k)} x_k^{p_k-1} e^{-ax_k} & \text{for } x_k \in \mathcal{A}_k \\ 0 & \text{otherwise,} \end{cases}$$

where a and p_k are positive constants. (See KOTLARSKI [2]).

EXAMPLE 4. - Put

$$\mathcal{A}_k = (0, \infty), \quad \Psi(y) = \log(1 + y), \quad \Psi_k(x) = \log(1 + x), \quad k = 0, 1, \dots, n.$$

Then we get the characterization of the beta distribution of the second kind given by the density.

$$f_k(x) = \begin{cases} \frac{1}{B(p_k, a - p_k)} \frac{x^{p_k-1}}{(1 + x)^a} & x \in \mathcal{A}_k \\ 0 & \text{otherwise,} \end{cases}$$

where $p_k > 0$ and $a > \max(p_0, p_1, \dots, p_n)$.

REFERENCES

- [1] J. ACZÉL, *Lectures on functional equations and their applications*, Academic Press New York and London 1966.
 - [2] I. I. KOTLARSKI, *Una caratterizzazione della distribuzione gamma per mezzo di statistiche indipendenti*, Rendiconti di matematica e delle sue applicazioni, 1969, (3-4) Vol. 2, Serie VI, pp. 1-5.
 - [3] M. V. TAMHANKAR, *A characterization of normality*, The Annals of Mathematical Statistics, Vol. 38 No. 6, December 1967, pp. 1924-1927.
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