On solutions of algebraic differential equations whose coefficients are entire functions of finite order (*).

STEVEN BANK (Urbana, U.S.A.) (**)

Summary. - *We determine bounds for the growth of entire solutions of first order equations whose coefficients are entire functions of finite order.*

1. - Introduction.

In this paper we investigate the rate of growth of entire functions which are solutions of first order algebraic differential equations whose coefficients are arbitrary entire functions of finite order (i.e. equations of the form

$$\Omega(z,\ y,\ dy/dz) = 0, \quad \text{where } \Omega(z,\ y,\ dy/dz) = \sum_{k,\, j \geq 0} f_{kj}(z) y^k (dy/dz)^j$$

is a polynomial in y and dy/dz, whose coefficients $f_{kj}(z)$ are entire functions of finite order).

In [4], VALIRON treated the special case where the coefficients $f_{kj}(z)$ are polynomials, and in this case, it was shown that any entire solution must be of finite order. (In fact, VALIRON showed, in the case of polynomial coefficients, that for an entire transcendental solution $g(z)$, with maximum modulus $M(r;\ g)$, there are positive constants k and b, with b rational, such that $\lim_{r \to +\infty} (\log M(r;\ g)/kr^b) = 1$).

In the general case where the coefficients $f_{kj}(z)$ are arbitrary entire functions of finite order, clearly such equations can possess entire solutions of infinite order (for example, $\exp(\exp z)$, $\sin(\cos z)$), but our main result here (§ 2 below) shows that the growth of an entire solution $h(z)$ of such an equation in the general case, is restricted in the following natural way: There exist positive constants r_0 and σ such that $M(r;\ h) \leq \exp(\exp r^{\sigma})$ for all $r > r_0$. In fact we show that for any real number λ which is greater than

(*) Entrata in Redazione il giorno 8 gennaio 1969.

(**) This research was supported in part by the National Science Foundation (GP 7374).

the order of each coefficient $f_{kj}(z)$, the inequality $M(r; h) \leq \exp(\exp r^{\lambda})$ holds for all r greater than some number $r_0(\lambda)$. In § 4, we show the necessity of λ being *greater* than the order of each coefficient, by constructing a simple example where if λ is taken to be equal to the maximum order of the coefficients, then for no constant K will the inequality $M(r; h) \leq \exp(\exp(Kr^{\lambda}))$ hold for all sufficiently large r.

The principal tools used in the proof of the main result consist of:

(i) the WIMAN-VALIRON theory of the maximum term ([6, 7, 8] or for a complete discussion [3, Chapt. 9]);

(ii) Certain results ([6; pp. 239-240]) concerning the relation between the maximum modulus and the maximum term of an entire function, and

(iii) Certain results from the theory of entire functions of finite order.

2. - We now state our main result:

THEOREM. - Let

$$\Omega(z, y, dy/dz) = \sum_{k, j \geq 0} f_{kj}(z) y^k (dy/dz)^j$$

be a polynomial in y and dy/dz, where each coefficient $f_{kj}(z)$ is an entire function of finite order. Let $h(z)$ be any entire function which satisfies $\Omega(z, h(z), h'(z)) \equiv 0$, and let $M(r; h) = \max_{|z|=r} |h(z)|$. Then for any real number λ which is greater than the order of each coefficient $f_{kj}(z)$, there exists a positive real number $r_0(\lambda)$ such that $M(r; h) \leq \exp(\exp r^{\lambda})$ for all $r > r_0(\lambda)$.

3. - PROOF OF THE THEOREM. - If $h(z)$ is a polynomial, clearly the result holds. Hence we may assume that

(1) h is an entire transcendental function. Let $\sum_{n=0}^{\infty} c_n z^n$ be the power series expansion of $h(z)$, and let $M(r) = \max_{|z|=r} |h(z)|$. For each $r \geq 0$, let $\nu(r)$ be the central index [1; p. 183] of h (i.e., $\nu(r)$ is the maximum j such that $|c_j| r^j = \max_{m \geq 0} |c_m| r^m$). Then in view of (1),

(2) $\nu(r)$ is an unbounded increasing function of r, and it is proved in [3; pp. 198, 210] (and also in [5; pp. 95, 103]) that there exists $\alpha \in (0, 1)$ such that if we exclude from the interval $(1, +\infty)$ an infinite sequence of exceptional finite open intervals (W_s, W'_s) for which

(3) $\sum_{s=1}^{\infty} (\log W'_s - \log W_s)$ converges,

and for which we may assume

(4) $W'_s < W_{s+1}$ for all s, and $\lim_{s\to\infty} W'_s = +\infty$, then in the remaining set $(1, +\infty) - E$, where $E = \bigcup_{s=1}^{\infty} (W_s, W'_s)$, the following are true: There exists a number $R_1 \geq 1$ such that for $r > R_1$ and $r \notin E$, we have

(5) $\log M(r) > c(\nu(r))^{\alpha}$, where c is a positive constant independent of r, and if z is any point on $|z| = r$ at which $|h(z)| = M(r)$, then

(6) $h'(z) = (\nu(r)/z)(1 + \varepsilon(z))h(z)$, where $|\varepsilon(z)| < (\nu(r))^{-\delta}$ for some fixed $\delta > 0$. (The elements, of $(1 + \infty) - E$ are called *ordinary* values of index α in [3, 4, 5]).

Now each coefficient $f_{kj}(z)$ is an entire function of finite order (and of course there are only finitely many non-zero f_{kj}). Let d be the maximum of the orders of the coefficients f_{kj}, and let λ be any number greater than d. Define,

(7) $b = (\lambda + d)/2$ and $\sigma = (\lambda - d)/5$. Thus,

(8) $b > d \geq 0$, $\sigma > 0$ and $b + 2\sigma < \lambda$.

Since $b > d$, clearly there exists $R_2 > 1$ such that when $r > R_2$,

(9) $|f_{kj}(z)| \leq \exp(r^b)$ on $|z| = r$ for all k, j.

Now let,

(10) $p = \max\{k + j : f_{kj} \not\equiv 0\}$ and $m = \max\{j : f_{p-j,j} \not\equiv 0\}$, and consider the coefficient $f_{p-m,m}(z)$. Let $a_1, a_2, \ldots$ be the non-zero roots (if any) of $f_{p-m,m}(z)$, and let D be the domain obtained by removing from the plane all the disks $|\zeta - a_n| < |a_n|^{-b}$. Then since b is greater than the order of $f_{p-m,m}(z)$ (by (8)), it is proved in [2; p. 328] that

(11) $\sum_{n\geq 1} |a_n|^{-b}$ converges, and it is proved in [2; p. 336] (by using the representation for $f_{p-m,m}$ given by the Hadamard Factorization Theorem) that there exists $R_3 > 1$ such that,

(12) $|f_{p-m,m}(z)| \geq \exp(-r^b)$ for $z \in D$ and $|z| = r > R_3$. Thus if we let F be the union of all the open intervals, $(|a_n| - |a_n|^{-b}, |a_n| + |a_n|^{-b})$ for $n = 1, 2, \ldots$, then in view of (12),

(13) $|f_{p-m,m}(z)| \geq \exp(-r^b)$ on $|z| = r$ if $r > R_3$ and $r \notin F$. In view of (11), it is clear that the set F can be written as the union of a sequence of finite open intervals (T_s, T'_s) such that,

(14) $T'_s < T_{s+1}$ for all s, and $\sum_{s\geq 1} (T'_s - T_s)$ converges.

In view of (3), (4), and (14), clearly we may write $E \cup F$ as the union of a sequence of finite open intervals,

(15) $E \cup F = \bigcup_{s=1}^{\infty} (U_s, U'_s)$, where

(16) $U'_s < U_{s+1}$ for all s and $\lim_{s \to \infty} U'_s = +\infty$

and

(17) $\sum_{s=1}^{\infty} (\log U'_s - \log U_s)$ converges.

Now define

(18) $A = \{r \mid r > 1 \text{ and } \nu(r) > \exp(r^{b+\sigma})\}$ (where σ is as in (7)).

We now prove,

LEMMA A. - There exists a number $r^* > 1$ such that $A \cap (r^*, +\infty) \subset E \cup F$.

PROOF. - Assume the contrary. Then there exists a sequence of distinct values of r in $(1, +\infty$ tending to $+\infty$ such that

(19) $r \in A$ but $r \notin E \cup F$.

Let B be the set of values of r comprising this sequence. Now $h(z)$ satisfies the relation,

(20) $\sum f_{kj}(z)(h(z))^k (h'(z))^j \equiv 0$.

Let $r \in B$ and let z be a point on $|z| = r$ at which $|h(z))| = M(r)$. Then clearly $h(z) \neq 0$, and so by dividing equation (20) by $(h(z))^p$ (where p is as in (10)), we can write equation (20) in the form,

(21) $\sum_{j=0}^{m} f_{p-j,j}(z)(h'(z)/h(z))^j = - \sum_{k+j<p} f_{kj}(z((h'(z)/h(z))^j (h(z))^{k+j-p}$.

We will denote the left side of (21) by $\Lambda(z)$, and the right side by $\Phi(z)$.
We now assert that there exists a real number $r' > R_3$ such that if $r \in B$ and $r > r'$, then

(22) $|\Phi(z)| \leq (M(r))^{-1/2} \exp(r^b)$,

at each point of $|z| = r$ at which $|h(z)| = M(r)$.
To prove (22), we recall first from (19) that if $r \in B$ then $r \notin E$ and $r \notin F$. Since $\nu(r) \to +\infty$ as $r \to +\infty$ (by (2)), we see that $\varepsilon(z)$ (in (6)) tends to zero as $r \to +\infty$ in B. Since $M(r)$ also tends to $+\infty$, there exists $R_4 > 1$ such that for $r \in B$ and $r > R_4$, we have

(23) $M(r) > 1$, $\nu(r) > 1$ and $|\varepsilon(z)| < 1/2$,

at every point of $|z| = r$ at which $|h(z)| = M(r)$. Let $R_5 = \max_{1 \leq j \leq 4} R_j$ and let r be any element of B such that $r > R_5$.
Let z be any point on $|z| = r$ at which $|h(z)| = M(r)$. We refer to the right side of (21). If $k + j < p$. then $p - (k + j) \geq 1$ so $|h(z)|^{k+j-p} = (M(r))^{k+j-p} \leq \leq M(r))^{-1}$ (since $M(r) > 1$). Since $|\varepsilon(z)| < 1/2$ (by (23)), we have by (6) that $|h'(z)/h(z)| < 2\nu(r)/r < 2\nu(r)$ since $r > 1$. Thus, $|h'(z)/h(z)|^j < 2^p(\nu(r))^p$ if $k + j < p$ (since $j < p$ and $\nu(r) > 1$). In view of the above estimates and (9), it is clear that

(24) $\quad |\Phi(z)| \leq K(\nu(r))^p(M(r))^{-1} \exp(r^b)$

where K is a positive constant independent of r. Now since $r \in B$ and $r > R_5$, we have by (5) that $\nu(r) < (c^{-1} \log M(r))^{1/\alpha}$, and so from (24),

(25) $\quad |\Phi(z)| \leq \psi(r)(M(r))^{-1/2} \exp(r^b)$,

where $\psi(r) = K(c^{-1} \log M(r))^{p/\alpha} (M(r))^{-1/2}$. Since $M(r) \to +\infty$ as $r \to +\infty$, it is clear that $\psi(r) \to 0$ as $r \to +\infty$. Hence there exists $r' > R_5$ such that $\psi(r) < 1$ for $r > r'$. Thus (22) follows from (25).

We now consider $\Lambda(z)$ (i.e. the left side of (21)).

CASE I: $m = 0$. Then $\Lambda(z) = f_{p-m,\, m}(z)$. Since $\Lambda(z) = \Phi(z)$ (by (21)), we have by (13) and (22) that, $\exp(-r^b) \leq (M(r))^{-1/2} \exp(r^b)$ if $r \in B$ and $r > r'$. Thus,

(26) $\quad M(r) \leq \exp(4r^b)$ if $r \in B$ and $r > r'$.

But if $r \in B$ then $r \in A$ and so $\nu(r) > \exp(r^{b+\sigma})$ by (18). Hence by (5), $\log M(r) > c \exp(\alpha r^{b+\sigma})$, and so by (26),

(27) $\quad \exp(c \exp(\alpha r^{b+\sigma}) - 4r^b) \leq 1$ if $r \in B$ and $r > r'$. But since $c > 0$, $\alpha > 0$, $b > 0$, and $\sigma > 0$, it is clear that the left side of (27) tends to $+\infty$ as $r \to +\infty$. Thus (27) is impossible (since by our assumption (19), there exist r-values in B tending to $+\infty$). This contradiction proves Lemma A in the case $m = 0$.

CASE II: $m > 0$. By (23), if $r \in B$ and $r > r'$ then $|\varepsilon(z)| < 1/2$, and so at each point of $|z| = r$ at which $|h(z)| = M(r)$, we have by (6) that $|h'(z)/h(z)| \geq \geq (1 - |\varepsilon(z)|)\nu(r)/r \geq (2r)^{-1}\nu(r)$. But if $r \in B$ then $r \in A$ and so $\nu(r) > \exp(r^{b+\sigma})$. Thus if $r \in B$ and $r > r'$, then

(28) $\quad |h'(z)/h(z)| \geq (2r)^{-1} \exp(r^{b+\sigma})$

at each point of $|z| = r$ at which $|h(z)| = M(r)$.

We now assert that there exists $r^{\#} > r'$ such that if $r \in B$ and $r > r^{\#}$, then

(29) $\quad |\Lambda(z)| \geq \exp((m/2)r^{b+\sigma})$,

at each point of $|z| = r$ at which $|h(z)| = M(r)$.

To prove (29), we note first that $\Lambda(z)$ may be written in the form,

(30) $\Lambda(z) = f_{p-m,\, m}(z)(h'(z)/h(z))^m(1 + \sum_{j=0}^{m-1} \Psi_j(z))$,

where

(31) $\Psi_j(z) = (f_{p-j,\, j}(z)/f_{p-m,\, m}(z))(h'(z)/h(z))^{j-m}$

for $j = 0, 1, \ldots, m-1$. We consider the quotients $\Psi_j(z)$ at points on $|z| = r$ at which $|h(z)| = M(r)$, where $r \in B$ and $r > r'$. Now for $0 \leq j \leq m-1$, it follows from (28) that, $|h'(z)/h(z)|^{m-j} \geq (2r)^{-m} \exp(r^{b+\sigma})$, and so using (9) and (12), we obtain

(32) $|\Psi_j(z)| \leq (2r)^m \exp(2r^b - r^{b+\sigma})$.

Since $b > 0$ and $\sigma > 0$, it is clear that the right side of (32) tends to zero as $r \to +\infty$. Hence there exists $R_6 > r'$ such that for $r \in B$ and $r > R_6$, we have

(33) $|\Psi_j(z)| \leq 1/(m+1)$ for $j = 0, 1, \ldots, m-1$.

Now by (30), $|\Lambda(z)| \geq |f_{p-m,\, m}(z)|\, |h'(z)/h(z)|^m(1 - \sum_{j=0}^{m-1} |\Psi_j(z)|)$, and so by (13), (28) and (33), we obtain for $r \in B$ and $r > R_6$,

(34) $|\Lambda(z)| \geq (1/(m+1))(2r)^{-m} \exp(mr^{b+\sigma} - r^b)$,

at every point of $|z| = r$ at which $|h(z)| = M(r)$, Now clearly the function $\varphi(r) = (1/(m+1))(2r)^{-m} \exp((m/2)r^{b+\sigma} - r^b)$ tends to $+\infty$ as $r \to +\infty$, so there exists $r^{\#} > R_6$ such that $\varphi(r) > 1$ for $r > r^{\#}$. In view of (34) and the definition of $\varphi(r)$, we obtain (29).

Since $\Lambda(z) = \Phi(z)$ (by (21)), we have by (22) and (29) that if $r \in B$ and $r > r^{\#}$, then $\exp((m/2)r^{b+\sigma}) \leq (M(r))^{-1/2} \exp(r^b)$, and so $M(r) \leq \exp(2r^b - mr^{b+\sigma})$. Hence,

(35) $M(r) \leq 1$ if $r \in B$ and $r > \max\{r^{\#}, (2/m)^{1/\sigma}\}$.

But $M(r) \to +\infty$ as $r \to +\infty$, so (35) is impossible (since by our assumtpion (19) there exist r-values in B tending to $+\infty$). This contradiction proves Lemma A in Case II and so the proof of Lemma A is complete.

We now prove,

LEMMA B - There exists a real number $r_1 > 1$ such that $\nu(r) < \exp(2r^{b+\sigma})$ for all $r \geq r_1$.

PROOF. - By Lemma A, there exists $r^* > 1$ such that

(36) $A \cap (r^*, +\infty) \subset E \cup F$.

Now by (15), (16), and (17), $E \cup F$ is the union of a sequence of open intervals (U_s, U'_s) where $U'_s < U_{s+1}$, $\lim_{s\to\infty} U'_s = +\infty$ and $\sum_{s=1}^{\infty} \log (U'_s/U_s)$ converges. Since the series converges, $\lim_{s\to\infty} (U'_s/U_s) = 1$ and so $\lim_{s\to\infty} (U'_s/U_s)^{b+\sigma} = 1$. Since also $\lim_{s\to\infty} U'_s = +\infty$, clearly there exists s_0 such that

(37) $U'_s > r^*$ and $(U'_s/U_s)^{b+\sigma} < 2$ for all $s \geq s_0$.

We will show that if r_1 is taken to be U'_{s_0} then the conclusion of the lemma holds. Let $r \geq r_1$.

If $r \notin A$, then (by (18)), $\nu(r) \leq \exp(r^{b+\sigma}) < \exp(2r^{b+\sigma})$.

If $r \in A$, then by (36), $r \in E \cup F$. Hence for some q, $r \in (U_q, U'_q)$. Since $U'_{s_0} \leq r < U'_q$, clearly (by (16)), $q > s_0$, so (37) holds for q. Now the endpoint U'_q is clearly not in $E \cup F$, so by (36), $U'_q \notin A$. Hence $\nu(U'_q) \leq \exp((U'_q)^{b+\sigma})$, and so by (37) (for $s = q$), we have $\nu(U'_q) \leq \exp(2(U_q)^{b+\sigma})$. Since ν is increasing, we thus have $\nu(r) \leq \nu(U'_q) \leq \exp(2(U_q)^{b+\sigma})$. But $U_q < r$ and so we obtain $\nu(r) < \exp(2r^{b+\sigma})$ which proves Lemma B completely.

Now for each $r \geq 0$, let $\mu(r)$ be the maximum term [3, p. 193] of h (i.e. if $\sum_{n=0}^{\infty} c_n z^n$ is the power series expansion of h, then $\mu(r) = \max_{n\geq 0} |c_n| r^n$). It is proved in [3; p. 195] that for any $r_0 > 0$, we have

(38) $$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} (\nu(x)/x)dx.$$

By Lemma B, we have $\nu(x) \leq \exp(2x^{b+\sigma})$ for all $x \geq r_1$. Applying (38) with r_0 equal to r_1, and observing that for $r_1 \leq x \leq r$, $\nu(x)/x \leq \nu(x) \leq \nu(r)$ (since $r_1 > 1$ and ν is increasing), we obtain

(39) $\log \mu(r) \leq \log \mu(r_1) + (r - r_1)\exp(2r^{b+\sigma})$ for $r \geq r_1$.

Hence clearly, there exists $r_2 > r_1$ such that

(40) $\log \mu(r) \leq \exp(3r^{b+\sigma})$ for all $r > r_2$.

Now it is proved in [5; p. 106] that the following relation holds between $M(r)$ and $\mu(r)$: For any $\varepsilon' > 0$, there exists $r_3 > r_2$ such that if we remove from $(r_3, +\infty)$ an infinite sequence of exceptional finite open intervals (V_s, V'_s) for which

(41) $\sum_{s=1}^{\infty} (\log V'_s - \log V_s)$ converges,

and for which we may assume that

(42) $V'_s < V_{s+1}$ for all s and $\lim_{s\to\infty} V'_s = +\infty$,

then for all r in the set $(r_3, +\infty) - G$, where $G = \bigcup_{s=1}^{\infty} (V_s, V_s')$, we have

(43) $M(r) < \mu(r)(\log \mu(r))^{\varepsilon' + (1/2)}$.

We apply the above with $\varepsilon' = 1/2$, and using (40) we obtain for $r > r_3$ and $r \notin G$,

(44) $M(r) < (\exp(\exp(3r^{b+\sigma}))\exp(3r^{b+\sigma})$.

Hence clearly there exists $r_4 > r_3$ such that

(45) $M(r) < \exp(\exp(4r^{b+\sigma}))$ for $r > r_4$ and $r \notin G$.

We now prove,

LEMMA C. - There exists $r_5 > 1$ such that $M(r) < \exp(\exp(8r^{b+\sigma}))$ for all $r > r_5$.

PROOF. - By the convergence of the series (41), clearly $\lim_{s\to\infty} (V_s'/V_s) = 1$, $\lim_{s\to\infty} (V_s'/V_s)^{b+\sigma} = 1$. Since also, $\lim_{s\to\infty} V_s' = +\infty$ (by (42)), there exists t_0 such that

(46) $V_t' > r_4$ (where r_4 is as in (45)) and $(V_t'/V_t)^{b+\sigma} < 2$, for $t \geq t_0$. We show that if r_5 is taken to be V_{t_0}', then the conclusion of Lemma C holds. Let $r > r_5$.

If $r \notin G$, then by (45), $M(r) < \exp(\exp(8r^{b+\sigma}))$.

If $r \in G$, then there exists t_1 such that $r \in (V_{t_1}, V_{t_1}')$. Since $V_{t_0}' < r < V_{t_1}'$, clearly $t_1 > t_0$ (by (42)), so (46) holds for $t = t_1$. Now clearly the endpoint V_{t_1}' is not in G and so by (45), $M(V_{t_1}') < \exp(\exp(4(V_{t_1}')^{b+\sigma}))$. By (46), we thus obtain, $M(V_{t_1}') < \exp(\exp(8(V_{t_1})^{b+\sigma}))$. But $V_{t_1} < r$ so $M(V_{t_1}') < \exp(\exp(8r^{b+\sigma}))$, and hence since M is increasing, we have $M(r) < \exp(\exp(8r^{b+\sigma}))$. This proves Lemma C.

Since $M(r) < \exp(\exp(8r^{b+\sigma}))$ for $r > r_5$ by Lemma C, it clearly follows that there exists $r_0 > 1$ such that $M(r) < \exp(\exp(r^{b+2\sigma}))$ for $r > r_0$. But by choice of σ (i.e., (8)), we have $b + 2\sigma < \lambda$, so $M(r) < \exp(\exp(r^{\lambda}))$ for $r > r_0$, This concludes the proof of the theorem stated in § 2.

4. - REMARK. - If d is the maximum of the orders of the coefficients f_{kj} of $\Omega(z, y, y') = 0$, then by our theorem, we know that for any entire solution h and any $\lambda > d$, the inequality $M(r; h) \leq \exp(\exp(r^{\lambda}))$ is valid for all sufficiently large r. We give here an example where this inequality cannot be replaced (for sufficiently large r) by an inequality of the form $M(r; h) \leq \leq \exp(\exp(Kr^{d}))$ where K is a constant. To see this, let f be an entire function of finite order $d > 0$ and maximal type [2; p. 324]. (For example, we can take $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is Euler's Gamma function, and in this case $d = 1$). Then $f'(z)$ will also be of order d and maximal type. Let $h(z) = \exp(f(z))$.

Then h is an entire solution of $y' - f'(z)y = 0$, and by our theorem, we know that for $\lambda > d$, $M(r;\ h) \leq \exp(\exp r^{\lambda})$ for all sufficiently large r. We assert that there is no constant K such that the inequality $M(r;\ h) \leq \exp(\exp(Kr^d))$ holds for all sufficiently large r. If we assume the contrary and let $A(r) = \max_{|z|=r} Re(f(z))$, then clearly we would have $\exp(A(r)) = M(r;\ h) \leq \exp(\exp(Kr^d)$ for all sufficiently large r. But by a theorem of BOREL [5; p. 19], for all sufficiently large r, we have $M(r;\ f) \leq (R/(R-r))[4A(R) + 3|f(0)|]$ if $R > r$. Taking $R = 2r$, we clearly would obtain, $M(r;\ f) \leq \exp((K2^d + 1)r^d)$ for all sufficiently large r, which contradicts the fact that f is of maximal type. Hence no such K can exist.

BIBLIOGRAPHY

[1] E. HILLE, *Analytic Function Theory*. Volume II, Ginn and Company, Boston, 1962.

[2] S. SAKS and A. ZYGMUND, *Analytic Functions*, Warsaw, 1952.

[3] G. VALIRON, *Fonctions Analytiques*, Presses Universitaires de France, Paris, 1954.

[4] ——, *Sur les fonctions entieres vérifiant une classe d'équations différentielles*, Bull. Soc. Math. France, 51 (1923), 33-45.

[5] ——, *Lectures on the general theory of integral functions*, Edouuard Privat, Toulouse, 1923.

[6] ——, *Les théorèmes généraux de M. Borel dans la théorie des fonctions entières*, Ann. Ecole norm., t. 37 (1920), 219-253.

[7] A. WIMAN, *Ueber den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem grössten Gliede der zugehörigen Taylorschen Reihe*, Acta Math., 37 (1914), pp. 305-326.

[8] ——, *Ueber den Zusamenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem grössten Betrage bei gegebenem Argumente der Funktion*, Acta Math., 41 (1916), pp. 1-28.