Commutators of two operators one of which is unbounded and semi-normal.

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Summary. - *The concept of semi-normality and hyponormality are defined for unbounded operators in Hilbert Space and Commutators $AB-BA$, with A unbounded and semi-normal and with B bounded, are studied. Four theorems are proved. Two of them are generalizations of results of C.R. Putman.*

1. - Introduction.

In this paper, problems concerning commutators of (Linear not necessarily bounded) operators defined in a separable infinite dimensional Hilbert Space H will be treated (**).

The domain of definition and the range of an operator A will be denoted by $\mathfrak{D}_A$ and $\mathfrak{R}_A$, respectively. The fact that B is an extension of A will be denoted by $A \subset B$. The bounded operators will be considered as defined on the whole space H .

We quote a theorem of C.R. PUTNAM concerning bounded operators.

(a) If A is a semi-normal bounded operator and B and C are bounded operators such that

$$AB - BA = C$$

then

$$\inf_{\|x\|=1} |(Cx, x)| = 0$$

(see [2] p. 128).

In particular, this theorem is *true* when A is hyponormal or normal.

We recall that a bounded operator A is called hyponormal if $\|A^*x\| \leq \|Ax\|$ and is called semi-normal when A or A^* is hyponormal. In section 2, the notions of hyponormality and semi-normality for linear, not necessarily bounded, operators will be defined and an example of a not normal semi-normal unbounded operator will be given.

In section 3, theorem (a) will be extended to various cases in which A and C are not necessarily bounded. In Theorem 1 the case in which A^* is

(*) Entrata in Redazione il 5 dicembre 1968.

(**) This paper forms part of a thesis in partial fulfillment of the requirements for the degree of Doctor of Science at the Technion-Israel Institute of Technology. The author wishes to thank Professor M. REICHAU for his help in preparing the paper.

hyponormal will be considered. Theorem 1 will be used in the proof of Theorem 2 which is a generalization of Theorem (a). The case in which A is normal will be considered in Theorem 3.

Finally, Theorem 4 is a generalization of theorem 2.10.1 in [3].

2. - Now we define the notions of hyponormality and semi-normality for not necessarily bounded operators.

DEFINITION 1. - An operator A is called hyponormal if it is closed, densely defined and satisfies the conditions

$$\mathfrak{D}_A = \mathfrak{D}_{A^*},$$

$$\|A^*x\| \leq \|Ax\| \quad \text{for } x \in \mathfrak{D}_A.$$

DEFINITION 2. - An operator A is called semi-normal if it is closed, densely defined and if A or A^* is hyponormal.

It is clear that every normal operator is hyponormal and therefore semi-normal. Moreover the above definitions are extensions of the definitions of hyponormality and semi-normality for bounded operators. The following is a simple example of a semi-normal unbounded operator which is not normal. Let $H = l_2$ and let A be the infinite matrix.

$$(1) \quad (a_{ij})_{i,j=1}^{\infty} \quad \text{where}$$

$$a_{i,i+1} = i \quad \text{for } i = 1, 2, \dots \text{ and } a_{ij} = 0 \quad \text{for } j \neq i+1$$

We will show that A^* is hyponormal. It is clear that A is closed and densely defined. It remains therefore to show that $\mathfrak{D}_A = \mathfrak{D}_{A^*}$ and $\|Ax\| \leq \|A^*x\|$ for each x in $\mathfrak{D}_A$ and that there exists vector in $\mathfrak{D}_A$ for which $\|Ax\| < \|A^*x\|$. We have

$$\mathfrak{D}_A = \{x = (\xi_1, \xi_2, \dots); \sum_{i=2}^{\infty} (i-1)^2 |\xi_i|^2 < \infty\}$$

$$\mathfrak{D}_{A^*} = \{x = (\xi_1, \xi_2, \dots); \sum_{i=1}^{\infty} i^2 |\xi_i|^2 < \infty\}.$$

For each $x \in \mathfrak{D}_{A^*}$ $x \neq 0$ we have

$$\|Ax\|^2 = \sum_{i=2}^{\infty} (i-1)^2 |\xi_i|^2 < \sum_{i=1}^{\infty} i^2 |\xi_i|^2 = \|A^*x\|^2.$$

Moreover, since $i^2 \leq 2^2(i-1)^2$ for $i \geq 2$, the inequality

$$\sum_{i=2}^{\infty} (i-1)^2 |\xi_i|^2 < \infty \quad \text{implies} \quad \sum_{i=1}^{\infty} i^2 |\xi_i|^2 < \infty.$$

The semi-normality of A is proved.

3. - In the following Theorem 1 $\text{sp } A$ stands for «the spectrum of A ».

THEOREM 1. - Let A be a closed operator whose adjoint is hyponormal, B a bounded operator and C such that

$$(2) \quad AB - BA \leq C.$$

If $\text{sp } A$ is a non-trivial subset of the complex plane (i.e. it is not the whole plane and not the empty set) and if

$$(3) \quad B(\mathfrak{D}_A) \subset \mathfrak{D}_A$$

then

$$\inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| = 0$$

PROOF. - Let $\mathfrak{F}$ be the complex plane. By assumption $\text{sp } A$ is a non-trivial closed subset of $\mathfrak{F}$. Let α be a point of the boundary of $\text{sp } A$ and α_n a sequence in $\mathfrak{F} - \text{sp } A$ converging to α . By (2) and (3)

$$(AB - BA)x = Cx$$

for each x in $\mathfrak{D}_A$. Hence

$$(4) \quad (A - \alpha_n I)Bx - B(A - \alpha_n I)x = Cx$$

for each x in $\mathfrak{D}_A$ and for $n = 1, 2, \dots$. Since $\alpha_n \in \mathfrak{F} - \text{sp } A$, $A - \alpha_n I$ has a bounded inverse $(A - \alpha_n I)^{-1}$. Let y be a vector in H with $\|y\| = 1$. Since A is closed and A^* is hyponormal we have

$$x = [(A - \alpha_n I)^*]^{-1}y \in \mathfrak{D}_{A^*} = \mathfrak{D}_{A^{**}} = \mathfrak{D}_A.$$

This vector may therefore be substituted in (4) and we get

$$\begin{aligned} (A - \alpha_n I)B((A - \alpha_n I)^*)^{-1}y - B(A - \alpha_n I)((A - \alpha_n I)^*)^{-1}y = \\ = C((A - \alpha_n I)^*)^{-1}y. \end{aligned}$$

Applying $(A - \alpha_n I)^{-1}$ to both sides we obtain

$$(A - \alpha_n I)^{-1}C(A - \alpha_n I)^*y = B((A - \alpha_n I)^*)^{-1}y - (A - \alpha_n I)^{-1}B(A - \alpha_n I)((A - \alpha_n I)^*)^{-1}y.$$

Since, as easily seen, hyponormality of A^* implies hyponormality of $(A - \alpha_n I)^*$ it follows that

$$\begin{aligned} \|(A - \alpha_n I)^{-1}C((A - \alpha_n I)^*)^{-1}y\| &\leq \|B((A - \alpha_n I)^*)^{-1}y\| + \|(A - \alpha_n I)^{-1}B\| \\ \|(A - \alpha_n I)^{-1}C((A - \alpha_n I)^*)^{-1}y\| &\leq \|B\| \|((A - \alpha_n I)^*)^{-1}\| \|y\| + \|(A - \alpha_n I)^{-1}\| \|B\| \|y\|. \end{aligned}$$

Hence $(A - \alpha_n I)^{-1} C((A - \alpha_n I)^*)^{-1}$ is bounded and

$$(5) \quad \|(A - \alpha_n I)^{-1} C((A - \alpha_n I)^*)^{-1}\| \leq 2 \| (A - \alpha_n I)^{-1} \| \| B \|.$$

On the other hand, for each y , $\|y\| = 1$, $y \in H$

$$\begin{aligned} \|(A - \alpha_n I)^{-1} C((A - \alpha_n I)^*)^{-1} y\| &\geq |(A - \alpha_n I)^{-1} C((A - \alpha_n I)^*)^{-1} y, y| = \\ &= |C((A - \alpha_n I)^*)^{-1} y, (A - \alpha_n I)^{-1} y| = \frac{(C((A - \alpha_n I)^*)^{-1} y, (A - \alpha_n I)^{-1} y)}{\|(A - \alpha_n I)^{-1} y\|^2} \\ &\quad \|(A - \alpha_n I)^{-1} y\|^2. \end{aligned}$$

Hence

$$\|(A - \alpha_n I)^{-1} C((A - \alpha_n I)^*)^{-1} y\| \geq \inf_{\substack{x \neq 0 \\ x \in \mathfrak{D}_C}} \frac{|(Cx, x)|}{\|x\|^2} \|(A - \alpha_n I)^{-1} y\|^2$$

and

$$\begin{aligned} \|(A - \alpha_n I)^{-1} C((A - \alpha_n I)^*)^{-1}\| &\geq \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| \|(A - \alpha_n I)^{-1}\|^2 \\ &= \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| \|(A - \alpha_n I)^{-1}\|^2. \end{aligned}$$

Therefore, by (5) we have

$$(6) \quad \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| \leq 2 \|B\| \|(A - \alpha_n I)^{-1}\| = \frac{2 \|B\|}{\|(A - \alpha_n I)^{-1}\|}.$$

Since α_n is a sequence in $\mathfrak{S}$ -sp A converging to $\alpha \in \text{sp } A$ it follows that $\frac{1}{\alpha - \alpha_n} \in \text{sp } (A - \alpha_n I)^{-1}$ (see [1] p. 600). Hence $\frac{1}{|\alpha - \alpha_n|} \leq \|(A - \alpha_n I)^{-1}\|$.

This inequality and (6) imply

$$\inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| \leq 2 \|B\| |\alpha - \alpha_n|.$$

Since $\alpha_n - \alpha \rightarrow 0$ for $n \rightarrow \infty$ it follows that

$$\inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| = 0.$$

Theorem 1 is proved.

THEOREM 2. - Let A be a semi normal operator, B a bounded operator and C such that

$$AB - BA \in C.$$

If $\text{sp } A$ is a non-trivial subset of the complex plane and the conditions

$$(7) \quad B(\mathfrak{D}_A) \subset \mathfrak{D}_A$$

$$(8) \quad B^*(\mathfrak{D}_A) \subset \mathfrak{D}_A$$

are fulfilled, then

$$(9) \quad \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| = 0.$$

PROOF. - If A^* is hyponormal then the theorem follows from theorem 1. We will suppose, therefore that A is hyponormal. Let us define

$$C_1 = AB - BA \quad D = A^*B^* - B^*A^*.$$

Then

$$(10) \quad C_1 \in C \quad D \in C^*$$

(see [4] p. 300]. Since (8) holds and $\mathfrak{D}_A = \mathfrak{D}_{A^*}$ we have by theorem 1

$$(11) \quad \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_A \subset \mathfrak{D}_D}} |(Dx, x)| = 0.$$

By (10) we get

$$\inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_A}} |(Dx, x)| = \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_A}} |(C_1^*x, x)| = \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_A}} |(C_1x, x)| \geq \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)|$$

and the theorem follows from (11).

REMARK 1. - It is easy to find examples of normal unbounded operators A and bounded operators B satisfying the conditions of the theorem. Let us give now an example of an unbounded semi-normal not normal operator A having a non trivial spectrum and a bounded operator B satisfying the conditions (7) and (8). Let $H = K \oplus K$ where K is a Hilbert Space.

Take

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \quad B = \begin{pmatrix} M & N \\ S & T \end{pmatrix}$$

where P is a self adjoint unbounded operator in K , Q a bounded semi-normal operator on K which is not normal, M , N and S are bounded operators on K satisfying

$$M(\mathfrak{D}_p) \subset \mathfrak{D}_p \quad M^*(\mathfrak{D}_p) \subset \mathfrak{D}_p$$

$$\mathfrak{R}_N \subset \mathfrak{D}_p \quad \mathfrak{R}_{S^*} \subset \mathfrak{D}_p$$

and T is an arbitrary bounded operator on K . It is easily seen that the operator matrices A and B have the desired properties.

REMARK 2. - Without the assumption that $\text{sp } A$ is a non-trivial subset of the complex plane Theorem 2 is not true. In fact, let A be the semi-normal operator defined in (1) and $B = (b_{ij})_{i,j=1}$ where $b_{j+1,j} = 1$ for $j = 1, 2, \dots$ and $b_{ij} = 0$ for $i \neq j + 1$.

It is easily seen that every complex number is an eigenvalue of A .

The conditions (7) and (8) are satisfied since

$$\sum_{i=2}^{\infty} i^2 |\xi_i|^2 \leq \sum_{i=2}^{\infty} 2^2 (i-1)^2 |\xi_i|^2$$

and

$$\sum_{i=3}^{\infty} (i-2)^2 |\xi_i|^2 < \sum_{i=2}^{\infty} (i-1)^2 |\xi_i|^2.$$

However, one may show easily that $AB - BA \neq I$ where I denotes the identity operator which does not satisfy (9).

Let us note that in the above remark A is not a normal operator. In case that A is normal we have

THEOREM 3. - Let A be a normal operator, B a bounded operator and C such that

$$(12) \quad AB - BA = C.$$

If

$$(13) \quad B(\mathfrak{D}_A) \subset \mathfrak{D}_A$$

then

$$(14) \quad \inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(Cx, x)| = 0.$$

PROOF. - Since the spectrum of a normal operator is not empty the theorem follows from theorem 1 in case that $\text{sp } A$ is not the whole plane.

Let us suppose that $\text{sp } A$ is the whole plane. Let

$$A = \int \lambda dE_\lambda$$

be the spectral representation of A and let a_n be a sequence of positive numbers converging to zero. Define

$$L_n = \{\lambda; \lambda = c + id, \text{ where } c, d \in [0, a_n]\}$$

and

$$\Phi_n(\lambda) = \begin{cases} \lambda & \lambda \in L_n \\ 0 & \lambda \notin L_n. \end{cases}$$

Let

$$F_n = \int \Phi_n(\lambda) dE_\lambda$$

and

$$(15) \quad A_n = A - F_n = \int (\lambda - \Phi_n(\lambda)) dE_\lambda.$$

Since F_n is bounded $\mathfrak{D}_{A_n} = \mathfrak{D}_A$ for $n = 1, 2, \dots$. Therefore, (13) implies $B(\mathfrak{D}_{A_n}) \subset \mathfrak{D}_{A_n}$.

It is easily seen that $\text{sp } A_n$ is a nontrivial subset of the plane for $n = 1, 2, \dots$. Moreover, by (12) and (15) we have

$$A_n B - B A_n = (A - F_n) B - B(A - F_n) = AB - BA + BF_n - F_n B C_n$$

where

$$C_n = C + BF_n - F_n B.$$

Therefore, A_n , B and C_n satisfy the conditions of theorem 1. Hence

$$(16) \quad \inf_{\substack{\|x\| \\ x \in \mathfrak{D}_{C_n}}} |(C_n x, x)| = 0$$

for $n = 1, 2, \dots$.

Since $\|F_n\| \leq a_n \sqrt{2} \rightarrow 0$ for $n \rightarrow \infty$ it follows that $BF_n - F_n B \rightarrow 0$ when $n \rightarrow \infty$. Thus, for each x , $\|x\| = 1$, $x \in \mathfrak{D}_{C_n} = \mathfrak{D}_C$ $(C_n x, x) \rightarrow (C x, x)$ uniformly relative to x . Hence, by (16)

$$\inf_{\substack{\|x\|=1 \\ x \in \mathfrak{D}_C}} |(C x, x)| = 0$$

Theorem 3 is proved.

Note that if (13) is not satisfied Theorem 3 does not hold. This is shown by the following example:

Let $H = L_2[a, b]$, $A = i \frac{d}{dx}$, $\mathfrak{D}_A$ be the set of absolutely continuous complex functions f in $[a, b]$ with $f' \in L_2$ and $f(a) = f(b)$. Let $B = \Phi(x)$ where $\Phi(x)$ is any absolutely continuous complex function in $[a, b]$ satisfying

$$\Phi(a) \neq \Phi(b) \quad \Re \Phi'(x) > \alpha, \quad \Im \Phi'(x) > \beta$$

for two positive numbers α and β .

Here A is self adjoint and (12) holds for $C = i\Phi'$ and C does not satisfy (14).

In this example (13) is not satisfied i.e. the equality

$$(17) \quad (AB - BA)x = Cx$$

does not hold for each x in $\mathfrak{D}_A$.

However (17) holds on the set Ω of all functions of $\mathfrak{D}_A$ vanishing at a and at b , Ω is obviously a dense subset of H . We also remark that $A(\Omega)$ is not dense. The following theorem (some ideas of which are similar to those of theorem 2.10.1 in [3]) shows that the fact that $A(\Omega)$ is not dense is not incidental.

THEOREM 4. - Let A be a self-adjoint operator, B be a bounded operator and let C be such that (17) holds on a linear subset Ω of H . If

$$(18) \quad \inf_{\|x\|=1, x \in \mathfrak{D}_C} |(Cx, x)| = \alpha \neq 0.$$

Then $A(\Omega)$ is not dense in H .

PROOF. - Suppose to the contrary that $A(\Omega)$ is dense in H . By (17) and (18) we have

$$\begin{aligned} \alpha \|x\|^2 &\leq |(Cx, x)| = |(ABx, x) - (BAx, x)| \leq |(Bx, Ax)| + |(Ax, B^*x)| \leq \\ &\leq 2\|B\| \|Ax\| \|x\| \end{aligned}$$

for each x in Ω . Hence

$$(19) \quad \|x\| \leq \frac{2\|B\|}{\alpha} \|Ax\|.$$

Let $y \in H$. Since $A(\Omega)$ is dense there exists a sequence $\{x_n\}$ in $A(\Omega)$ such that $Ax_n \rightarrow y$. It follows, by (19) that there exists a vector x in H such that $x_n \rightarrow x$. Since A is closed $x \in \mathfrak{D}_A$ and $Ax = y$. Hence $\mathfrak{R}_A = H$. Therefore, A has a bounded self-adjoint inverse A^{-1} .

Let $h \in A(\Omega)$. Putting in (17) $x = A^{-1}h$ we get

$$ABA^{-1}h - Bh = CA^{-1}h.$$

Thus

$$(20) \quad BA^{-1}h - A^{-1}Bh = A^{-1}CA^{-1}h.$$

Let $\mu \neq 0$, $\mu \in \text{sp } A^{-1}$. Since A^{-1} is bounded and self-adjoint there exists a sequence y_n in H with $\|y_n\| = 1$ such that

$$(21) \quad A^{-1}y_n - \mu y_n \rightarrow 0.$$

Let

$$1 > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$$

be a sequence converging to zero. Since $A(\Omega)$ is dense there exists a sequence t_n in $A(\Omega)$ such that

$$(22) \quad \|t_n - y_n\| < \varepsilon_n.$$

Thus, by (21)

$$(23) \quad A^{-1}t_n - \mu t_n \rightarrow 0.$$

Therefore by $\|y_n\| = 1$ and by (22)

$$(24) \quad \|t_n\| < 1 + \varepsilon_n < 2$$

and

$$(25) \quad \|t_n\| > 1 - \varepsilon_n > 1 - \varepsilon_1 > 0.$$

By (24)

$$\begin{aligned} |(BA^{-1}t_n, t_n) - (A^{-1}Bt_n, t_n)| &= |(BA^{-1}t_n, t_n) - (Bt_n, A^{-1}t_n)| = \\ &= |((BA^{-1} - \mu B)t_n, t_n) - (Bt_n, A^{-1}t_n - \mu t_n)| \leq \\ &\leq 2\|BA^{-1} - \mu A\| \|t_n\| + 2\|B\| \|A^{-1}t_n - \mu t_n\|. \end{aligned}$$

By (23) both terms of the last expression tend to zero. As $n \rightarrow \infty$. Hence

$$(BA^{-1}t_n, t_n) - (A^{-1}Bt_n, t_n) \rightarrow 0$$

as $n \rightarrow \infty$ and by (20)

$$(CA^{-1}t_n, A^{-1}t_n) = (A^{-1}CA^{-1}t_n, t_n) \rightarrow 0.$$

But by (18) we have

$$\alpha \|A^{-1}t_n\|^2 \leq |(CA^{-1}t_n, A^{-1}t_n)|$$

for $n = 1, 2 \dots$. Hence $A^{-1}t_n \rightarrow 0$ and by (23) $t_n \rightarrow 0$. This contradicts (25).
The theorem is proved.

REMARK. - Theorem 4 is a generalization of theorem 2.10.1 in [3].

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